manuscripta math. 77, 169 - 189 (1992)

manuscripta
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ISOPERIMETRIC INEQUALITIES ON MINIMAL SUBMANIFOLDS OF SPACE FORMS

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For a domain U on a certain k-dimensional minimal submanifold of S^n or H^n, we introduce a "modified volume" $M(U)$ of U and obtain an optimal isoperimetric inequality for U $k^k\omega_k M(D)^{k-1} \leq Vol(\partial D)^k$, where ω_k is the volume of the unit ball of $\mathbf{R}^k$. Also, we prove that if D is any domain on a minimal surface in S^n_+ (or H^n respectively), then D satisfies an isoperimetric inequality $2\pi A \leq L^2 + A^2$ ($2\pi A \leq L^2 - A^2$ respectively). Moreover, we show that if U is a k-dimensional minimal submanifold of H^n, then $(k-1)\,Vol(U) \leq Vol(\partial U)$.

Let C be a simple closed curve in the plane, bounding the domain D. Let the length of C be L and the area of D be A. Then the classical isoperimetric inequality states that

$$4\pi A \leq L^2$$

with equality if and only if C is a circle. If D is a domain on the sphere, the sharp isoperimetric inequality for D was given by F. Bernstein [2]:

$$4\pi A \leq L^2 + KL^2,$$

where K is the Gauss curvature of the sphere. In fact, in this form it is valid both for the sphere and for the plane, where $K \equiv 0$. One might guess that it would hold equally for the hyperbolic plane H^2, where $K \equiv -1$. Schmidt [9] showed that this turns out to be the case.

*Supported in part by KME and GARC

Considering that minimal surfaces are viewed as generalized planes, it is tempting to conjecture that these sharp isoperimetric inequalities should still hold for domains on any immersed minimal surfaces in space forms, $\mathbf{R}^n, S^n, H^n$. The truth of this conjecture is not yet known, except in special cases. For example, it was shown for minimal surfaces in $\mathbf{R}^n$ with radially connected boundary in [4] and later for analogous minimal surfaces in H^n in our companion paper [5].

In this paper we introduce a *modified volume* $M(D)$ of a domain D and obtain an isoperimetric inequality without the additional term that Bernstein and Schmidt had:

$$k^k \omega_k M(D)^{k-1} \leq \mathit{Volume}(\partial D)^k,$$

where ω_k is the volume of the unit ball of $\mathbf{R}^k$, and D is a domain on a certain k-dimensional minimal submanifold of a space form. This modified isoperimetric inequality is proved under any of the following conditions: (Theorem 4) ∂D lies on a geodesic sphere of the space form; (Theorem 2) $k = 2$, $D \subset S^n_+$ (the hemisphere) and ∂D is weakly connected; (Theorem 1) $k = 2$, $D \subset S^n_+$ and ∂D is radially connected about a point of D. Observe that D is *not* assumed to have minimum volume. The inequality is sharp for a geodesic ball in a totally geodesic submanifold Σ, and appears to be a plausible conjecture for any minimal submanifold of S^n_+ or $\mathbf{R}^n$.

While we need an assumption of radial connectivity for a partial solution of the above conjectures, L. Simon (see [3, p.318], [8, Theorem 5.3]) made no connectedness assumption and obtained a weaker isoperimetric inequality

$$2\pi A \leq L^2$$

for any minimal surface in $\mathbf{R}^n$. In the second part of this paper we extend his method and derive (Theorem 5)

$$2\pi A \leq L^2 + A^2$$

for a minimal surface in S^n, and

$$2\pi A \leq L^2 - A^2$$

for a minimal surface in H^n. We further extend this method and show (Theorem 6) that if Σ is a k-dimensional minimal submanifold of H^n, then

$$(k-1)\ \mathit{Volume}(\Sigma) \leq \mathit{Volume}(\partial\Sigma).$$

This is an exact extension of Yau's result for a domain in H^k [10]. More isoperimetric inequalities of this kind are proved. Finally, to each isoperimetric inequality of these types, we obtain corresponding Sobolev-type inequalities for smooth nonnegative functions with compact support on minimal surfaces (see Corollaries 1, 2, and 3).

We would like to thank Walter Wei for bringing Simon's work to our attention.

1 Modified volume

We assume throughout this paper that D and Σ are differentiable up to its boundary, and that their closures are compact.

Definition 1 Let p be a point in the n-dimensional sphere S^n and let $r(x)$ be the distance from p to x in S^n. Given a k-dimensional submanifold Σ in S^n, the *modified volume* of Σ with center at p is defined to be

$$M_p(\Sigma) = \int_\Sigma \cos r.$$

Similarly for Σ in the n-dimensional hyperbolic space H^n, we define the *modified volume* of Σ by

$$M_p(\Sigma) = \int_\Sigma \cosh r.$$

Obviously we have in S^n

$$M_p(\Sigma) \leq \mathit{Volume}(\Sigma),$$

and in H^n

$$M_p(\Sigma) \geq \mathit{Volume}(\Sigma).$$

Lemma 1 *(a) Suppose that S^n is embedded in $\mathbf{R}^{n+1}$ with p the north pole $(0,\ldots,0,1)$. Then the modified volume of a domain $U \subset S^n$ is the Euclidean volume of the orthogonal projection of U into the horizontal hyperplane $x_{n+1} = 0$, counting orientation. (b) In case U is a domain in H^n, embed H^n isometrically onto the hypersurface S, $x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1$, $x_{n+1} > 0$, of $\mathbf{R}^{n+1}$ with the Minkowski metric $ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2$ such that p becomes the point $(0,\ldots,0,1) \in S$. Then $M_p(U)$ equals the Euclidean volume of the projection of U onto the hyperplane $x_{n+1} = 0$.*

Proof. Part (a) follows from the fact that $\cos r$ is the Jacobian of the projection of S^n into $x_{n+1} = 0$. For part (b), since the metric on H^n and the distance function r are rotationally symmetric about p, it suffices to consider only the case where $U \subset H^1$. Let $x = (x_1, x_2) \in H^1$, $x_1^2 - x_2^2 = -1$. Then $x_1 dx_1 = x_2 dx_2$ and hence

$$\begin{aligned} r(x) &= \int_p^x ds = \int_0^{x_1} \sqrt{dx_1^2 - dx_2^2} = \int_0^{x_1} \sqrt{1 - (x_1/x_2)^2}\, dx_1 \\ &= \int_0^{x_1} \frac{1}{\sqrt{1 + x_1^2}} dx_1 = \log\left(x_1 + \sqrt{1 + x_1^2}\right). \end{aligned}$$

Hence $x_1 = \sinh r$. Therefore

$$dx_1 = \cosh r dr,$$

which gives the desired result.

