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A note on Hayman's theorem on the bass note of a drum

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Recently Hayman settled a long-outstanding problem in the theory of vibrating membranes [9]. He showed, in effect, that in order for a drum to produce an arbitrarily deep note it is necessary that it include an arbitrarily large circular drum.

We shall discuss later the background of this problem. To state Hayman's result precisely, let D be a plane domain; the *inradius* ρ of D is the maximum radius of a disk included in D. Define $\Lambda \geq 0$ by

$$\Lambda^2 = \inf_{f \in F} \frac{\iint_D |\nabla f|^2}{\iint_D f^2} \tag{1}$$

where F is the family of smooth functions $f \not\equiv 0$ with compact support in D. Hayman showed that if D is simply connected, then

$$\Lambda > \frac{1}{30\rho}. \tag{2}$$

If D has a smooth boundary, then Λ is equal (under suitable normalization) to the lowest frequency of a vibrating homogeneous membrane in the shape of D, fixed along the boundary of D, and (2) shows that this frequency has a uniform lower bound in terms of ρ, independent of the total size and shape of D.

One also denotes Λ^2 by λ_1, the smallest eigenvalue of the boundary-value problem

$$\Delta u + \lambda u \quad \text{in} \quad D, \qquad u = 0 \quad \text{on} \quad \partial D. \tag{3}$$

The purpose of this note is to derive an improved form of Hayman's inequality. Our argument also allows a number of extensions to multiply-connected domains, and to domains on surfaces. On the other hand, Hayman's method generalizes to higher dimensions, which does not seem to be the case for ours.

The result is the following

THEOREM. *Let D be a two-dimensional Riemannian manifold. Denote by K its Gauss curvature, and by ρ the supremum of the distance to the boundary of points of D. If Λ is defined by* (1), *then*

(a) *if D is a simply-connected or doubly-connected plane domain with the euclidean metric, then*

$$\Lambda \geq \frac{1}{2\rho}; \tag{4}$$

(b) *if D is a plane domain of connectivity $k \geq 2$ with the euclidean metric, then*

$$\Lambda \geq \frac{1}{k\rho}; \tag{5}$$

(c) *the inequality* (4) *also holds if D is simply connected, and if $K \leq 0$ on D, or more generally, if $\iint_D K^+ \leq 2\pi$; in particular, if D lies on a hemisphere;*

(d) *the stronger inequality*

$$\Lambda \geq \frac{\alpha}{2 \tanh \alpha\rho} \tag{6}$$

holds if D is a simply-connected domain with $K \leq -\alpha^2$, $\alpha > 0$.

The proof of the theorem uses a slight modification of Cheeger's estimate for λ_1 [4], together with appropriate isoperimetric inequalities. We consider the latter first.

LEMMA 1. *Let M be a two-dimensional manifold, and let D be a domain on M bounded by a finite number of smooth curves. Let A be the area of D, L the total length of its boundary curves, ρ the inradius, and K the Gauss curvature of D. Then*

(i) *if D is a simply-connected plane domain with the euclidean metric, then*

$$\rho L \geq A + \pi\rho^2; \tag{7}$$

(ii) *if D is a plane domain of connectivity $k \geq 2$ with the euclidean metric, then*

$$\rho L \geq 2A/k; \tag{8}$$

(iii) *if D is simply connected, then*

$$\rho L \geq A + \left(\pi - \frac{1}{2}\iint_D K^+\right)\rho^2; \tag{9}$$

(iv) *if D is a simply-connected domain with* $K \leq -\alpha^2$, $\alpha > 0$, *then*

$$L \geq \frac{\alpha}{\tanh \alpha\rho} A + \frac{2\pi}{\alpha} \tanh \frac{\alpha\rho}{2}. \tag{10}$$

COROLLARY. *If D is either a doubly-connected plane domain with the euclidean metric, or else a simply-connected domain satisfying* $\iint_D K^+ \leq 2\pi$, *then*

$$\frac{L}{A} \geq \frac{1}{\rho}. \tag{11}$$

Remarks. 1. Inequality (11) is sharp for doubly-connected domains, with equality for a circular annulus. For simply-connected domains, inequality (11) is strict. However, as Santaló has noted ([17], p. 155) no inequality of the same form with a constant greater than 1 on the right side can hold for all simply-connected domains, as one sees by considering long thin rectangles.

