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Valuations on free resolutions and higher geometric invariants of groups

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Dedicated to Beno Eckmann on the occasion of his seventieth birthday.

1. Introduction

1.1. Let G be a group. A G-module A is said to be *of type* $(FP)_m$ if A admits a resolution

$$\cdots \to F_i \to F_{i-1} \to \cdots \to F_1 \to F_0 \to A \to 0 \tag{1.1}$$

by free G-modules F_i which are finitely generated for all $i \le m$. If the trivial G-module $\mathbb{Z}$ is of type $(FP)_m$ – and this is indeed the most interesting situation – we say also that *the group G is of type* $(FP)_m$.

If a G-module A is of type $(FP)_m$ then subgroups $U \le G$ may or may not have the property that A is of type $(FP)_m$ when regarded as an U-module. Our paper aims to shed some light on the distribution of the subgroups U with respect to this dichotomy. We find that the situation is rather complex but not totally out of control if we assume that U contains the commutator subgroup G' of G. The main results have been announced in [6].

1.2. Our approach is based on and extends the "geometric invariant" Σ_A which was originally introduced by Ralph Strebel and the first author for modules A over finitely generated *Abelian* groups Q[3], [4]. Σ_A is a subset of the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$, where n is the $\mathbb{Z}$-rank of Q, and it was designed to contain the information as to whether a group G, which is an extension of Q by A, admits a finite presentation. Under joint effort with Walter D. Neumann [5] much of the theory grew up to the case when G is an arbitrary finitely generated group and A a normal subgroup containing G' and acted on by conjugation.

The present paper adds a generalization in a new direction. We introduce, for an arbitrary finitely generated group G and any G-module A, a chain of *higher* *geometric invariants*

$$S^{n-1} \supseteq \Sigma^0(G;A) \supseteq \Sigma^1(G;A) \supseteq \cdots \supseteq \Sigma^k(G;A) \supseteq \cdots$$

containing the previous invariants as the special cases $\Sigma^0(G;A)$ and $\Sigma^1(G;\mathbb{Z})$. $\Sigma^k(G;A)$ contains complete information as to which subgroups $U \leq G$ containing G' have the property that A is of type $(FP)_k$ over U.

1.3. We briefly give the definition of the higher invariants. By a *character* of G we mean a non-zero homomorphism $\chi: G \to \mathbb{R}$ into the additive group of the reals. Two characters are equivalent if they coincide up to multiplication by a positive real number. The equivalence class of a character $\chi: G \to \mathbb{R}$ thus is the straight ray from 0 through χ in $\operatorname{Hom}(G, \mathbb{R}) \cong \mathbb{R}^n$. Hence the set of all equivalence classes $[\chi]$ of characters χ has the structure of a sphere which we denote by $S(G)$. Attached to every point $[\chi] \in S(G)$ we consider the submonoid $G_\chi = \{g \mid \chi(g) \geq 0\}$ of G. Then, if A is an arbitrary left G-module and m an integer ≥ 0, we put

$$\Sigma^m(G;A) = \{[\chi] \mid A \text{ is of type } (FP)_m \text{ over } \mathbb{Z}G_\chi\}. \tag{1.2}$$

The precise relationship with the invariants of [3] and [5] is the following. 1) To say that a module A is of type $(FP)_0$ means simply that A is finitely generated. Hence $\Sigma^0(G;A)$ coincides with the invariant Σ_A of [3] by definition. 2) The invariant $\Sigma_N(G)$ of [5] is defined for an arbitrary finitely generated group G and a right G-operator group N: it consists of all points $[\chi] \in S(G)$ with the property that N is finitely generated as an operator group over a finitely generated submonoid of G_χ. It turns out that if N is the commutator subgroup G' of G, acted on by conjugation from the right,

$$\Sigma^1(G;\mathbb{Z}) = -\Sigma_{G'}(G). \tag{1.3}$$

(see Proposition 6.1). The funny sign stems from the fact that $\mathbb{Z}$, on the left hand side, is a left module, whereas in [5] we have been using right action. It would disappear if one only could agree to consistent action.

1.4. The main results of our paper are extensions of [5], Theorems A and B.

THEOREM A. *$\Sigma^m(G;A)$ is an open subset of $S(G)$ for every finitely generated group G and every G-module A.*

THEOREM B. *Let G be a finitely generated group, N a subgroup of G containing G', and A a G-module. Then A is of type $(FP)_m$ over N if and only if $\Sigma^m(G;A)$ contains the great subsphere $S(G,N) = \{[\chi] \in S(G) \mid \chi(N) = 0\}$.*

The conjunction of Theorems A and B allows a similar application as in [5]. We note that the set $\mathfrak{N}$ of all subgroups N with $G' \leq N \leq G$ and $rk_{\mathbb{Z}}(G/N) = j$ admits a natural map into the Grassmann space $\mathbb{G}_{n,j}$ of all j-dimensional linear subspaces of Hom $(G, \mathbb{R}) \cong \mathbb{R}^n$; thus $\mathfrak{N}$ carries the topology induced by $\mathfrak{N} \to \mathbb{G}_{n,j}$. If A is of type $(FP)_m$ over N then $S(G,N) \subseteq \Sigma^m(G;A)$ by Theorem B. But then, as $\Sigma^m(G;A)$ is open, it will also contain the subspheres $S(G,N_1)$ for $N_1 \in \mathfrak{N}$ sufficiently close to N. Hence we have

COROLLARY AB. *The set of all $N \in \mathfrak{N}$ with the property that A is of type $(FP)_m$ over N is open in $\mathfrak{N}$.*

In particular, *the set of all groups of type $(FP)_m$ in $\mathfrak{N}$ is open in $\mathfrak{N}$*. For $m = 2$ this is closely related to a result of Fried and Lee. Indeed, groups of type $(FP)_2$ can also be characterized by the property that they admit presentations with finitely many generators and finitely generated relation module. Thus every finitely presented group is of type $(FP)_2$ – whether or not, conversely, every group of type $(FP)_2$ is finitely presented is an open problem. The Fried–Lee result [9] asserts that the set of all *finitely presented* groups in $\mathfrak{N}$ is open in $\mathfrak{N}$.

1.5. The crucial tools for the proof of both Theorems A and B are two descriptions of $\Sigma^m(G;A)$ in terms of a free resolution of the G-module A. One of these extends (and perhaps explains) the somewhat technical "equational definition" of $\Sigma_{G'}(G)$ in [5], Section 2.

Before we give a brief sketch of these descriptions we make the following observation: the group ring $\mathbb{Z}G$ is the ascending union of the free cyclic G_χ-modules $\mathbb{Z}G_\chi g^k$, $0 \leq k \in \mathbb{Z}$, where g is an arbitrary element of G with $\chi(g) < 0$. From this we infer that $\mathbb{Z}G$ is flat as a G_χ-module and that $\mathbb{Z}G \otimes_{G_\chi} A$ is isomorphic to A for every G-module A. Consequently, we can apply the tensor product $\mathbb{Z}G \otimes_{G_\chi}$- to a finitely generated G_χ-free resolution of A in order to obtain a finitely generated G-free resolution of A. This shows that if $\Sigma^m(G;A)$ is non-empty then A is of type $(FP)_m$ over G.