In the following lemma, as usual, the notation Vf, a vector (field) V followed by a function f, implies the directional derivative of f in the direction V.

Lemma 2 *Let f be a smooth function on an n-dimensional Riemannian manifold M and Σ a k-dimensional submanifold of M. Let $\overline{\nabla}$ and $\overline{\Delta}$ be the connection and Laplacian on M respectively, and Δ the Laplacian on Σ. If H is the mean curvature vector of Σ in M, then*

$$\Delta(f|\Sigma) = (\overline{\Delta} f)|\Sigma + Hf - \sum_{\alpha=k+1}^{n} \overline{\nabla}^2 f(\bar{e}_\alpha, \bar{e}_\alpha),$$

where $\bar{e}_{k+1}, \ldots, \bar{e}_n$ are orthonormal vectors which are perpendicular to Σ.

Proof. Let $e_1, \ldots, e_k$ be orthonormal vector fields on a neighborhood of a point of Σ. These vector fields can be smoothly extended to orthonormal vector fields $\bar{e}_1, \ldots, \bar{e}_n$ on M such that $\bar{e}_i = e_i$, $1 \le i \le k$, on Σ. Then $\sum_{i=1}^k \overline{\nabla}_{\bar{e}_i} \bar{e}_i = H + \sum_{i=1}^k \nabla_{e_i} e_i$, and hence we have

$$\begin{aligned} \Delta f &= \sum_{i=1}^{k} (e_i e_i f - \nabla_{e_i} e_i f) \\ &= \sum_{i=1}^{n} \bar{e}_i \bar{e}_i f - \sum_{\alpha=k+1}^{n} \bar{e}_\alpha \bar{e}_\alpha f - \sum_{i=1}^{n} \overline{\nabla}_{\bar{e}_i} \bar{e}_i f + Hf + \sum_{\alpha=k+1}^{n} \overline{\nabla}_{\bar{e}_\alpha} \bar{e}_\alpha f \\ &= \overline{\Delta} f + Hf - \sum_{\alpha=k+1}^{n} \overline{\nabla}^2 f(\bar{e}_\alpha, \bar{e}_\alpha) \end{aligned}$$

Lemma 3 *Let $\overline{\nabla}$ and g be the connection and metric tensor respectively of S^n or H^n. On S^n, $\overline{\nabla}^2 \cos r = -(\cos r)g$. On H^n, $\overline{\nabla}^2 \cosh r = (\cosh r)g$.*

Proof. Let p be the point from which the distance r is measured. Let $\frac{\partial}{\partial r}, V_1, \ldots, V_{n-1}$ be perpendicular vector fields on a neighborhood of $q \in S^n$ such that $|V_i| = \sin r$, that $\overline{\nabla}_{V_i} V_j(q)$ is parallel to $\frac{\partial}{\partial r}$, and that V_i are Jacobi fields along any geodesic through p. Clearly $\overline{\nabla}_{V_i} \frac{\partial}{\partial r}$ is parallel to V_i and $[V_i, \frac{\partial}{\partial r}] = 0$. Hence

$$\langle \overline{\nabla}_{V_i} \frac{\partial}{\partial r}, V_i \rangle = \langle \overline{\nabla}_{\partial/\partial r} V_i, V_i \rangle = \frac{1}{2} \frac{\partial}{\partial r} \langle V_i, V_i \rangle = \sin r \cos r,$$

and $\overline{\nabla}_{V_i} \frac{\partial}{\partial r} = \cot r V_i$. So

$$\langle \overline{\nabla}_{V_i} V_j, \frac{\partial}{\partial r} \rangle = -\langle V_j, \overline{\nabla}_{V_i} \frac{\partial}{\partial r} \rangle = -\delta_{ij} \sin r \cos r,$$

where δ_{ij} is the Kronecker delta. Hence $\overline{\nabla}_{V_i} V_j(q) = -\delta_{ij} \sin r \cos r \frac{\partial}{\partial r}$. Therefore

$$\begin{aligned} \overline{\nabla}^2 \cos r(V_i, V_j)(q) &= V_i V_j \cos r - \overline{\nabla}_{V_i} V_j \cos r \\ &= -\delta_{ij} \sin^2 r \cos r = -\cos r g(V_i, V_j). \end{aligned}$$

Moreover,

$$\begin{aligned} \overline{\nabla}^2 \cos r(\frac{\partial}{\partial r}, V_j)(q) &= \frac{\partial}{\partial r} V_j \cos r - \overline{\nabla}_{\frac{\partial}{\partial r}} V_j \cos r \overline{\nabla}_{V_j} \frac{\partial}{\partial r} \cos r \\ &= \sin r \cos r V_j (\cos r) = 0 = -\cos r g\left(\frac{\partial}{\partial r}, V_j\right). \end{aligned}$$

Also,

$$\begin{aligned} \overline{\nabla}^2 \cos r(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) &= \frac{d^2}{dr^2} \cos r - \overline{\nabla}_{\partial/\partial r} \frac{\partial}{\partial r} \cos r \\ &= -\cos r = -\cos r g(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}). \end{aligned}$$

Thus we have the proof for S^n. A similar proof holds for H^n.

Lemma 4 *(a) If $\Sigma \subset S^n$ is a k-dimensional minimal submanifold or a cone, and r is the distance in S^n from a fixed point, then*

$$\Delta \cos r = -k \cos r,$$

where Δ is the Laplacian on the submanifold Σ and, in case Σ is a cone, r is the distance from the center of Σ.

(b) Suppose Σ is a k-dimensional minimal submanifold or a cone in H^n. Then

$$\Delta \cosh r = k \cosh r.$$

Here again, in case of a cone Σ, r is the distance from the center of Σ.

Proof. The proof for a minimal submanifold follows from *Lemma 2* and *Lemma 3*. On a cone $\Sigma \subset H^n$, we see from *Lemma 2* that

$$\begin{aligned}\Delta \cosh r &= \overline{\Delta} \cosh r + H \cosh r - \sum_{\alpha=k+1}^{n} \overline{\nabla}^2 \cosh r(\bar{e}_\alpha, \bar{e}_\alpha) \\ &= n \cosh r + \sinh r H r - (n-k) \cosh r = k \cosh r,\end{aligned}$$

since H is perpendicular to $\partial/\partial r$ on the cone; similarly for a cone $\Sigma \subset S^n$.

Given a $(k-1)$-dimensional submanifold M of S^n or H^n and a point p in S^n or H^n, the k-dimensional *cone* $p \ast\!\ast M$ with center at p is defined to be the set of all minimizing geodesics from p to the points of M.