2. Inequality (7) has a long and curious history. It was first stated explicitly by Bonnesen in 1921 ([2], p. 222), although an equivalent inequality appears earlier in Chisini ([7], p. 296). Both Bonnesen and Chisini gave proofs only for convex domains. The first proof of (7) for arbitrary (simply-connected) domains is contained implicitly in a review by Scherk [18] of a paper by Santaló [17]. Santaló proved (11), and Scherk pointed out that the proof actually gives a stronger inequality, which turns out to be equivalent to (7). Later Besicovitch [1], apparently unaware that (7) was known, gave a new proof; he also gave the first characterization of domains where equality holds. Finally, Burago and Zalgaller proved (iii), which of course includes (i) as a special case, but they were apparently unaware of all the previous work in this direction.(1)

3. Part (iv) follows from Theorem 1 in Ionin [12]. Thus the only new part of the lemma is (ii). We shall give a proof of (ii) by a method which also proves (i).

Let D be a plane domain of connectivity k. Let $A(t)$ be the area of the

[1] (Added March 8, 1977). The history is in fact still longer and more curious. Mention should also be made of papers by H. Hadwiger and F. Fiala in Commentarii Math. Helvetici 13 (1940/41). Hadwiger, on page 199, proves yet another inequality equivalent to (7); Fiala, on page 336, proves not only (7), but also (9), for the case of analytic Jordan curves, assuming that the metric is also analytic. These and related matters will be discussed in more detail in a forthcoming paper of the author on Bonnesen-type inequalities.

subdomain of D consisting of points whose distance to the boundary is less than t. Then one knows that $A'(t)$ exists for almost all t, $A'(t) \leq L + 2\pi(k-2)t$, and

$$A = \int_0^\rho A'(t)\, dt \leq \rho L + \pi(k-2)\rho^2. \tag{12}$$

(see Sz-Nagy [19], p. 46, Hartman [8], p. 722, Ionin [12], Lemma 1.)

When D is simply-connected, $k = 1$, and (12) reduces to (7), thus proving (i).

When D is doubly-connected, (12) reduces to (8) for the case $k = 2$.

When $k > 2$, we consider two cases.

Case 1. $A \geq \pi k \rho^2$. Then by (12),

$$\frac{\rho L}{A} \geq 1 - \frac{\pi(k-2)\rho^2}{A} \geq 1 - \frac{\pi(k-2)\rho^2}{\pi k \rho^2} = \frac{2}{k}.$$

Case 2. $A \leq \pi k \rho^2$. Using the fact that D includes a disk of radius ρ, and that the outer boundary curve of D surrounds this disk, one has $L \geq 2\pi\rho$. Thus (8) holds in both cases, and the lemma is proved.

We next prove Cheeger's inequality, in the form that we shall need.

LEMMA 2. *Let D be a plane domain of connectivity k endowed with a Riemannian metric. Let F_k be the family of relatively compact subdomains of D having smooth boundary and connectivity at most k. Let*

$$h = \inf_{D' \in F_k} \frac{L'}{A'}, \tag{13}$$

where A' is the area of D', and L' the length of its boundary. Then if Λ is defined by (1),

$$\Lambda \geq \frac{h}{2}. \tag{14}$$

Proof. First of all, it follows immediately from the definitions (1) and (13), that if one proves (14) for all domains in a regular exhaustion of D, then (14) will also hold for D. Since every finitely-connected domain has a regular exhaustion by domains of the same connectivity, we may assume that D has a smooth boundary. Then the boundary-value problem (3) has a solution f corresponding to $\lambda = \lambda_1$, and one has

$$\Lambda^2 = \lambda_1 = \frac{\iint_D |\nabla f|^2}{\iint_D f^2}. \tag{15}$$

Furthermore f cannot change sign in D, and we may assume that $f \geq 0$.