So we may assume that we are given a free $\mathbb{Z}G$-resolution $\mathbf{F} \twoheadrightarrow A$ as in (1.1). For each $i \geq 0$ we pick a specific basis $X_i \subseteq F_i$ (finite for $0 \leq i \leq m$) and, without loss of generality, we may assume that $\partial x \neq 0$ for all $x \in X_i$. In Section 2 we show how one can then associate to every character $\chi: G \to \mathbb{R}$ a certain map $v: \mathbf{F} \to \mathbb{R} \cup \{\infty\}$ which formally behaves similar to a valuation on a ring and which we therefore call the valuation on $\mathbf{F}$ extending χ (with respect to the bases X_i). It is then natural to consider the "valuation complex"

$$\mathbf{F}_v = \{c \in \mathbf{F} \mid v(c) \geq 0\}.$$

Since $v(\partial c) \geq v(c)$, for all $c \in \mathbf{F}$, $\mathbf{F}_v$ is a subcomplex; but it is not, in general, exact. Its deviation from exactness in dimension $j \geq 0$ is measured by the quantity

$$D_j = \sup_z \inf_c \{v(z) - v(c) \mid 0 \neq z \in F_j,\ c \in F_{j+1},\ \partial c = z\}.$$

It is convenient to extend this definition to the case $j = -1$ by using the augmentation map $\varepsilon : F_0 \twoheadrightarrow A$,

$$D_{-1} = \sup_a \inf_c \{-v(c) \mid 0 \neq a \in A,\ c \in F_0,\ \varepsilon(c) = a\}.$$

THEOREM C. *The following three conditions are equivalent for a non-negative integer m.*

(I) $[\chi] \in \Sigma^m(G; A)$
(II) $D_j < \infty$ *for each* $-1 \leq j < m$;
(III) *The identity on A can be lifted to a chain endomorphism* $\varphi : \mathbf{F} \to \mathbf{F}$ *with the property that* $v(\varphi(x)) > v(x)$ *for all* $x \in X_i$, $0 \leq i \leq m$.

Among the three descriptions of $\Sigma^m(G; A)$ in Theorem C, Criterion (III) seems to be the most powerful one. In particular, Theorem A, the openness of $\Sigma^m(G; A)$, is an immediate consequence. For the chain endomorphism $\varphi : \mathbf{F} \to \mathbf{F}$, asserted to exist if $[\chi] \in \Sigma^m(G; A)$, will also do for every point sufficiently close to $[\chi]$.

The charm of Criterion (II), on the other hand, lies in the fact that it generalizes to a statement in terms of a *projective* resolution of A (cf. Section 3.3); whence the consequence

COROLLARY D. *If the G-module A admits a projective resolution of length* $\leq d$ *then*

$$\Sigma^m(G; A) = \Sigma^d(G; A)$$

for every $m \geq d$.

1.6. Some readers will probably find topological versions of Criteria (II) and (III) more attractive, and we do share these feelings. The topological version of (II) is stated in Proposition 6.1, and we use it to establish (1.3).

The topological translation of (III), or rather of Theorem 4.2, is crucial for a homotopical version $^*\Sigma^m(G)$ of the invariant, investigated by the second author. This will appear separately (see Section 6.5 and [12]).

1.7. EXAMPLES. In view of the ones given in [3], [4], and [5] there is certainly no shortage of examples for $\Sigma^m(G;A)$ with $m=0$ or 1. As to $m \geq 2$ our computations of examples are still rather incomplete and technical; therefore we prefer here to confine with a few easy remarks, based on our general results, and hope to come back to a more systematic treatment of examples elsewhere.

A point of the sphere $S(G)$ is said to be *rational* if it can be represented by a character $\chi: G \to \mathbb{R}$ with $\chi(G) \subseteq \mathbb{Z}$. If Σ is a subset of $S(G)$ we write Σ_{rat} for the set of all rational points in Σ. Information on $\Sigma^m(G;A)_{\text{rat}}$ is often easily available from Theorem B.

a) Let G be the *fundamental group of a 3-manifold.* Then $\Sigma^1(G;\mathbb{Z})_{\text{rat}}$ coincides with its antipodal set (see [5]). So if $[\chi] \in \Sigma^1(G;\mathbb{Z})_{\text{rat}}$ then $\Sigma^1(G;\mathbb{Z})$ contains the subsphere $S(G,N) = \{\pm[\chi]\}$, $N = \ker\chi$; hence N is finitely generated. Since N is a 3-manifold group this implies that N is, in fact, of type $(FP)_\infty$. By Theorem B, it follows that $\pm[\chi] \in \Sigma^m(G;\mathbb{Z})$ for each $m \geq 1$. Whence

$$\Sigma^m(G;\mathbb{Z})_{\text{rat}} = \Sigma^1(G;\mathbb{Z})_{\text{rat}} \text{ for all } m \geq 1.*$$

b) We find the same behaviour for G a *one relator group*. In fact, we then have even

$$\Sigma^m(G;\mathbb{Z}) = \Sigma^1(G;\mathbb{Z}), \text{ for each } m \geq 1,$$

as was pointed out to us by Walter D. Neumann. Neumann's argument was based on K. S. Brown's explicit computation of $\Sigma_{G'}(G)$ for one relator groups [8]. In Section 7 we illustrate the techniques revolving around Theorem C, Condition III, by giving new proofs of both Brown's and Neumann's result.

c) One relator groups and fundamental groups of non-closed 3-dimensional manifolds are prominent examples of *groups of cohomological dimension* ≤ 2. Because of their parallel behaviour in a) and b) above, the reader might wonder whether the assertion of Corollary D holds even for $m = d-1$, perhaps at least

* W. D. Neumann has shown that this holds without the restriction to rational points.

for $A = \mathbb{Z}$. This is *not* the case. Indeed, recall that a group N is of type $(FP)_1$, if and only if N is finitely generated. Thus the assertion $\Sigma^2(G; \mathbb{Z}) = \Sigma^1(G; \mathbb{Z})$ implies, by Theorem B, that every finitely generated subgroup $N \leq G$ containing the commutator subgroup G' is also of type $(FP)_2$. Now, let $G = F(a, b) \times F(x, y)$ be the direct product of two free groups of rank two on the exhibited generators. Then the subgroup N of G generated by $\{a, xb, yb\}$ is normal with $G/N \cong \mathbb{Z}$. But $H_2(N; \mathbb{Z})$ is not finitely generated (N can be constructed by taking the free product of two free groups of rank two, amalgamated over a free subgroup of infinite rank [13]); hence N is not of type $(FP)_2$. This shows that $\Sigma^2(G; \mathbb{Z}) \neq \Sigma^1(G; \mathbb{Z})$. Straightforward calculation along the lines of Section 7 shows that $\Sigma^2(G; \mathbb{Z})$ is, in fact, empty.

1.8. We are indebted to Ralph Strebel for a number of comments and an extended discussion on a preliminary version of this paper which have influenced our exposition. In particular, we use his comment that our original definition of $[\chi] \in \Sigma^m$ (via Criterion (II) of Theorem C) is equivalent to the $(FP)_m$-property over the submonoid G_χ. The present concise version of (III), Theorem C, and the idea of extending our techniques to projective resolutions, in order to prove Corollary D, came up in the course of that discussion. We are also grateful to Walter Neumann and Ken Brown for discussions on the case of a one relator group, and to Ross Geoghegan for tutorials on his work with Michael Mihalik [10], which stimulated this research at an early stage.

2. Valuations on modules and resolutions

2.1. Throughout this section $\chi : G \to \mathbb{R}$ is a fixed character of the group G. We write $\mathbb{R}_\infty$ for the reals supplemented with an auxiliary element ∞ which, by definition, is greater than every real number and satisfies $r + \infty = \infty = \infty + r$ for every $r \in \mathbb{R}_\infty$.