Proposition 1 *For any minimal submanifold Σ with boundary in S^n or H^n, and any point $p \in S^n$ or H^n,*

$$M_p(\Sigma) \leq M_p(p \ast\!\ast \partial\Sigma).$$

Proof. Let ν and η be the unit conormals to $\partial\Sigma$ on Σ and $p \ast\!\ast \partial\Sigma \subset S^n$ respectively. Then it is easy to see, as in [4, *Proposition 1*], that

$$\frac{\partial r}{\partial \nu} \leq \frac{\partial r}{\partial \eta}.$$

Hence

$$\begin{aligned}M_p(\Sigma) &= -\frac{1}{k}\int_\Sigma \Delta \cos r = \frac{1}{k}\int_{\partial\Sigma} \sin r \frac{\partial r}{\partial \nu} \leq \frac{1}{k}\int_{\partial\Sigma} \sin r \frac{\partial r}{\partial \eta} \\ &= -\frac{1}{k}\int_{p \ast\!\ast \partial\Sigma} \Delta \cos r = M_p(p \ast\!\ast \partial\Sigma).\end{aligned}$$

The same argument applies to $\Sigma \subset H^n$.

Lemma 5 *(a) On S^n, $\overline{\nabla}^2 r = \cot r(g - dr \otimes dr)$. On H^n, $\overline{\nabla}^2 r = \coth r(g - dr \otimes dr)$. (b) Let Σ be a k-dimensional minimal submanifold or a cone from p in S^n or H^n. Then $\Delta r = \cot r(k - |\nabla r|^2)$ for $\Sigma \subset S^n$, and $\Delta r = \coth r(k - |\nabla r|^2)$ for $\Sigma \subset H^n$.*

Proof. (a) Using *Lemma 3*, we compute

$$\begin{aligned}-(\cos r)g &= \overline{\nabla}^2 \cos r = -\cos r \overline{\nabla} r \otimes \overline{\nabla} r - \sin r \overline{\nabla}^2 r, \\ (\cosh r)g &= \overline{\nabla}^2 \cosh r = \cosh r \overline{\nabla} r \otimes \overline{\nabla} r + \sinh r \overline{\nabla}^2 r,\end{aligned}$$

on S^n and H^n respectively. The results follow immediately.

(*b*) From *Lemma 4*, we have

$$\begin{aligned} -k\cos r &= \Delta\cos r = div(-\sin r\nabla r) \\ &= -\cos r|\nabla r|^2 - \sin r\Delta r, \end{aligned}$$

which gives the result; similarly for $\Sigma \subset H^n$.

Lemma 6 *Let $G(x)$ be Green's function of S^k (H^k respectively), whose derivative is $\sin^{1-k} x$ for $0 < x < \pi$ ($\sinh^{1-k} x$ for $0 < x < \infty$ respectively). If Σ is a k-dimensional minimal submanifold of S^n (H^n respectively), then $G \circ r$ is subharmonic on $\Sigma - \{\pm p\} \subset S^n$ (on $\Sigma - \{p\} \subset H^n$ respectively). (Note : When $k = 2$, $G(r) = G \circ r = \log\tan\frac{r}{2}$, and when $k = 3$, $G(r) = -\cot r$ on $\Sigma \subset S^n$.)*

Proof.

$$\begin{aligned} \Delta G(r) &= div(\sin^{1-k} r\nabla r) \\ &= (1-k)\sin^{-k} r \cdot \cos r|\nabla r|^2 + \sin^{-k} r \cdot \cos r(k - |\nabla r|^2) \\ &= k\sin^{-k} r \cdot \cos r(1 - |\nabla r|^2) \geq 0. \end{aligned}$$

The proof for $\Sigma \subset H^n$ is similar.

Definition 2 Let $C \subset S^n$ be a k-dimensional rectifiable set and p a point in S^n such that $dist(p,q) < \pi$ for all $q \in C$. The k-dimensional *angle* $A^k(C,p)$ of C viewed from p is defined by setting

$$A^k(C,p) = \sin^{-k} t \cdot Volume[(p \ast\!\!\ll C) \cap S(p,t)],$$

where $S(p,t)$ is the geodesic sphere of radius $t < dist(p,C)$ centered at p, and the volume is measured counting multiplicity. Clearly the angle does not depend on t. There is obviously an analogous definition for the angle of $C \subset H^n$ viewed from $p \in H^n$.

Note that

$$A^k(C,p) = (k+1)\omega_{k+1}\Theta^{k+1}(p \ast\!\!\ll C, p),$$

where $\Theta^{k+1}(p \ast\!\!\ll C, p)$ is the $(k+1)$-dimensional density of $p \ast\!\!\ll C$ at p.

Proposition 2 *Let Σ be a k-dimensional compact minimal submanifold with boundary in S^n or H^n, and let p be an interior point of Σ. In case of $\Sigma \subset S^n$, we assume that $dist(p,q) \leq \frac{\pi}{2}$ for all $q \in \Sigma$. Then*

$$A^{k-1}(\partial\Sigma, p) \geq k\omega_k.$$

Equality holds if and only if Σ is a domain on a totally geodesic submanifold that is star-shaped with respect to p.

Proof. Let $B^n(p,t) \subset S^n$ be the geodesic ball of radius t centered at p, $t < dist(p, \partial\Sigma)$, $\Sigma_t = \Sigma \sim B^n(p,t)$, and $C_t = \Sigma \cap S(p,t)$. *Lemma 6* says that $G(r)$, $r = dist(p,\cdot)$, is subharmonic. Hence

$$\begin{aligned} 0 &\leq \int_{\Sigma_t} \Delta G(r) = \int_{\Sigma_t} div(\sin^{1-k} r \nabla r) \\ &= -\int_{\partial\Sigma_t} \sin^{1-k} r \cdot \frac{\partial r}{\partial \nu} + \int_{\partial\Sigma} \sin^{1-k} r \cdot \frac{\partial r}{\partial \nu}, \end{aligned}$$

where ν on $\partial\Sigma_t$ is the unit normal to $\partial\Sigma_t$ away from p on $\Sigma \cap B^n(p,t)$. Since $\frac{\partial r}{\partial \nu} \to 1$ and $\frac{\sin t}{t} \to 1$ on $\partial\Sigma_t$ as $t \to 0$, we have

$$\begin{aligned} k\omega_k &= \lim_{t\to 0} \int_{\partial\Sigma} \sin^{1-k} r \cdot \frac{\partial r}{\partial \nu} \leq \int_{\partial\Sigma} \sin^{1-k} r \cdot \frac{\partial r}{\partial \nu} \\ &\leq \int_{\partial\Sigma} \sin^{1-k} r \cdot \frac{\partial r}{\partial \eta} = A^{k-1}(\partial\Sigma, p), \end{aligned}$$

where η is the same as in *Proposition 1*. Equality holds if and only if $\Delta G(r) = 0$, $\Theta^k(\Sigma, p) = 1$, and $\nu = \eta$ if and only if Σ is a star-shaped minimal cone with density at the center equal to 1. Since S^{k-1} is the only $(k-1)$-dimensional minimal submanifold in S^n with volume $k\omega_k$, we conclude that Σ lies in a totally geodesic submanifold of S^n. A similar proof holds for $\Sigma \subset H^n$.