Following Cheeger, we set $g=f^2$. Then $|\nabla g|=2f|\nabla f|$, and by Schwarz's inequality,

$$\left[\iint_D |\nabla g|\right]^2 \leq 4\iint_D f^2 \cdot \iint_D |\nabla f|^2.$$

From (15) we deduce that

$$\Lambda \geq \frac{1}{2}\frac{\displaystyle\iint_D |\nabla g|}{\displaystyle\iint_D g}. \tag{16}$$

For regular values t of the function g, we set

$$D_t = \{p \in D : g(p) > t\}.$$

What we must show is that the connectivity of D_t is at most k. Since t is a regular value of g, the boundary C_t of D_t consists of a finite number, say m, of smooth curves along which $\nabla g \neq 0$. If m were greater than k, then some component C' of C_t would not contain any component of the boundary of D in its interior, so that the interior D' of C' would lie entirely in D. But the function f satisfies $\Delta f = -\lambda_1 f \leq 0$ in D, hence f is superharmonic. Since $f=\sqrt{t}$ on C', it follows that $f \geq \sqrt{t}$ in D'. But this contradicts the fact $g<t$ in the part of D' near C'.

Thus $m \leq k$, and by the definition (13),

$$L(t) \geq hA(t) \tag{17}$$

for all regular values of t, where $L(t)$ is the length of C_t and $A(t)$ the area of D_t. Since the set of singular values of g is a closed set of measure zero, its complement is a countable union of open intervals I_n. The domain E_n defined by

$$E_n = \{p \in D : g(p) \in I_n\}$$

is foliated by the level lines of g. Using arc length along these level lines and along their orthogonal trajectories as parameters, one finds

$$\iint_{E_n} |\nabla g| = \int_{I_n} L(t)\, dt. \tag{18}$$

On the other hand, one has

$$\iint_D g = \int_0^\infty A(t)\,dt, \tag{19}$$

since both sides of (19) represent the volume of the three-dimensional domain

$$G = \{(x, y, z) : (x, y) \in D, \qquad 0 < z < g(x, y)\},$$

$A(t)$ being the area of the cross-section of G with the plane $z = t$.

Combining (17), (18) (19) yields

$$\begin{aligned}\iint_D |\nabla g| &\geq \sum_{n=1}^{\infty} \iint_{E_n} |\nabla g| = \sum_{n=1}^{\infty} \int_{I_n} L(t)\,dt \\ &\geq h \sum_{n=1}^{\infty} \int_{I_n} A(t)\,dt = h \int_0^\infty A(t)\,dt = h \iint_D g.\end{aligned} \tag{20}$$

In view of (16), the lemma is proved.

Finally, combining the various parts of Lemma 1 and its corollary with Lemma 2, together with the observation that for all subdomains D' of D, $\rho(D') \leq \rho(D)$, one obtains the statement of the Theorem.

Concerning the proof of this theorem, we note that Cheng [5] has made similar use of Cheeger's inequality together with the isoperimetric inequality (9) of Burago and Zalgaller to obtain a lower bound for the first positive eigenvalue of a compact surface of non-negative curvature. He neglects to mention the need of some modification of Cheeger's result, such as Lemma 2 above, to make the argument complete.

We note next a number of related results.

First of all, by the definition of Λ, it is immediate that $\Lambda(D) \leq \Lambda(D')$ whenever $D' \subset D$. Since the value of Λ for a disk of radius ρ is known explicitly to be j/ρ, where j is the first zero of the Bessel function J_0, one has a trivial upper bound for $\Lambda\rho$ for arbitrary plane domains. Combined with (4), this gives

$$\tfrac{1}{2} < \rho\Lambda \leq j \sim 2.4 \tag{21}$$

for all simply-connected or doubly-connected plane domains.

In the case of *convex* plane domains, Hersch [10] obtained the sharp bounds

$$\frac{\pi}{2} < \rho\Lambda \leq j, \tag{22}$$

where the left-hand side is the limiting value of $\rho\Lambda$ for long thin rectangles.