DEFINITION. Let A be a G-module. A map $v : A \to \mathbb{R}_\infty$ is said to be a *valuation on A extending χ* if the following axioms hold

$$v(a+b) \geq \min\{v(a), v(b)\}, \quad \text{all } a, b \in A, \tag{2.1}$$

$$v(ga) = \chi(g) + v(a), \quad \text{all } g \in G,\ a \in A, \tag{2.2}$$

$$v(-a) = v(a), \quad \text{all } a \in A, \tag{2.3}$$

$$v(0) = \infty. \tag{2.4}$$

Remark. 1) As usual one can deduce that (2.1) is an equality if $v(a) \neq v(b)$. Indeed, if $v(a) < v(b)$ then $v(a+b) \geq v(a)$ by (2.1); on the other hand, (2.1) and (2.3) applied to $(a+b)-b$ yields $v(a) \geq v(a+b)$.

2) If $\varphi: A \to B$ is a homomorphism of G-modules, then every valuation v on B extending χ induces a valuation $v^* = v \cdot \varphi$ on A extending χ.

2.2. Let F be a free G-module with a fixed basis X. Given an arbitrary map $v: X \to \mathbb{R}$, there is an easy way to extend v to a valuation $v: F \to \mathbb{R}_\infty$ extending χ. We put

$$v(0) = \infty,$$

$$v(gx) = \chi(g) + v(x), \quad \text{for} \quad g \in G, x \in X, \quad \text{and}$$

$v(f) = \min\{v(y) \mid n_y \neq 0\}$, if $f = \Sigma n_y y$ is the unique expansion of $0 \neq f \in F$ in terms of the $\mathbb{Z}$-basis GX, $n_y \in \mathbb{Z}$.

If we wish to express the dependence on the basis X we shall write $v_X: F \to \mathbb{R}_\infty$ for the valuation v. v_X is thus defined relative to a choice of $v_X(x)$ for all $x \in X$. As we only consider cases where $v_X(X) \subseteq \mathbb{R}$, our valuations v_X will always have the feature that

$$v_X(f) = \infty \Leftrightarrow f = 0. \tag{2.5}$$

As a special case we have $F = \mathbb{Z}G$ with basis $X = \{1\}$. Choosing $v_1(1) = 0$ yields the valuation $v_1\mathbb{Z}G \to \mathbb{R}_\infty$ which is a valuation on the group ring in the usual sense provided $\mathbb{Z}G$ is a domain.

We shall repeatedly need the following.

LEMMA 2.1. *Let F and F' be free G-modules on X and X' respectively and let $\varphi: F \to F'$ be a G-homomorphism. Then*

$$v_{X'}(\varphi(f)) \geq v_X(f) + \inf_{x \in X} \{v_{X'}(\varphi(x)) - v_X(x)\}$$

for every $f \in F$.

Proof. The statement is obvious for $f \in X$. For $f = gx \in GX$ and $f = \Sigma n_y y$ it follows by using the definition of v_X above.

2.3. We call two valuations v, $v' : A \to \mathbb{R}_\infty$ *equivalent* if there are real numbers r, r' such that $v'(a) \le v(a) + r$ and $v(a) \le v'(a) + r'$, for all $a \in A$. As a consequence of Lemma 2.1 we have

COROLLARY 2.2. *If F is a finitely generated free G-module, then the equivalence class of the valuation $v_X : F \to \mathbb{R}_\infty$, defined in 2.2 does not depend on the choice of the basis X nor on the values $v_X(x)$, $x \in X$.*

Proof. A different choice of X amounts to composing v with an automorphism $\varphi : F \to F$. The corollary thus follows from Lemma 2.1 applied to φ and φ^{-1}.

A similar argument opens the possibility to define a canonical equivalence class of valuations on every *finitely generated projective module P*. Let $\iota : P \rightarrowtail F$ be a split embedding of P into a free G-module F of finite rank, and choose a basis X of F. Then define $v : P \to \mathbb{R}_\infty$ by putting $v = v_X \circ \iota$.

LEMMA 2.3. *The equivalence class of $v : P \to \mathbb{R}_\infty$ is independent of the choice of F, ι, X, and $v(X)$.*

Proof. Let F', $\iota' : P \rightarrowtail F'$, and $X' \subseteq F'$ be a second choice, and let $\pi : F \twoheadrightarrow P$ be a splitting of ι. Then $\iota' = \varphi\iota$, where $\varphi : F \to F'$ is the homomorphism $\iota'\pi$. By Lemma 2.1 we obtain for every $p \in P$,

$$\begin{aligned} v_{X'}(\iota'(p) &= v_{X'}(\varphi\iota(p)) \\ &\ge v_X(\iota(p)) + \inf_X \{v_{X'}(\varphi(x)) - v_X(x)\}. \end{aligned}$$

The inf term is independent of p and $<\infty$. Interchanging the rôle of F, ι, X with F', ι', X' thus yields the result.

2.4. We extend the notion of valuations on a (free) module F to free resolutions $\mathbf{F} \twoheadrightarrow A$ of a G-module A. We shall always assume that the resolution $\mathbf{F}$ is *admissible*, by which we mean that it has the following additional feature: For every $i \ge 0$ the free G-module F_i is endowed with a specific basis $X_i \subseteq F_i$, and for this basis we have $\partial x \ne 0$ for every $x \in X_i$ (here ∂_0 is to be interpreted as the augmentation map $F_0 \twoheadrightarrow A$). Of course, every G-module A admits admissible free resolutions.

We find it convenient to think of $\mathbf{F}$ as the free G-module $\bigoplus_{i \ge 0} F_i$ on the basis $X = \bigcup_{i \ge 0} X_i$. And we write $\mathbf{F}^{(m)}$ for the m-skeleton $\mathbf{F}^{(m)} = \bigoplus_{i=0}^{m} F_i$ which is free with basis $X^{(m)} = \bigcup_{i=0}^{m} X_i$.

The resolution $\mathbf{F}$ is now equipped with the following valuation $v:\mathbf{F}\to\mathbb{R}_\infty$. For each $i\geq 0$, the valuation v restricted to F_i, denoted $v_i:F_i\to\mathbb{R}$, is the valuation v_{X_i} of Section 2.2, where the values of v_i on X_i are chosen inductively by putting, for each $x\in X_i$,

$$v_i(x)=\begin{cases}0, & \text{if } i=0\\ v_{i-1}(\partial x), & \text{if } i>0.\end{cases}$$

One can then define the value on arbitrary elements of $\mathbf{F}$ by taking the minimum value on its homogeneous components. The reader can easily verify that the so defined map $v:\mathbf{F}\to\mathbb{R}_\infty$ satisfies (2.1)–(2.5) and has the additional property that

$$v(\partial c)\geq v(c)\quad \text{for every } c\in\mathbf{F}. \tag{2.6}$$

Observe also that $v(\mathbf{F})\subseteq\chi(G)\cup\{\infty\}$.

2.5. We mention a rather useful alternative description of $v:\mathbf{F}\to\mathbb{R}_\infty$.

For every element $c\in\mathbf{F}$ we define the support $\mathrm{supp}_X\, c$ of c with respect to X. $\mathrm{supp}_X\, c$ is a finite subset of G defined by the following inductive procedure.