2 Isoperimetric inequality for modified volume

For a minimal surface Σ in $\mathbf{R}^n$, the isoperimetric inequality $4\pi A \leq L^2$ has been proved whenever $\partial\Sigma$ is connected, or more generally, whenever there is a point $p \in \mathbf{R}^n$ such that no sphere centered at p separates $\partial\Sigma$ [4]. We shall prove an analogous result for modified area in S^n_+:

Theorem 1 *Let Σ be a compact minimal surface with boundary in S^n. Assume that r is the distance in S^n from a point p of Σ and that $r \leq \frac{\pi}{2}$ on Σ. If $\partial\Sigma$ is radially*

connected from p, that is, $\{s : s = dist(p,q),\ q \in \partial\Sigma\}$ *is a connected interval, then* $4\pi M_p(\Sigma) \leq Length(\partial\Sigma)^2$. *Equality holds if and only if* Σ *is a totally geodesic disk with center at p.*

Proof. By *Proposition 1*, $M_p(\Sigma) \leq M_p(p*\partial\Sigma)$. For each geodesic sphere $S(p,t)$ centered at p , one has a local isometry between the curve $S(p,t) \cap (p*\partial\Sigma)$ and a great circle on $S(p,t)$. Therefore, developing $p*\partial\Sigma$ on a great sphere $S^2 \subset S^n$, one can find a curve C (not necessarily closed) in S^2 and a local isometry from $p*\partial\Sigma$ into $\bar{p}*C$ where $\bar{p}$ is the north pole of S^2. Clearly we have $M_p(p*\partial\Sigma) = M_{\bar{p}}(\bar{p}*C)$, $Length(\partial\Sigma) = Length(C)$, and $A^1(\partial\Sigma, p) = A^1(C, \bar{p})$. Moreover, if we let $\bar{q}$ and $\bar{\bar{q}}$ be the end points of C, then $dist(\bar{p},\bar{q}) = dist(\bar{p},\bar{\bar{q}})$. First, let us assume that p is an interior point of Σ. We see from *Proposition 2* that $A^1(\partial\Sigma, p) \geq 2\pi$ and that C has a self-intersection point. Now, from *Lemma 1*, it follows that $M_{\bar{p}}(\bar{p}*C)$ equals the Euclidean area of the standard projection of $\bar{p}*C$ onto the plane containing the equator of S^2. Let C' be the image of C under the projection and 0 the origin of the plane. Then $M_{\bar{p}}(\bar{p}*C) = Area(0*C')$, $A^1(C,\bar{p}) = A^1(C',0)$ and $Length(C) \geq Length(C')$. The last inequality follows from the fact that the projection is a length-decreasing map. Furthermore, if q' and q'' are the end points of C', then $dist(0,q') = dist(0,q'')$. Thus we can apply *Lemma 1* of [4] and conclude that

$$4\pi Area(0*C') \leq Length(C')^2,$$

and equality holds if and only if C' is the boundary of a circle. Therefore

$$4\pi M_p(\Sigma) \leq Length(\partial\Sigma)^2,$$

and equality holds if and only if $Length(C') = Length(\partial\Sigma)$ and $A^1(\partial\Sigma, p) = 2\pi$, or equivalently, Σ is a totally geodesic disk centered at p. Second, if p is a boundary point of Σ, then we apply *Corollary 1* instead of *Lemma 1* of [4].

Remark 1 The same inequality is false for minimal surfaces in H^n. In fact, among domains in H^2 with prescribed boundary length, the modified area has no upper bound. Our proof fails in H^n because the projection in Minkowski space $\mathbf{R}^{n+1}$ from H^n onto the hyperplane $x_{n+1} = 0$ is a length-increasing map. In *Theorem 4* below, however, we shall prove the inequality in the special case that $\partial\Sigma$ lies in a sphere of H^n centered at p.

Li, Schoen and Yau [7] have proved the isoperimetric inequality for a minimal surface Σ in $\mathbf{R}^n$, provided that $\partial\Sigma$ is weakly connected, that is, no coordinate hy-

perplane separates $\partial\Sigma$. The analogous hypothesis for Σ^2 in S^n suffices to prove an isoperimetric inequality for modified area:

Theorem 2 *Let Σ be a compact minimal surface with boundary in $S^n \subset \mathbf{R}^{n+1}$ and r the distance in S^n from the north pole $p = (0,\dots,0,1) \in S^n$. Assume without loss of generality that the center of mass of $\partial\Sigma$ in $\mathbf{R}^{n+1}$ lies on the line $l = \{(x_1,\dots,x_{n+1}) : x_1 = \cdots = x_n = 0\}$. Assume also that $\partial\Sigma$ is weakly connected : there exists a rectangular coordinate system $\{y_1,\dots,y_n\}$ in the hyperplane $x_{n+1} = 0$ such that, $\mathbf{R}^{n+1}$ being equipped with the coordinate system $\{y_1,\dots,y_n,x_{n+1}\}$, no hyperplanes of $\mathbf{R}^{n+1}$ $y_i = const$, $1 \le i \le n$, ever separate $\partial\Sigma$. Then $4\pi M_p(\Sigma) \le Length(\partial\Sigma)^2$.*

Proof. Let us modify Li,Schoen and Yau's argument in [7]. Assume $x_i = y_i$ $(1 \le i \le n)$. For any fixed $1 \le i \le n$, it follows from the weak connectedness of $\partial\Sigma$ that each component of $\partial\Sigma$ can be translated in a proper direction perpendicular to $\partial/\partial x_i$, so that after translation, their union forms a connected one-dimensional simplicial complex C_i. Observe here that

$$\int_{\partial\Sigma} x_i = \int_{C_i} x_i, \quad \int_{\partial\Sigma} x_i^2 = \int_{C_i} x_i^2, \quad \int_{\partial\Sigma} \left(\frac{dx_i}{ds}\right)^2 = \int_{C_i} \left(\frac{dx_i}{ds}\right)^2,$$

and $Length(\partial\Sigma) = Length(C_i)$.