It is possible that (22) holds for all simply-connected plane domains, but it is not true for simply-connected domains with vanishing Gauss curvature.[(2)] For example, the domain D defined in polar coordinates by

$$D : \pi < r < 2\pi, \qquad 0 < \theta < 2\pi,$$

can be considered as a domain lying on the Riemann surface of $\log z$. Its inradius is $\rho = \pi/2$, and its lowest frequency is $\Lambda = 1$, corresponding to the eigenfunction

$$f = \frac{\sin r}{\sqrt{r}} \sin \frac{\theta}{2}.$$

Thus $\rho\Lambda = \pi/2$. Extending D past $\theta = 2\pi$, for example by adding a semicircle of radius $\pi/2$, one obtains a larger domain D' with same value of ρ, for which $\rho\Lambda < \pi/2$.

The left-hand side of (22) is also considerably too large for doubly-connected domains. For example, if D_ε is the annulus $\varepsilon < r < 1$, and D the unit disk, then

$$\Lambda(D_\varepsilon) \to \Lambda(D) = j \quad \text{as} \quad \varepsilon \to 0, \tag{23}$$

whereas $\rho(D_\varepsilon) = (1-\varepsilon)/2 \to \frac{1}{2}$, so that

$$\inf \rho\Lambda \leq \frac{j}{2} \sim 1.2$$

for doubly-connected domains.

The validity of (23) is a special case of the following lemma, whose statement and proof were suggested by Walter Hayman.

LEMMA. *Let D be a plane domain, and D_ε the domain obtained by removing from D a finite number of disjoint circular disks of radius ε centred at a fixed set E of points in D. Then*

$$\lim_{\varepsilon \to 0} \Lambda(D_\varepsilon) = \Lambda(D). \tag{24}$$

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[2] The referee has kindly pointed out that even for simply-connected plane domains, the constant on the left of (22) must be reduced. Namely, the function $f = r^{-1/2} \sin r \sin \frac{1}{2}\theta$ is also an eigenfunction corresponding to the eigenvalue $\lambda_1 = 1$ in the domain $0 < r < \pi$, $0 < \theta < 2\pi$, for which $\rho = \pi/2$. Extending this domain to the right preserves ρ and decreases Λ.

Proof. Using the definition (1) of Λ, we show that for any competing function f with compact support in D, we can construct corresponding functions f_ε with support in D_ε, such that

$$\limsup_{\varepsilon \to >0} \frac{\iint_{D_\varepsilon} |\nabla f_\varepsilon|^2}{\iint_D f_\varepsilon^2} \le \frac{\iint_D |\nabla f|^2}{\iint_D f^2}. \tag{25}$$

Since $D_\varepsilon \subset D \Rightarrow \Lambda(D_\varepsilon) \ge \Lambda(D)$, (24) follows from (25).

It is sufficient to construct f_ε which are piecewise smooth rather than smooth. To do so, choose $r_2 > r_1 > \varepsilon$ such that the disks of radius r_2 with centers at the points of E are still disjoint. Define a function u equal to 1 outside the disks of radius r_2, equal to zero inside the disks of radius r_1 and of the form

$$\log \frac{r}{r_1} \Big/ \log \frac{r_2}{r_1}$$

in each of the annular domains in between. Let $f_\varepsilon = uf$. Then in each of the annular domains one has the estimates

$$\begin{gathered} |\nabla f_\varepsilon|^2 \le |\nabla f|^2 + \left[\frac{2}{r}|f|\,|\nabla f| \log \frac{r}{r_1} + \frac{f^2}{r^2}\right] \Big/ \left(\log \frac{r_2}{r_1}\right)^2, \\ \iint_{r_1 \le r \le r_2} |\nabla f_\varepsilon|^2 \le \pi M^2 \left[(r_2^2 - r_1^2) + \frac{2(r_2 - r_1)}{\log (r_2/r_1)} + \frac{2}{\log (r_2/r_1)}\right], \end{gathered} \tag{26}$$

where $M \ge \{\max_D |f|, \max_D |\nabla f|\}$. If the dependence of r_1, r_2 on ε is such that $r_2 \to 0$ and $r_2/r_1 \to \infty$ as $\varepsilon \to 0$ (for example, $r_1 = 2\varepsilon$, $r_2 = 2\sqrt{\varepsilon}$, for small ε), then since $f_\varepsilon \equiv f$, $\nabla f_\varepsilon \equiv \nabla f$ on D_{r_2}, (25) follows from (26), and the lemma is proved.