If $c=\Sigma n_y y$ is the unique expansion of c in terms of the $\mathbb{Z}$-basis $Y=GX$, $n_y\in\mathbb{Z}$, $y\in Y$, then

$$\mathrm{supp}_X\, c=\bigcup_{n_y\neq 0}\mathrm{supp}_X\, y. \tag{2.7}$$

If $c=y\in GX_i$, with $i>0$, then

$$\mathrm{supp}_X\, y=\mathrm{supp}_X\,(\partial y). \tag{2.8}$$

If $c=gx\in GX_0$ then

$$\mathrm{supp}_X\,(gx)=\{g\}. \tag{2.9}$$

Remark. Note that this definition includes the case of a free G-module F (concentrated in dimension 0). In particular, for $F=\mathbb{Z}G$ and $X=\{1\}$ one obtains the usual notion of support in the group ring.

We leave the proof of the following formal properties and Lemma 2.4 as an exercise:

$$\operatorname{supp}_X(c + c') \subseteq (\operatorname{supp}_X c) \cup (\operatorname{supp}_X c'), \qquad c, c' \in \mathbf{F}, \tag{2.10}$$

$$\operatorname{supp}_X(gc) = g \operatorname{supp}_X c, \qquad g \in G,\ c \in \mathbf{F}, \tag{2.11}$$

$$\operatorname{supp}_X(\partial c) \subseteq \operatorname{supp}_X c, \qquad c \in \mathbf{F}, \tag{2.12}$$

$$\operatorname{supp}_X c = \emptyset \Leftrightarrow c = 0. \tag{2.13}$$

LEMMA 2.4. *The valuation* $v : \mathbf{F} \to \mathbb{R}_\infty$ *defined in Section* 2.4 *can also be described by*

$$v(c) = \min \chi(\operatorname{supp}_X c), \qquad 0 \neq c \in \mathbf{F}.$$

3. The valuation subcomplex

3.1. We retain the notation and conventions of Section 2. In particular, $\mathbf{F}$ is an admissible free resolution of the G-module A and $v : \mathbf{F} \to \mathbb{R}_\infty$ the valuation defined in 2.4. Then we consider the valuation subcomplex $\mathbf{F}_v \leq \mathbf{F}$ defined by

$$\mathbf{F}_v = \{c \in \mathbf{F} \mid v(c) \geq 0\}.$$

It is immediate from (2.1)–(2.6) that $\mathbf{F}_v$ is a G_χ-subcomplex of $\mathbf{F}$.

LEMMA 3.1. *The* G_χ*-module* F_{iv} *is free of rank equal to the* G*-rank of* F_i, $i \geq 0$.

Proof. For every $x \in X_i$ the value $v(x) \in \mathbb{R}$ is attained on a group element (here we use admissibility!). So pick $g_x \in G$ with $\chi(g_x) = v(x)$ and put $X_i' = \{g_x^{-1}x \mid x \in X_i\}$. Then $v(x') = 0$ for every $x' \in X_i'$, and it is easy to see that $F_{iv} = \mathbb{Z}G_\chi X_i'$ is free on X_i'.

3.2. The situation becomes particularly interesting when the complex $\mathbf{F}_v \to A \to 0$ is exact and hence provides a free resolution of A as a G_χ-module. The deviation from exactness can be measured as follows: For every $j \geq -1$ we consider the reduced cycles

$$Z_j = \begin{cases} \ker(F_j \to F_{j-1}) & \text{if } j \geq 1 \\ \ker(F_0 \twoheadrightarrow A) & \text{if } j = 0 \\ A & \text{if } j = -1 \end{cases}$$

Moreover, we think of Z_{-1} as equipped with the trivial "valuation" $v: A \to \mathbb{R}_\infty$ ($v(a) = 0$, for all $0 \neq a \in A$). Then we define the *deviation* of a cycle $0 \neq z \in Z_j$ by

$$d_j(z) = v(z) - \sup v(\partial^{-1} z). \tag{3.1}$$

Note that $d_j(z) \geq 0$ for all $j \geq 0$ by (2.6); $d_{-1}(z)$ is, in general, not bounded below. Clearly $\mathbf{F}_v \to A \to 0$ is exact in dimension j if and only if $d_j(z) \leq 0$ for all $0 \neq z \in Z_j$.

DEFINITION. We say that $\mathbf{F}_v \to A \to 0$ is essentially exact in dimension j ($j \geq -1$), if the function $d_j: Z_j \backslash \{0\} \to \mathbb{R}$ has an upper bound.

Occasionally it is useful to have an explicit value for this upper bound; so we put

$$D_j = \sup \{d_j(z) \mid 0 \neq z \in Z_j\}. \tag{3.2}$$

THEOREM 3.2. *Let $\mathbf{F} \twoheadrightarrow A$ be an admissible free resolution with finitely generated m-skeleton. Then $[\chi] \in \Sigma^m(G; A)$ if and only if $\mathbf{F}_v \to A \to 0$ is essentially exact in all dimensions j with $-1 \leq j < m$ (in other words, $D_j < \infty$ for all $-1 \leq j < m$).*

Remarks. 1) In [6] we introduced the invariants $\Sigma^m(G; A)$ (for $A = \mathbb{Z}$) in terms of essentially exact valuation subcomplexes. The striking fact that our definition can be rephrased in terms of the $(FP)_m$-property over G_χ was pointed out to us by Ralph Strebel.

2) Theorem 3.2 establishes, in particular, that whether $\mathbf{F}_v$ is essentially exact in all dimensions $<m$ is independent of the choice of $\mathbf{F}$.

3) It is useful to observe that Theorem 3.2 remains valid if one replaces the valuation v by an arbitrary valuation $w: \mathbf{F} \to \mathbb{R}_\infty$ which, when restricted to the m-skeleton, is equivalent to v. The fact that $\mathbf{F}_w = \{c \in \mathbf{F} \mid w(c) \geq 0\}$ is, in general, not a subcomplex need not concern us.

4) We shall see later that if $[\chi] \in \Sigma^m(G; A)$ then there exist admissible resolutions $\mathbf{F} \twoheadrightarrow A$ with finitely generated m-skeleton such that the valuation

subcomplex $\mathbf{F}_v \to A \to 0$ is, in fact, exact, (see 4.5 Remark 1). This would yield a constructive proof of Theorem 3.2. The non-constructive proof below, however, is much simpler.

Proof (of Theorem 3.2). Let $g \in G$ with $\chi(g) < 0$ and put $\mathbf{E}_k = g^k \mathbf{F}_v$, $k = 0, 1, 2, \ldots$. Since $gG_\chi = G_\chi g$, $\mathbf{E}_k$ is a G_χ-subcomplex of $\mathbf{F}$ and is isomorphic to $\mathbf{F}_v$. By Lemma 3.1. $\{\mathbf{E}_k\}$ is a filtration of $\mathbf{F}$ by finitely generated free subcomplexes. It is convenient to write $\tilde{\mathbf{F}}_v$ and $\tilde{\mathbf{E}}_k$ for the chain complexes $\mathbf{F}_v \to A \to 0$ and $\mathbf{E}_k \to A \to 0$, respectively. The condition that $\tilde{\mathbf{F}}_v$ is essentially exact in some dimension $j \geq -1$ amounts to saying that for every $k \in \mathbb{N}$ there is some $k' \geq k$ with the property that the homomorphism $H_j(\tilde{\mathbf{E}}_k) \to H_j(\tilde{\mathbf{E}}_{k'})$ is zero. In this situation a variant on K. S. Brown's $(FP)_m$-criterion [7], Theorem 2.2 applies, asserting that this is equivalent to the condition that A be of type $(FP)_m$ over G_χ.

Appendix

Because we are in a slightly more general but at the same time much easier situation than [7], Theorem 2.2, (arbitrary modules A but only free action on **F**) we repeat Brown's argument for the convenience of the reader.