By hypothesis,

$$\int_{\partial\Sigma} x^i = 0, \quad i = 1,\dots,n.$$

Hence the Poincaré inequality applied to C_i says that

$$\int_{C_i} x_i^2 \le \frac{1}{4\pi^2} Length(C_i)^2 \int_{C_i} \left(\frac{dx_i}{ds}\right)^2,$$

where s is the arclength parameter of C_i. Thus

$$\begin{aligned}
M_p(\Sigma) &= -\int_\Sigma \frac{1}{2}\Delta \cos r = \frac{1}{2}\int_{\partial\Sigma} \sin r \frac{\partial r}{\partial \nu} \\
&\le \frac{1}{2}\int_{\partial\Sigma} |\sin r| \le \frac{1}{2}\left[\int_{\partial\Sigma} 1\right]^{1/2}\left[\int_{\partial\Sigma} \sin^2 r\right]^{1/2} \\
&= \frac{1}{2} Length(\partial\Sigma)^{1/2}\left[\int_{\partial\Sigma} \sum_{i=1}^n x_i^2\right]^{1/2} \\
&\le \frac{1}{2} Length(\partial\Sigma)^{1/2}\left[\frac{1}{4\pi^2}\sum_{i=1}^n Length(C_i)^2 \int_{C_i}\left(\frac{dx_i}{ds}\right)^2\right]^{1/2} \\
&\le \frac{1}{4\pi} Length(\partial\Sigma)^{3/2}\left[\int_{\partial\Sigma}\sum_{i=1}^n \left(\frac{dx_i}{ds}\right)^2\right]^{1/2}
\end{aligned}$$

$$\leq \frac{1}{4\pi} Length(\partial\Sigma)^{3/2} Length(Proj(\partial\Sigma))^{1/2}$$
$$\leq \frac{1}{4\pi} Length(\partial\Sigma)^2.$$

Closely related to isoperimetric inequalities on a minimal submanifold Σ is the monotonicity property: the volume of $\Sigma \cap B(p,r)$, divided by the corresponding volume for a totally geodesic manifold of the same dimension, is a nondecreasing function of r. This property has been proved for $\mathbf{R}^n$ and for H^n [1], but not for S^n (see however [6], p.353). Our next result shows that modified volume enjoys the monotonicity property in all three cases.

Theorem 3 *Let Σ be a k-dimensional minimal submanifold in S^n (H^n respectively) and r the distance in S^n (H^n respectively) from p. Then $\sin^{-k} r \cdot M_p(\Sigma \cap B^n(p,r))$ ($\sinh^{-k} r \cdot M_p(\Sigma \cap B^n(p,r))$ respectively) is a monotonically nondecreasing function of r for $0 < r < \min(\frac{\pi}{2}, dist(p,\partial\Sigma))$ ($0 < r < dist(p,\partial\Sigma)$ respectively).*

Proof. Define $\Sigma_r = \Sigma \cap B^n(p,r) \subset S^n$. Then

$$M_p(\Sigma_r) = -\frac{1}{k}\int_{\Sigma_r} \Delta \cos r$$
$$= \frac{1}{k}\int_{\partial\Sigma_r} \sin r \frac{\partial r}{\partial \nu} = \frac{1}{k}\sin r \int_{\partial\Sigma_r} |\nabla r|.$$

Denote the volume forms on Σ and $\partial\Sigma_r$ by dv and dS_r. Then we have $dv = \frac{1}{|\nabla r|} dS_r dr$. So

$$\frac{d}{dr}\int_{\Sigma_r} \cos r |\nabla r|^2 dv = \frac{d}{dr}\int_0^r \int_{\partial\Sigma_r} \cos r |\nabla r| dS_r dr = \cos r \int_{\partial\Sigma_r} |\nabla r|.$$

Therefore

$$M_p(\Sigma_r) = \frac{1}{k}\frac{\sin r}{\cos r}\cos r \int_{\partial\Sigma_r} |\nabla r| = \frac{1}{k}\frac{\sin r}{\cos r}\frac{d}{dr}\int_{\Sigma_r} \cos r |\nabla r|^2$$
$$\leq \frac{1}{k}\frac{\sin r}{\cos r}\frac{d}{dr}\int_{\Sigma_r} \cos r = \frac{1}{k}\frac{\sin r}{\cos r}\frac{d}{dr} M_p(\Sigma_r).$$

In the above inequality we used the fact that $r \leq \frac{\pi}{2}$ and $|\nabla r|^2 \leq 1$ on Σ. Hence

$$\frac{d}{dr}\log\left[\sin^{-k} r \cdot M_p(\Sigma_r)\right] \geq 0.$$

Thus $\sin^{-k} r \cdot M_p(\Sigma_r)$ is nondecreasing; similarly for $\Sigma \subset H^n$.

A very special case of the hypothesis of radially connected boundary occurs when $\partial\Sigma$ lies on a geodesic sphere. In this case, the conclusion of *Theorem 1* may be extended to hyperbolic space, and the minimal submanifold may have any dimension.

Theorem 4 *Let Σ be a k-dimensional minimal submanifold in S^n or H^n. Assume that $\partial\Sigma$ lies on a geodesic sphere centered at a point p of Σ and that r is the distance in S^n or H^n from p. Furthermore, in case of $\Sigma \subset S^n$, assume $r \leq \frac{\pi}{2}$ on Σ. Then*

$$k^k \omega_k M_p(\Sigma)^{k-1} \leq Volume(\partial\Sigma)^k.$$

Equality holds if and only if Σ is a totally geodesic ball centered at p.

Proof. Assume $\Sigma \subset H^n$ and let R be the radius of the geodesic sphere in which $\partial\Sigma$ lies. Then

$$\begin{aligned} M_p(\Sigma) &= \frac{1}{k}\int_\Sigma \Delta \cosh r = \frac{1}{k}\int_{\partial\Sigma} \sinh r \frac{\partial r}{\partial \nu} \\ &= \frac{1}{k}\sinh R \int_{\partial\Sigma} \frac{\partial r}{\partial \nu} \leq \frac{1}{k}\sinh R \cdot Volume(\partial\Sigma). \end{aligned}$$

Since $\lim_{r\to 0} \sinh^{-k} r \cdot M_p(\Sigma \cap B^n(p,r)) = \omega_k$, we obtain from *Theorem 3*,

$$\sinh^{-k} R \cdot M_p(\Sigma) \geq \omega_k.$$

Hence

$$M_p(\Sigma) \leq \frac{1}{k}\omega_k^{-1/k} M_p(\Sigma)^{1/k} Volume(\partial\Sigma)$$

and so the desired inequality follows. Obviously equality holds if and only if Σ is a cone with density at the center equal to 1, or equivalently, Σ is a totally geodesic ball. A similar proof holds for $\Sigma \subset S^n$.