As an application of the lemma, let D be the unit square $0 < x < 1$, $0 < y < 1$, and let E be the set of $(n-1)^2$ points $(r/n, s/n)$, r, $s = 1, \ldots, n-1$. Then

$$\Lambda(D_\varepsilon) \to \Lambda(D) = \pi\sqrt{2},$$

$$\rho(D_\varepsilon) \to \frac{\sqrt{2}}{2n},$$

while the connectivity k of D_ε is $k = (n-1)^2 + 1$. Thus, if F_k is the family of domains of connectivity k, then for $k \ge 2$,

$$\frac{1}{k} \le \inf_{D \in F_k} \rho\Lambda < \frac{\pi}{\sqrt{k}}, \tag{27}$$

where the left-hand side is from (5). This example shows that the right-hand side of (5) must tend to zero as $k \to \infty$, but it raises the question whether the bound can be improved to something of the order of $1/\sqrt{k}$.

Concerning part (d) of the Theorem, one has a result for arbitrary (not necessarily simply-connected) domains D in the hyperbolic plane

$$\frac{1}{2} \le \frac{1}{2\tanh R} \le \Lambda \le \frac{1}{2}\sqrt{\left[1+\left(\frac{2\pi}{\rho}\right)^2\right]} \tag{28}$$

where ρ is again the inradius, and R is the *circumradius:* the radius of the smallest geodesic disk including D. In fact, the left-hand side of (28) holds for domains D on any complete simply-connected surface with $K \le -1$, while the right-hand side holds when $K \ge -1$. The left-hand side follows by noting that the method of Yau ([20], p. 498) shows that for any subdomain D' of D, one has the inequality $L \ge A \coth R$. The right-hand side follows from results of Cheng ([6], p. 290 and 294). An equivalent form of (28), using $\lambda_1 = \Lambda^2$, is

$$\left(\frac{1}{\sinh R}\right)^2 \le \lambda_1 - \frac{1}{4} \le \left(\frac{\pi}{\rho}\right)^2. \tag{29}$$

Not directly connected with the inradius, but in a spirit related to the methods of this paper, are the inequalities

$$\frac{\pi}{4}\frac{L}{A} \le \Lambda < \frac{\pi}{2}\frac{L}{A} \tag{30}$$

for certain plane domains D of area A and boundary length L. The left-hand inequality holds for convex domains, and was proved by Makai [14] using Hersch's inequality (22). The right-hand side was proved by Pólya [15] for convex domains, but the same argument can also be used for arbitrary simply-connected or doubly-connected domains (see Hersch [11], p. 134). Both constants are best possible, the left as the limiting case of a narrow circular sector, and the right as the limiting case of a long thin rectangle.

Finally, instead of the inradius ρ, one often considers the *maximum inner radius* $\dot{r}$ of D, defined via conformal mapping (see Pólya–Szegö [16], p. 2). It follows immediately from its definition, using Schwarz's Lemma, that $\dot{r} \ge \rho$. Thus it follows from (4) that

$$\Lambda \dot{r} > \tfrac{1}{2} \tag{31}$$

for arbitrary simply-connected domains in the plane. In the book of Pólya and Szegö ([16], Table 1.21 on page 17) it is asserted that $A\dot{r}$ has a positive lower bound k for convex domains, and it is conjectured that $k = 2$. On the bottom of page 16 they say "Nothing is asserted about the lower bound for unrestricted plane domains; not even a conjecture is offered." Hayman's theorem settles the existence of a positive lower bound, and inequality (31) gives what may be a candidate for the best constant.