To say that the maps $H_j(\tilde{\mathbf{E}}_k) \to H_j(\tilde{\mathbf{E}}_{k'})$ are zero, for $k' - k$ sufficiently large, is equivalent with saying that $\varinjlim_k \prod H_j(\tilde{\mathbf{E}}_k) = 0$ for arbitrary direct powers Π. We prefer to interpret this in terms of $\mathbf{E}_k$, and the translation is given by the short exact sequence of chain complexes $A \rightarrowtail \tilde{\mathbf{E}}_k \twoheadrightarrow \mathbf{E}_k$ (A concentrated in dimension -1). This gives rise to the isomorphisms $H_j(\tilde{\mathbf{E}}_k) \cong H_j(\mathbf{E}_k)$ for $j > 0$ and the exact sequence

$$0 \to H_0(\tilde{\mathbf{E}}_k) \to H_0(\mathbf{E}_k) \xrightarrow{\Delta} A \to H_{-1}(\tilde{\mathbf{E}}_k) \to 0.$$

Hence the condition that $\tilde{\mathbf{F}}_v$ is essentially exact in dimension j, for some $j \geq -1$, is equivalent to the conditions

$$\varinjlim_k \prod H_j(\mathbf{E}_k) = 0, \quad \text{if} \quad j \geq 1,$$

$$\Delta \text{ induces a monomorphism } \varinjlim_k \prod H_0(\mathbf{E}_k) \rightarrowtail \Pi A, \quad \text{if} \quad j = 0,$$

$$\Delta \text{ induces an epimorphism } \varinjlim_k \prod H_0(\mathbf{E}_k) \twoheadrightarrow \Pi A, \quad \text{if} \quad j = -1.$$

Now,

$$\begin{aligned}\operatorname{Tor}_j^{\mathbb{Z}G_\chi}(\Pi\mathbb{Z}G_\chi, A) &= H_j((\Pi\mathbb{Z}G_\chi)\otimes_{G_\chi}\mathbf{F})\\ &\cong \varinjlim H_j((\Pi\mathbb{Z}G_\chi)\otimes_{G_\chi}\mathbf{E}_k)\\ &\cong \varinjlim H_j(\Pi\mathbf{E}_k), \quad \text{if} \quad j<m,\\ &\cong \varinjlim \prod H_j(\mathbf{E}_k),\end{aligned}$$

where we have used that $\mathbf{F}\twoheadrightarrow A$ is a flat resolution over G_χ, that $\otimes$ and H_j commute with $\varinjlim$, that $\prod$ commutes with the tensor product by a finitely generated free module, and that $\prod$ commutes with H_j. Hence the condition that $\mathbf{F}_v\to A\to 0$ be essentially exact in *all* dimensions $-1\le j<m$ is equivalent to the condition

$(\Pi\mathbb{Z}G_\chi)\otimes_{G_\chi} A\twoheadrightarrow \prod A$ is epimorphic, if $m=0$,

or,

$(\Pi\mathbb{Z}G_\chi)\otimes_{G_\chi} A\twoheadrightarrow \prod A$ is an isomorphism and
$\operatorname{Tor}_j^{\mathbb{Z}G_\chi}(\prod\mathbb{Z}G_\chi, A)=0$ for all $1\le j<m$, if $m\ge 1$.

This is precisely the Tor-criterion for type $(FP)_m$, see e.g. [1] or [2].

3.3. It is useful to reinterpret Theorem 3.2 in terms of a *projective* resolution $\mathbf{P}\twoheadrightarrow A$ with finite m-skeleton. Let us assume that $\mathbf{P}$ is *admissible* in the sense that for all $i\ge 0$, $\partial P_i\ne 0$ unless $P_i=0$ (with ∂_0 interpreted as the augmentation map). By carefully choosing projective complements Q_i for P_i we find an exact admissible projective complex $\mathbf{Q}$ such that $\mathbf{P}\oplus\mathbf{Q}=\mathbf{F}$ is a free resolution of A with finitely generated m-skeleton and retains the admissibility condition above. Then it is also easy to choose suitable bases $X_i\subseteq F_i$ such that $\mathbf{F}$ is admissible in the sense of Section 2.4. Let us consider the valuation subcomplexes $\mathbf{P}_v=\mathbf{P}\cap\mathbf{F}_v$ and $\mathbf{Q}_v=\mathbf{Q}\cap\mathbf{F}_v$.

It is easy to observe that $\mathbf{F}_v$ is essentially exact in dimension j if and only if both $\mathbf{P}_v$ and $\mathbf{Q}_v$ are essentially exact in dimension j. We claim that $\mathbf{F}_v$ is essentially exact in all dimensions $<m$ if and only if $\mathbf{P}_v$ is essentially exact in all dimensions $<m$. And to prove this we have to show that $\mathbf{Q}_v$ is *always* essentially exact in dimensions $<m$.

Now, $\mathbf{Q}$ can be regarded as a projective resolution of the trivial module 0. Again we find an admissible projective complement $\mathbf{R}$ such that $\mathbf{Q}\oplus\mathbf{R}=\mathbf{E}$ is an admissible free resolution of 0 with finitely generated m-skeleton. Let $w:\mathbf{E}\to\mathbb{R}_\infty$ denote the corresponding valuation. Then Theorem 3.2 asserts that $\mathbf{E}_w$ is essentially exact in all dimensions $<m$, hence so is $\mathbf{Q}_w = \mathbf{Q} \cap \mathbf{E}_w$. But by Lemma 2.3 v and w, when restricted to $\mathbf{Q}^{(m)}$, are equivalent. Hence $\mathbf{Q}_v$ is essentially exact in all dimensions $<m$. Thus we have proved that whether $\mathbf{F}_v$ is essentially exact in all dimensions $<m$ can be read off from $\mathbf{P}_v$.

We summarize:

THEOREM 3.3. *Let $\mathbf{P} \twoheadrightarrow A$ be an admissible projective resolution with finite m-skeleton; consider a valuation $v : \mathbf{P}^{(m)} \to \mathbb{R}_\infty$ by choosing valuations v_i on each finitely generated projective module P_i, $i \leq m$, as in Section 2.3. Then $\mathbf{P}_v^{(m)}$ is essentially exact in all dimensions $<m$ if and only if $[\chi] \in \Sigma^m(G; A)$.*

The effort to establish Theorem 3.2 for projective resolutions is rewarded by the following application: Let A be a G-module of projective dimension $\leq d$. If $\Sigma^d(G; A)$ is not empty then A must be of type $(FP)_d$ and so has a projective resolution which is both finitely generated and of finite length $\leq d$. Then $\mathbf{P}_v$ is obviously exact in all dimensions $\geq d$. Whence

COROLLARY 3.4. *If the G-module A has a projective resolution of finite length $\leq d$ then*

$$\Sigma^m(G; A) = \Sigma^d(G; A)$$

for every $m \geq d$.

4. Criteria for $\Sigma^m(G; A)$

4.1. We keep the notation and conventions of Section 2; in particular, $\mathbf{F}$ is an admissible free resolution of the G-module A, and $v : \mathbf{F} \to \mathbb{R}_\infty$ is the valuation extending $\chi : G \to \mathbb{R}$ defined in Section 2.4 (or 2.5).