3 Isoperimetric inequalities via Fubini's theorem

All the theorems of preceding section require that $\partial\Sigma$ satisfy some hypotheses related to connectivity. Without such hypotheses in $\mathbf{R}^n$, the strongest inequality known for minimal surfaces is due to Leon Simon. Simon used Fubini's theorem to obtain a weaker isoperimetric inequality for a minimal surface in $\mathbf{R}^n$ (see [3, p.318], [8, *Theorem 5.3*]) :

$$2\pi Area(\Sigma) \leq Length(\partial\Sigma)^2.$$

The key idea, rephrased in our setting, is to integrate the angle of $\partial\Sigma$ from the points all over Σ. In this section we extend his method from $\mathbf{R}^n$ to S^n and H^n, and derive various isoperimetric inequalities in space forms. Let us first reprove Simon's theorem.

Lemma 7 (*a*) *Let D and g be the Euclidean connection and the metric respectively, and let r be the distance in $\mathbf{R}^n$ from a fixed point p. Then*

$$D^2 r = \frac{1}{r}(g - dr \otimes dr).$$

(*b*) $D^2 r^2 = 2g$.

(*c*) *Let Σ be a k-dimensional minimal submanifold of $\mathbf{R}^n$. Then*

$$\Delta r^\alpha = \alpha r^{\alpha-2}(k + (\alpha - 2)|\nabla r|^2).$$

(*d*) $\Delta \log r = \frac{1}{r^2}(k - 2|\nabla r|^2)$.

Proof. Conclusions (a) and (b) are standard computations in orthogonal coordinates of $\mathbf{R}^n$. Conclusions (c) and (d) follow from *Lemma 2* and straightforward computations.

Proof of Simon's theorem. By the above lemma, $\log r$ is subharmonic on a minimal surface Σ. From [4, *Proposition 2*], or by following the same argument as in *Proposition 2* of this paper, we have

$$2\pi \leq A^1(\partial\Sigma, p),$$

where p is an interior point of Σ.
More precisely,

$$2\pi \leq \int_{\partial\Sigma} \frac{1}{r_p}\frac{\partial r}{\partial \nu} \leq \int_{\partial\Sigma} \frac{1}{r_p},$$

where r_p indicates that the distance r is measured from a fixed p. Letting p range over all points of Σ and integrating the above inequality over Σ, we get

$$2\pi Area(\Sigma) \leq \int_\Sigma \int_{\partial\Sigma} \frac{1}{r_p}.$$

Then by Fubini's theorem,

$$2\pi Area(\Sigma) \leq \int_{\partial\Sigma} \int_\Sigma \frac{1}{r_p}.$$

Now by *Lemma 7* (*c*) with $\alpha = 1$,

$$\Delta r \geq \frac{1}{r}.$$

Hence

$$2\pi Area(\Sigma) \leq \int_{\partial\Sigma}\int_\Sigma \Delta r = \int_{\partial\Sigma}\int_{\partial\Sigma} \frac{\partial r}{\partial \nu} \leq \int_{\partial\Sigma}\int_{\partial\Sigma} 1 = Length(\partial\Sigma)^2,$$

which completes the proof.

The key point in this proof is the nice relationship between the gradient of $\log r$ and the Laplacian of r. In order to apply this method to a minimal surface in a space form, we need to choose suitable functions of r and compute their Laplacians.

Lemma 8 *Let Σ be a k-dimensional minimal submanifold in S^n.*
(a) $\Delta \log \sin r = k \cot^2 r - \frac{|\nabla r|^2}{\sin^2 r}(1 + \cos^2 r)$.
(b) $\Delta \log \cos r = -k - |\nabla r|^2 \tan^2 r$.
Assume $\Sigma \subset H^n$.
(c) $\Delta \log \sinh r = k \coth^2 r - \frac{|\nabla r|^2}{\sinh^2 r}(1 + \cosh^2 r)$.
(d) $\Delta \log \cosh r = k - |\nabla r|^2 \tanh^2 r$.

Proof. *(a)*

$$\begin{aligned}
\Delta \log \sin r &= div\left(\frac{\cos r}{\sin r}\nabla r\right) \\
&= -\frac{1}{\sin^2 r}|\nabla r|^2 + \frac{\cos r}{\sin r}\Delta r \\
&= -\frac{1}{\sin^2 r}|\nabla r|^2 + \frac{\cos^2 r}{\sin^2 r}\left(k - |\nabla r|^2\right) \\
&= k\frac{\cos^2 r}{\sin^2 r} - \frac{|\nabla r|^2}{\sin^2 r}\left(1 + \cos^2 r\right).
\end{aligned}$$

Here we used *Lemma 5* *(b)* for Δr.
(b)

$$\begin{aligned}
\Delta \log \cos r &= div\left(-\frac{\sin r}{\cos r}\nabla r\right) = -\frac{1}{\cos^2 r}|\nabla r|^2 - \frac{\sin r}{\cos r}\Delta r \\
&= -k + |\nabla r|^2\left(1 - \frac{1}{\cos^2 r}\right).
\end{aligned}$$

A similar proof holds for $\Sigma \subset H^n$.

Lemma 9 *On a k-dimensional minimal submanifold Σ of S^n,*

$$\Delta \sin r = \frac{1}{\sin r}(k \cos^2 r - |\nabla r|^2).$$

Proof.

$$\begin{aligned}
\Delta \sin r &= div(\cos r \nabla r) = -\sin r|\nabla r|^2 + \cos r \Delta r \\
&= -\frac{\sin^2 r + \cos^2 r}{\sin r}|\nabla r|^2 + k\frac{\cos^2 r}{\sin r} \\
&= \frac{1}{\sin r}(k \cos^2 r - |\nabla r|^2).
\end{aligned}$$

Theorem 5 (*a*) *Let* Σ *be a minimal surface in* S^n *with* $diam(\Sigma) \leq \frac{\pi}{2}$. *Let* $A = Area(\Sigma)$ *and* $L = Length(\partial\Sigma)$. *Then*

$$2\pi A \leq L^2 + A^2.$$

(*b*) *For a minimal surface* Σ *in* H^n,

$$2\pi A \leq L^2 - A^2.$$

Proof. From *Lemma 8* (*a*) we have

$$\begin{aligned}\Delta \log \sin r &= \frac{2\cos^2 r}{\sin^2 r} - \frac{|\nabla r|^2}{\sin^2 r}(1+\cos^2 r) \\ &= \frac{\cos^2 r}{\sin^2 r} + \frac{1-\sin^2 r}{\sin^2 r} - \frac{|\nabla r|^2}{\sin^2 r}(1+\cos^2 r) \\ &= \frac{1-|\nabla r|^2}{\sin^2 r}(1+\cos^2 r) - 1 \geq -1.\end{aligned}$$