REFERENCES

[1] A. S. BESICOVITCH, *A variant of a classical isoperimetric problem*, Quarterly J. Math., Oxford Ser. *20* (1949), 84–94.

[2] T. BONNESEN, *Über eine Verschärfung der isoperimetrischen Ungleichheit des Kreises in der Ebene und auf der Kugeloberfläche nebst einer Anwendung auf eine Minkowskische Ungleichheit für konvexe Körper*, Math. Ann. *84* (1921), 216–227.

[3] J. D. BURAGO and V. A. ZALGALLER, Isoperimetric problems for regions on a surface having restricted width, pp. 100–108 of *Two-Dimensional Manifolds of Bounded Curvature*, Proc. Steklov Inst. Math. No. *76* (1965); English translation, Amer. Math. Soc., Providence 1967.

[4] J. CHEEGER, A lower bound for the smallest eigenvalue of the Laplacian, pp. 195–199 of *Problems in Analysis, A Symposium in Honor of Salomon Bochner*, ROBERT C. GUNNING, Editor, Princeton Univ. Press, Princeton, N.J. 1970.

[5] S.-Y. CHENG, Eigenfunctions and eigenvalues of Laplacian, pp. 185–193 of *Differential Geometry*, Proceedings of Symposia in Pure Mathematics, Vol. *27*, Part 2, Amer. Math. Soc., Providence 1975.

[6] S.-Y. CHENG, *Eigenvalue comparison theorems and its geometric applications*, Math. Z. *143* (1975), 289–297.

[7] O. CHISINI, Sulla teoria elementare degli isoperimetri, pp. 201–310 of Vol. *3* of Enriques: *Questioni riguardanti le matematiche elementari*, Bologna 1914.

[8] P. HARTMAN, *Geodesic parallel coordinates in the large*, Amer. J. Math. *86* (1964), 705–727.

[9] W. K. HAYMAN, *Some bounds for principal frequency* (to appear in Applicable Analysis).

[10] J. HERSCH, *Sur la fréquence fondamentale d'une membrane vibrante. Evaluation par défaut et principe de maximum*. Z. Angew. Math. Mech. *11* (1960), 387–413.

[11] J. HERSCH, Contribution to the method of interior parallels applied to vibrating membranes, pp. 132–139 of *Studies in Mathematical Analysis and Related Topics*, Essays in honor of George Pólya, Stanford University Press, Stanford 1962.

[12] V. K. IONIN, *Isoperimetric and various other inequalities on a manifold of bounded curvature*, Siberian Math. J. *10*, No. 2, (1969), 233–243.

[13] E. MAKAI, *Bounds for the principal frequency of a membrane and the torsional rigidity of a beam*, Acta Sci. Math. (Acta Univ. Szeged.) *20* (1959), 33–35.

[14] E. MAKAI, On the principal frequency of a membrane and the torsional rigidity of a beam, pp. 227–231 of *Studies in Mathematical Analysis and Related Topics*, Essays in honor of George Pólya, Stanford University Press, Stanford 1962.

[15] G. PÓLYA, *Two more inequalities between physical and geometrical quantities*, J. Indian Math. Soc. (N.S.) *24* (1960), 413–419.

[16] G. PÓLYA and G. SZEGÖ, *Isoperimetric Inequalities in Mathematical Physics*, Ann. of Math. Studies *27*, Princeton Univ. Press, Princeton, 1951.

[17] L. A. SANTALÓ, *Sobre el circulo de radio maximo contenido en un recinto*, Revisto Unión Mat. Argentina *10* (1945), 155–167.

[18] P. SCHERK, Review of [17] above, Math. Rev. *7* (1946), 168–169.
[19] B. SZ.-NAGY, *Über Parallelmengen nichtkonvexer ebener Bereiche*, Acta Sci. Math. (Acta Univ. Szeged.) *20* (1959), 36–47.
[20] S.-T. YAU, *Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold*, Ann. scient. École Norm. Sup. (4th Ser.) *8* (1975), 487–507.

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