The main technical result of this paper, which makes $\Sigma^m(G; A)$ to some extent accessible, is

THEOREM 4.1. *Assume that $\mathbf{F} \twoheadrightarrow A$ is an admissible free resolution with finitely generated m-skeleton $\mathbf{F}^{(m)}$. Then $[\chi] \in \Sigma^m(G; A)$ if and only if there is a chain endomorphism $\varphi : \mathbf{F} \to \mathbf{F}$, lifting the identity of A, such that $v(\varphi(x)) > v(x)$ for every basis element $x \in X^{(m)}$.*

Proof. Let us first assume that a chain endomorphism $\varphi : \mathbf{F} \to \mathbf{F}$ as mentioned in the Theorem, exists. Since φ lifts the identity of A we can choose a chain homotopy $\sigma : \varphi \simeq \mathrm{Id}_{\mathbf{F}}$. For $m > i \geq 0$ we consider the two real numbers

$$r = \min_{x \in X_{i+1}} \{v(\varphi(x)) - v(x)\},$$

$$s = \min_{x \in X_i} \{v(\sigma(x)) - v(x)\}.$$

By assumption we have $r > 0$. Let $z \in F_{i-1}$ be a cycle. Then $z = \partial c$ for some $c \in F_i$. We claim that c can always be chosen with $v(c) \geq v(z) + s$. Indeed, if $v(c) < v(z) + s$ we replace c by $c' = c + \partial\sigma(c) = \varphi(c) - \sigma(\partial c) = \varphi(c) - \sigma(z)$; and find

$$\begin{aligned} v(c') &\geq \min \{v(\varphi(c)), v(\sigma(z))\} \\ &\geq \min \{v(c) + r, v(z) + s\}, \quad \text{by Lemma 2.1.} \end{aligned}$$

Hence either $v(c') \geq v(z) + s$ or $v(c') \geq v(c) + r$. In the first case we are done; in the second case we have at least increased the value of $v(c)$ by the positive quantity r. Hence repeating the procedure will eventually produce $c \in F_i$ with $\partial c = z$ and $v(c) \geq v(z) + s$. By definition this means that $\mathbf{F}_v$ is essentially exact in dimension $i - 1$. Hence $[\chi] \in \Sigma^m(G; A)$ by Theorem 3.2.

Now we assume, conversely, that $[\chi] \in \Sigma^m(G; A)$. Then Theorem 3.2 asserts that D_j, as defined in (3.2), is finite for all $-1 \leq j < m$. Hence every real number $D > D_j$ has the property that for every $z \in Z_j$ there is $c \in F_{j+1}$ with $\partial c = z$ and $v(c) \geq v(z) - D$.

We pick an element $g \in G$ whose value $\chi(g) = l$ will be specified later. For each $x \in X_0$ we apply Theorem 3.2 for $j = -1$ to choose $c_x \in F_0$ with $\partial c_x = g^{-1} \partial x$ and $v(c_x) \geq -D_0$ (∂_0 is to be interpreted as the augmentation map). Putting $\varphi(x) = gc_x$ then yields a homomorphism $\varphi : F_0 \to F_0$ lifting the identity of A, with $v(\varphi(x)) \geq l - D_{-1}$ for every $x \in X_0$. Using Lemma 2.1 we deduce that even

$$\begin{aligned} v(\varphi(c)) &\geq v(c) + \inf_{X_0} (v(\varphi(x)) - v(x)) \\ &\geq v(c) + l - D_{-1}, \end{aligned}$$

for every $c \in F_0$.

Assume, inductively, that a chain map $\varphi : \mathbf{F}^{(j)} \to \mathbf{F}^{(j)}$ has been constructed with the property that

$$v(\varphi(c)) \geq v(c) + l - \sum_{i=-1}^{j-1} D_i,$$

for every $c \in F_j$. We apply Theorem 3.2 again, in order to find, for each $x \in X_{j+1}$, a chain $c_x \in F_{j+1}$ with $\partial c_x = \varphi(\partial x)$ and with $v(c_x) \geq v(\varphi(\partial x)) - D_j$. Then, putting $\varphi(x) = c_x$ yields a chain endomorphism $\varphi: \mathbf{F}^{(j+1)} \to \mathbf{F}^{(j+1)}$ with the property that for every $x \in X_{j+1}$

$$\begin{aligned} v(\varphi(x)) &\geq v(\varphi(\partial x)) - D_j \\ &\geq v(\partial x) + \inf_{X_{j-1}} (v(\varphi(x)) - v(x)) - D_j \\ &\geq v(x) + l - \sum_{i=-1}^{j} D_i, \end{aligned}$$

where we have used Lemma 2.1, (2.6), and the induction hypothesis. Using Lemma 2.1 again we find

$$v(\varphi(c)) \geq v(c) + l - \sum_{i=-1}^{j} D_i,$$

for every $c \in F_{j+1}$.

It suffices now to choose $l > D_{-1} + D_0 + \cdots + D_{m-1}$. Since $D_i \geq 0$ for all $i \geq 0$ we then have $l > D_{-1} + D_0 + \cdots + D_{j-1}$, for all j with $0 \leq j < m$, whence $v(\varphi(x)) > v(x)$ for every $x \in X^{(m)}$, as asserted in the theorem.

4.2. We shall also need a variant on Theorem 4.1 which makes a stronger conclusion at the expense of modifying the given free resolution $\mathbf{F}$ by elementary expansions in the sense of simple homotopy theory of J. H. C. Whitehead.

We recall that an elementary expansion $\tilde{\mathbf{F}}$ depends on the choice of an element $u \in F_j$, $j \geq 1$, and is defined as follows: adjoin a new basis element e to X_j and define $\partial e = \partial u$. Then, in order to kill the j-dimensional homology created by the first move, adjoin a new basis element e' to X_{j+1} and define $\partial e' = e - u$. It is easy to check that $\tilde{\mathbf{F}}$ is again a free resolution (of the same G-module), and if $\partial u \neq 0$ and $\mathbf{F}$ is admissible, so is $\tilde{\mathbf{F}}$.

THEOREM 4.2. *Let A be a G-module of type $(FP)_m$. Then $[\chi] \in \Sigma^m(G, A)$ if and only if there exists an admissible free resolution $\mathbf{F} \twoheadrightarrow A$ with finitely generated m-skeleton, a chain endomorphism $\varphi: \mathbf{F} \to \mathbf{F}$, and a chain homotopy $\sigma: \varphi \simeq \mathrm{Id}_{\mathbf{F}}$, such that*

$$v(\varphi(x)) > v(x) \quad \textit{for every } x \in X^{(m)} \tag{4.1}$$

and

$$\sigma(X_i) \subseteq X_{i+1} \cup \{0\} \quad \textit{for every } 0 \leq i \leq m. \tag{4.2}$$

The resolution $\mathbf{F}$ *is obtained by performing a finite sequence of elementary expansions on an arbitrary admissible free resolution of* A *with finitely generated* m*-skeleton.*

4.3. Before we prove Theorem 4.2, we draw some consequences of (4.1) and (4.2) which will be needed both for the inductive proof and for later applications.

LEMMA 4.3. *Under the assumption of Theorem* 4.2 *there is a real number* $t > 0$ *with the property that we have for every chain* $c \in \mathbf{F}^{(m)}$

$$\operatorname{supp}_X \sigma(c) \subseteq \operatorname{supp}_X c \cup G_{v(c)+t}, \tag{4.3}$$

where G_r, *for any* $r \in \mathbb{R}$, *stands for the set* $\{g \in G \mid \chi(g) \geq r\}$. *Consequently* $v(\sigma(c)) \geq v(c)$. *Moreover, the largest possible value for* t *is*

$$t = \min_{x \in X^{(m)}} \{v(\varphi(x)) - v(x)\} \tag{4.4}$$

Proof. Assume, for the moment, that (4.3) holds for all $c = x \in X^{(m)}$. Then, as is clear from (2.11) and (2.2), it holds also for $c = y \in GX^{(m)}$. And using (2.10) and (2.1) one obtains the assertion for arbitrary $c \in \mathbf{F}^{(m)}$.