Therefore, if we choose $r = r_p$ where $p \in \Sigma$, we find

$$\begin{aligned}-A &= \int_\Sigma -1 \leq \int_\Sigma \Delta \log \sin r \\ &= -\lim_{t\to 0} \int_{\partial\Sigma_t} \frac{\cos r}{\sin r}\frac{\partial r}{\partial \nu} + \int_{\partial\Sigma} \frac{\cos r}{\sin r}\frac{\partial r}{\partial \nu},\end{aligned}$$

where ν on $\partial\Sigma_t$ is the outer conormal on $\Sigma \cap B^n(p,t)$. Note that on $\partial\Sigma_t$,

$$r\frac{\cos r}{\sin r} \to 1 \quad \text{and} \quad \frac{\partial r}{\partial \nu} \to 1 \quad \text{as } t \to 0.$$

Hence for $r \leq \frac{\pi}{2}$,

$$-A \leq -2\pi + \int_{\partial\Sigma} \frac{\cos r_p}{\sin r_p},$$

where r_p is the distance from a fixed point p. Let p range over Σ and integrate both sides of the inequality over Σ :

$$-A^2 \leq \int_\Sigma -2\pi + \int_\Sigma \int_{\partial\Sigma} \frac{\cos r_p}{\sin r_p} = -2\pi A + \int_{\partial\Sigma}\int_\Sigma \frac{\cos r_p}{\sin r_p}.$$

By *Lemma 5* (*b*),

$$\Delta r \geq \cot r.$$

Hence

$$2\pi A - A^2 \leq \int_{\partial\Sigma}\int_\Sigma \Delta r = \int_{\partial\Sigma}\int_{\partial\Sigma} \frac{\partial r}{\partial \nu} \leq L^2.$$

(*b*) *Lemma 8* (*c*) states that

$$\begin{aligned}\Delta \log \sinh r &= \frac{2\cosh^2 r}{\sinh^2 r} - \frac{|\nabla r|^2}{\sinh^2 r}(1+\cosh^2 r) \\ &\geq \frac{2\cosh^2 r - 1 - \cosh^2 r}{\sinh^2 r} = 1.\end{aligned}$$

Hence

$$A \leq \int_\Sigma \Delta \log \sinh r \leq -2\pi + \int_{\partial\Sigma} \frac{\cosh r}{\sinh r},$$

and

$$\begin{aligned} A(A+2\pi) &\leq \int_\Sigma \int_{\partial\Sigma} \frac{\cosh r}{\sinh r} = \int_{\partial\Sigma} \int_\Sigma \frac{\cosh r}{\sinh r} \\ &\leq \int_{\partial\Sigma} \int_\Sigma \Delta r = \int_{\partial\Sigma} \int_{\partial\Sigma} \frac{\partial r}{\partial \nu} \leq L^2. \end{aligned}$$

Remark 2 Evidently, one can rewrite Simon's theorem and *Theorem 5* $(a), (b)$ in a single form

$$2\pi A \leq L^2 + KA^2,$$

where K is the sectional curvature of the space form that the minimal surface Σ sits in.

Due to the analytic nature of the proof of *Theorem 5*, we can derive, applying the same argument, the Sobolev-type inequalities corresponding to the above isoperimetric inequalities. As is well known, one can recover the isoperimetric inequalities from the Sobolev-type inequalities, using smoothed characteristic functions as test functions.

Corollary 1 *Let f be a compactly supported smooth nonnegative function on a minimal surface Σ in S^n or H^n. In case $\Sigma \subset S^n$, assume also that the diameter in S^n of the support of f is $\leq \frac{\pi}{2}$. Then*

$$2\pi \int_\Sigma f^2 \leq \left(\int_\Sigma |\nabla f| \right)^2 + K \left(\int_\Sigma f \right)^2,$$

where $K = 1$ or -1 depending on whether $\Sigma \subset S^n$ or $\Sigma \subset H^n$.

Proof. For $\Sigma \subset S^n$, we have

$$div(f \nabla \log \sin r) \geq \langle \nabla f, \cot r \nabla r \rangle - f.$$

Integrating both sides, we see that for a fixed $y \in \Sigma$,

$$2\pi f(y) \leq \int_\Sigma |\nabla f(x)| \cdot \cot r_y(x) dA_x + \int_\Sigma f(x) dA_x,$$

where $r_y(x) = dist(x, y)$. Moreover

$$div(f \nabla r) \geq \langle \nabla f, \nabla r \rangle + f \cot r.$$

So

$$\int_\Sigma f(y)\cot r_x(y)dA_y \le \int_\Sigma |\nabla f(y)|dA_y.$$

Therefore

$$\begin{aligned} 2\pi\int_\Sigma f^2 &\le \int_\Sigma f(y)\left(\int_\Sigma |\nabla f(x)|\cdot\cot r_y(x)dA_x + \int_\Sigma f(x)dA_x\right)dA_y \\ &= \int_\Sigma |\nabla f(x)|\left(\int_\Sigma f(y)\cot r_x(y)dA_y\right)dA_x + \left(\int_\Sigma f\right)^2 \\ &\le \left(\int_\Sigma |\nabla f|\right)^2 + \left(\int_\Sigma f\right)^2. \end{aligned}$$

A similar proof is valid for $\Sigma \subset H^n$.

The inequality of part (b) of the following theorem was proved for domains in H^k by Yau [10, p.498]. Although it is not sharp, it may be observed that the inequality is asymptotically sharp, by considering geodesic balls in H^k of large radius.

Theorem 6 (*a*) *Let* Σ *be a domain in* S^k *or a* k*-dimensional minimal submanifold of* S^n. *Assume* $diam(\Sigma) \le d \le \frac{\pi}{2}$. *Then*

$$k\cdot Volume(\Sigma) \le \tan d\cdot Volume(\partial\Sigma).$$

(*b*) *Let* Σ *be a domain in* H^k *or a* k*-dimensional minimal submanifold of* H^n. *Then*

$$(k-1)\,Volume(\Sigma) \le Volume(\partial\Sigma).$$

Proof. (*a*) *Lemma 8* (*b*) says that

$$-\Delta\log\cos r = k - |\nabla r|^2\left(1 - \frac{1}{\cos^2 r}\right) \ge k.$$

Therefore

$$k\cdot Volume(\Sigma) \le \int_\Sigma -\Delta\log\cos r \le \int_{\partial\Sigma}\tan d = \tan d\cdot Volume(\partial\Sigma).$$

(*b*) From *Lemma 8* (*d*),

$$\Delta\log\cosh r = k + |\nabla r|^2\left(\frac{1}{\cosh^2 r} - 1\right) \ge k - |\nabla r|^2 \ge k-1.$$

So

$$(k-1)\,Volume(\Sigma) \le \int_\Sigma \Delta\log\cosh r \le \int_{\partial\Sigma}\frac{\sinh r}{\cosh r} \le \int_{\partial\Sigma} 1 = Volume(\partial\Sigma).$$

Corollary 2 (*a*) *Let f be a nonnegative smooth function with compact support on a k-dimensional minimal submanifold Σ in S^n. If $diam(sptf) \leq d \leq \frac{\pi}{2}$, then*

$$k \int_\Sigma f \leq \tan d \int_\Sigma |\nabla f|.$$

(*b*) *If $\Sigma^k \subset H^n$, then*

$$(k-1) \int_\Sigma f \leq \int_\Sigma |\nabla f|.$$

Proof. (*b*) From *Lemma 5(b)*, we have

$$div(f \nabla r) \geq \langle \nabla f, \nabla r \rangle + (k-1) f.$$

Integrating both sides over Σ gives the desired inequality. For (*a*), we observe

$$div(f \nabla \log \cos r) \leq -\langle \nabla f, \tan r \nabla r \rangle - kf.$$

Theorem 7 *Let Σ be a minimal surface in S^n.*
(*a*) *If $diam(\Sigma) \leq d \leq \frac{\pi}{2}$ and $c = \cos d$, then*

$$2\pi c A \leq L^2.$$

(*b*) *If $diam(\Sigma) \leq \frac{\pi}{2}$, then*

$$2\pi A \leq 2AL + L^2.$$

Proof. In the proof of *Proposition 2*, we obtained

$$2\pi \leq \int_{\partial\Sigma} \frac{1}{\sin r} \frac{\partial r}{\partial \nu} \leq \int_{\partial\Sigma} \frac{1}{\sin r}$$

if $dist(p, \partial\Sigma) \leq \frac{\pi}{2}$. From *Lemma 5* (*b*), we have for $r \leq d$

$$\Delta r \geq \frac{c}{\sin r}.$$

Hence, if $diam(\Sigma) \leq \frac{\pi}{2}$,

$$\begin{aligned} 2\pi A &\leq \int_\Sigma \int_{\partial\Sigma} \frac{1}{\sin r} = \int_{\partial\Sigma} \int_\Sigma \frac{1}{\sin r} \\ &\leq \int_{\partial\Sigma} \int_\Sigma \frac{1}{c} \Delta r \leq \int_{\partial\Sigma} \int_{\partial\Sigma} \frac{1}{c} = \frac{L^2}{c}. \end{aligned}$$

This proves (*a*). For (*b*), we use *Lemma 9* and obtain

$$\Delta \sin r \geq \frac{2\cos^2 r - 1}{\sin r} = \frac{1}{\sin r} - 2 \sin r \geq \frac{1}{\sin r} - 2.$$

Therefore

$$\begin{aligned} 2\pi A &\leq \int_{\partial\Sigma}\int_{\Sigma}\frac{1}{\sin r} \leq \int_{\partial\Sigma}\int_{\Sigma}(2+\Delta\sin r) \\ &\leq \int_{\partial\Sigma}\left(2A+\int_{\partial\Sigma}\cos r\right) \leq 2AL+\int_{\partial\Sigma}\int_{\partial\Sigma}1 \\ &= 2AL+L^2. \end{aligned}$$

Corollary 3 *Assume that Σ is a minimal surface in S^n and that $f \in C_0^\infty(\Sigma)$ and $f \geq 0$.*

(a) If $diam(sptf) \leq d \leq \frac{\pi}{2}$ and $c = \cos d$, then

$$2\pi c\int_{\Sigma} f^2 \leq \left(\int_{\Sigma}|\nabla f|\right)^2.$$

(b) If $diam(sptf) \leq \frac{\pi}{2}$, then

$$2\pi\int_{\Sigma} f^2 \leq \left(\int_{\Sigma}|\nabla f|\right)^2 + 2\left(\int_{\Sigma}|\nabla f|\right)\cdot\left(\int_{\Sigma} f\right).$$

Proof. (a) From *Lemma 5(b)* it follows that

$$\begin{aligned} div\left(f\frac{1}{\sin r}\nabla r\right) &= \langle\nabla f, \frac{1}{\sin r}\nabla r\rangle + 2f\frac{\cos r}{\sin^2 r}(1-|\nabla r|^2) \\ &\geq \langle\nabla f, \frac{1}{\sin r}\nabla r\rangle. \end{aligned}$$

Hence

$$2\pi f(y) \leq \int_{\Sigma}\frac{|\nabla f(x)|}{\sin r_y(x)}dA_x.$$

Similarly,

$$div(f\nabla r) \geq \langle\nabla f, \nabla r\rangle + \frac{cf}{\sin r},$$

and hence

$$\int_{\Sigma}\frac{cf(y)}{\sin r_x(y)}dA_y \leq \int_{\Sigma}|\nabla f(y)|dA_y.$$

Thus

$$\begin{aligned} 2\pi\int_{\Sigma} f^2 &\leq \int_{\Sigma} f(y)\left(\int_{\Sigma}\frac{|\nabla f(x)|}{\sin r_y(x)}dA_x\right)dA_y \\ &= \int_{\Sigma}|\nabla f(x)|\left(\int_{\Sigma}\frac{f(y)}{\sin r_x(y)}dA_y\right)dA_x \leq \frac{1}{c}\left(\int_{\Sigma}|\nabla f|\right)^2. \end{aligned}$$

(b) Since

$$div(f\nabla\sin r) \geq \langle\nabla f, \cos r\nabla r\rangle + f\left(\frac{1}{\sin r}-2\right),$$

we obtain

$$\int_\Sigma \frac{f(y)}{\sin r_x(y)} dA_y \leq \int_\Sigma |\nabla f(y)| dA_y + 2\int_\Sigma f(y) dA_y.$$

Therefore as in (a),

$$\begin{aligned} 2\pi \int_\Sigma f^2 &\leq \int_\Sigma |\nabla f(x)| \left(\int_\Sigma \frac{f(y)}{\sin r_x(y)} dA_y \right) dA_x \\ &\leq \int_\Sigma |\nabla f(x)| \left(\int_\Sigma |\nabla f(y)| dA_y + 2\int_\Sigma f(y) dA_y \right) dA_x \\ &= \left(\int_\Sigma |\nabla f| \right)^2 + 2 \left(\int_\Sigma |\nabla f| \right) \cdot \left(\int_\Sigma f \right). \end{aligned}$$

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(Received March 6, 1992)