It remains to prove (4.3) for $c = x \in X^{(m)}$ and we do this by induction on m. For every $x \in X^{(m)}$ we have

$$\begin{aligned} \operatorname{supp}_X \sigma(x) &= \operatorname{supp}_X \partial\sigma(x), \quad \text{by (4.2)} \\ &= \operatorname{supp}_X (\varphi(x) - x - \sigma(\partial x)) \\ &\subseteq \operatorname{supp}_X x \cup G_{v(x)+t} \bigcup \operatorname{supp}_X \sigma(\partial x), \\ &\quad \text{by (2.10) and (4.1)} \end{aligned}$$

For $x \in X_0$, $\sigma(\partial x)$ is to be interpreted as 0. The induction is now obvious.

4.4. *Proof* (of Theorem 4.2). Assume first that $\mathbf{F}$, $\varphi : \mathbf{F} \to \mathbf{F}$ and $\sigma : \varphi \simeq \mathrm{Id}_{\mathbf{F}}$ as in the theorem exist. Then we observe that the real number s, defined in the first part of the proof of Theorem 4.1, is now ≥ 0. The proof of Theorem 4.1 then shows that $\mathbf{F}_v \to A \to 0$ is, in fact, *exact* in all dimensions $<m$. Hence A is of type $(FP)_m$ over G_χ, that is, $[\chi] \in \Sigma^m(G; A)$.

Now we assume, conversely, that $[\chi] \in \Sigma^m(G; A)$. We start with an arbitrary admissible free resolution $\mathbf{F} \twoheadrightarrow A$ with finite m-skeleton, and aim to construct φ and σ step by step while modifying $\mathbf{F}$ in terms of elementary expansions.

First we follow the proof of Theorem 4.1 to find a homomorphism $\varphi: F_0 \to F_0$, lifting the identity of A, and such that (4.1) holds for all $x \in X_0$. Then we perform for each $x \in X_0$ an elementary expansion by adjoining, in first move, a new basis element e_x to X_1 with $\partial e_x = \varphi(x) - x$. Note that $\varphi(x) \neq x$ by (4.1), so that $\mathbf{F}$ remains admissible. Then we define $\sigma: F_0 \to F_1$ by putting $\sigma(x) = e_x$.

Now we assume, inductively, that we have already constructed $\mathbf{F}$, $\varphi: \mathbf{F} \to \mathbf{F}$ and $\sigma: \varphi \simeq \mathrm{Id}_F$ in dimensions $\leq m-1$ with the property that (4.1) and (4.2) hold in these dimensions. In order to construct $\varphi: F_m \to F_m$ we then consider the real number

$$r = \min_{x \in X_{m-1}} \{v(\varphi(x)) - v(x)\},$$

which is positive by assumption. Using Lemma 2.1 we find that $v(\varphi(\bar{c})) \geq v(\bar{c}) + r$, for every $\bar{c} \in F_{m-1}$. Hence we have for $c \in F_m$ and $k \in \mathbb{N}$,

$$\begin{aligned} v(\varphi^k(\partial c)) &\geq v(\varphi^{k-1}(\partial c)) + r \\ &\geq \cdots \\ &\;\;\vdots \\ &\geq v(\partial c) + kr. \end{aligned}$$

$\varphi^k(\partial c)$ is, of course, an $(m-1)$-cycle; and since $[\chi] \in \Sigma^m(G; A)$ we know, by Theorem 2.3, that there must be some m-chain $\bar{c} \in F_m$ with $\partial \bar{c} = \varphi^k(\partial c)$ and $v(\bar{c}) \geq v(\varphi^k(\partial c)) - D$, for any $D > D_{m-1}$. Hence, by choosing k larger than D_{m-1}/r, $v(\bar{c}) > v(\partial c)$. In this fashion we find for each $x \in X_m$ a chain $c_x \in F_m$ with $\partial c_x = \varphi^k(\partial x)$ and $v(c_x) > v(\partial x)$.

Now we put

$$\varphi(x) = c_x - \sigma(\varphi + \varphi^2 + \cdots + \varphi^{k-1})(\partial x), \qquad x \in X_m. \tag{4.5}$$

With this choice of $\varphi(x)$ we have, for $x \in X_m$,

$$\begin{aligned} v(\varphi(x)) &\geq \min\{v(c_x), v(\sigma\varphi^i(\partial x)) \mid 0 < i < k\} \\ &\geq \min\{v(c_x), v(\varphi^i(\partial x)) \mid 0 < i < k\}, \end{aligned}$$

by induction and Lemma 4.3. By induction, again, it follows that $v(\varphi(x)) > v(\partial x) \geq v(x)$, as required. The homotopy property of σ in dimensions $\leq m-1$ together with $\partial\varphi^i(\partial x) = 0$ yields

$$\varphi^{i+1}(\partial x) - \varphi^i(\partial x) = \partial\sigma\varphi^i(\partial x).$$

Hence, by (4.5),

$$\begin{aligned}\partial\varphi(x) &= \varphi^k(\partial x) - \partial\sigma(\varphi + \varphi^2 + \cdots + \varphi^{k-1})(\partial x)\\ &= \varphi(\partial x),\end{aligned}$$

which shows that $\varphi : \mathbf{F}^{(m)} \to \mathbf{F}^{(m)}$ is indeed a chain map satisfying (4.1).

It remains to perform, for each $x \in X_m$ with $\varphi(x) \neq x + \sigma(\partial x)$, an elementary expansion the first move of which being adjunction of e_x to X_{m+1}, with $\partial e_x = \varphi(x) - x - \sigma(\partial x)$. Then we define $\sigma : F_m \to F_{m+1}$ by putting $\sigma(x) = e_x$, if $\varphi(x) \neq x + \sigma(\partial x)$, and $\sigma(x) = 0$ otherwise. This completes the proof of Theorem 4.2.

4.5. *Remarks.* 1) The proof of Theorem 4.2 shows, in particular, that if $[\chi] \in \Sigma^m(G; A)$ then there is always an admissible free resolution $\mathbf{F} \twoheadrightarrow A$ whose valuation complex $\mathbf{F}_v \to A \to 0$ is exact. This yields a proof of Theorem 3.2 avoiding Brown's $(FP)_m$-criterion.

2) The proofs of Theorems 4.1 and 4.2 yield somewhat stronger necessary conditions for $[\chi] \in \Sigma^m(G; A)$ than the statements of the theorems, namely

PROPOSITION 4.4. *Let* $\mathbf{F} \twoheadrightarrow A$ *be an admissible free resolution with finitely generated m-skeleton. If* $[\chi] \in \Sigma^m(G; A)$ *then every chain endomorphism* $\varphi : \mathbf{F}^{(m-1)} \to \mathbf{F}^{(m-1)}$, *with* $v(\varphi(x)) - v(x) > D_{m-1}$ *for all* $x \in X_{m-1}$, *can be extended to a chain endomorphism* $\varphi : \mathbf{F}^{(m)} \to \mathbf{F}^{(m)}$, *with* $v(\varphi(x)) - v(x) > 0$ *for every* $x \in X_m$.

PROPOSITION 4.5. *Let* $\mathbf{F} \twoheadrightarrow A$ *be an admissible free resolution. Let* $\sigma : F_i \to F_{i+1}$, $0 \leq i < m$ *be a sequence of homomorphisms such that* $v(\sigma(x)) \geq v(x)$ *and* $v(x + \sigma(\partial x) + \partial\sigma(x)) > v(x)$ *for every* $x \in X^{(m-1)}$. *If* $[\chi] \in \Sigma^m(G; A)$ *then* σ *can be extended to* $\sigma : F_m \to F_{m+1}$ *such that* $v(x + \sigma(\partial x) + \partial\sigma(x)) > v(x)$ *for all* $x \in X_m$.

Proposition 4.5 is immediate from the proof of Theorem 4.2 and the observation that the chain endomorphism φ can always be expressed in terms of σ.

4.6. We shall now have to consider more than just one fixed character $\chi: G \to \mathbb{R}$ at a time. Thus, from now on we write v_χ for the valuation $v: \mathbf{F} \to \mathbb{R}_\infty$ extending χ defined in Section 2.4 (or by Lemma 2.4), in order to express its dependence on χ.

Using Lemma 2.4 we make the elementary but crucial observation that evaluation at an element $c \in \mathbf{F}$ yields a *continuous* map $\varepsilon: \mathrm{Hom}(G, \mathbb{R}) \to \mathbb{R}$, $\varepsilon(\chi) = v_\chi(c)$. This has, in particular, the consequence that if a chain endomorphism $\varphi: \mathbf{F} \to \mathbf{F}$, with the properties stated in Theorem 4.1, exists for some $\chi \in \mathrm{Hom}(G, \mathbb{R})$, then the very same φ will do for all characters sufficiently close to χ. Hence Theorem 4.1 has the immediate

COROLLARY 4.6. *$\Sigma^m(G; A)$ is an open subset of $S(G)$, for every G-module A and all $m \geq 0$.*

4.7. We close this section by extending Theorems 4.1 and 4.2 from the singleton $\{[\chi]\}$ to a compact subset of $S(G)$.

THEOREM 4.7. *Let $\mathbf{F} \twoheadrightarrow A$ be an admissible free resolution with finitely generated m-skeleton. Then the following three conditions are equivalent for a compact subset $\Gamma \subseteq S(G)$.*

(i) $\Gamma \subseteq \Sigma^m(G; A)$

(ii) *there is a finite set ϕ of chain endomorphisms $\varphi: \mathbf{F} \to \mathbf{F}$, lifting* Id_A, *with the property that for each point $[\chi] \in \Gamma$ there is some $\varphi \in \phi$ with*

$$v_\chi(\varphi(x)) > v_\chi(x), \quad \textit{for every } x \in X^{(m)}. \tag{4.6}$$

(iii) *After replacing* $\mathbf{F}$ *by a suitable admissible free resolution, obtained by performing on* $\mathbf{F}$ *a finite sequence of elementary expansions, we can find a set ϕ as in* (ii) *and for each $\varphi \in \phi$ a chain homotopy $\sigma: \varphi \simeq \mathrm{Id}_\mathbf{F}$ with $\sigma(X_i) \subseteq X_{i+1} \cup \{0\}$ for every i with $0 \leq i \leq m$.*

Proof. (i)$\Rightarrow$(iii) Following the proof of Theorem 4.2 we find for each point $[\chi] \in \Gamma$ a homomorphism $\varphi_\chi: F_0 \to F_0$, lifting Id_A, such that (4.6) holds for $\varphi = \varphi_\chi$ and $m = 0$. But if (4.6) holds for *some* φ_χ then the very same φ_χ can be used in an open neighbourhood of $[\chi]$. Hence, by compactness of Γ, there is a finite set ϕ_0 of φ_χ's such that for each $[\chi] \in \Gamma$ there is some $\varphi \in \phi_0$ satisfying (4.6) for $m = 0$.

Now we perform for each $x \in X_0$ and each $\varphi \in \phi_0$ an elementary expansion adjoining a new basis element to X_1 with $\partial e_{x,\varphi} = \varphi(x) - x$ (note that $\varphi(x) - x$ is a cycle in the sense that its augmentation image is zero). Then we replace $\mathbf{F}$ by the new resolution which is again admissible since $\varphi(x) \neq x$ for each $x \in X_0$. And we define $\sigma_\varphi: F_0 \to F_1$ by putting $\sigma_\varphi(x) = e_{x,\varphi}$ for every $x \in X_0$.

Assume now, inductively, that we have already constructed a finite set ϕ_{m-1} of chain endomorphisms of $\mathbf{F}^{(m-1)}$, such that for each $[\chi] \in \Gamma$ there is $\varphi \in \phi_{m-1}$ satisfying (4.6) for m replaced by $m-1$, and a chain homotopy $\sigma : \varphi \simeq \mathrm{Id}_{\mathbf{F}}$ with $\sigma(X_i) \subseteq X_{i+1} \cup \{0\}$, $0 \leq i \leq m-1$. According to Proposition 4.4, φ can then be extended to a chain endomorphism $\varphi_\chi : \mathbf{F}^{(m)} \to \mathbf{F}^{(m)}$ satisfying (4.6). But if this is so for some φ_χ, the very same φ_χ will do for an open neighbourhood of $[\chi]$. Hence, again by compactness of Γ, it follows that there is a finite set ϕ_m of φ_χ's (finitely many extensions of the endomorphisms in ϕ_{m-1}) such that for each $[\chi] \in \Gamma$ there is some $\varphi \in \phi_m$ satisfying (4.6).

It remains to perform for each $\varphi \in \phi_m$ and each $x \in X_m$ with $\varphi(x) \neq x + \sigma_\varphi(\partial x)$ an elementary expansion in order to extend the chain homotopy σ_φ to dimension m.

The implication (i) $\Rightarrow$ (ii) is similar and easier than (i) $\Rightarrow$ (iii) and can be left as an exercise. The converse implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) are obvious from Theorems 4.1 and 4.2. This completes the proof of Theorem 4.7.

5. Type $(FP)_m$ over normal subgroups

5.1. In this section we consider a finitely generated group G and a normal subgroup $N \lhd G$ with Abelian factor group $Q = G/N$. As before A will denote a G-module of type $(FP)_m$. Then A may or may not be of type $(FP)_m$ over N – the information as to whether or not it *is* of type $(FP)_m$ is contained in the invariant $\Sigma^m(G; A)$. For we shall prove

THEOREM 5.1. *A is of type $(FP)_m$ over N if and only if $S(G, N) \subseteq \Sigma^m(G; A)$.*

Here, $S(G, N)$ stands for the subsphere of $S(G)$ consisting of all points $[\chi] \in S(G)$ with $\chi(N) = 0$. Note that the canonical projection $\pi : G \twoheadrightarrow Q$ induces an embedding $\pi^* : S(Q) \rightarrowtail S(G)$ which maps $S(Q)$ isomorphically onto $S(G, N)$. Also, $S(G, N)$ remains unchanged if we replace N by a subgroup N_1 of finite index in N. Since type $(FP)_m$ over N is equivalent to type $(FP)_m$ over N_1 for the G-module A, we can thus replace N by the preimage of the torsion subgroup of Q and assume that Q is free Abelian of finite rank n.

5.2. We start with the easy direction of Theorem 5.1, which is the assertion that A of type $(FP)_m$ over N implies $[\chi] \in \Sigma^m(G; A)$ for every character $\chi : G \to \mathbb{R}$ with $\chi(N) = 0$. We first have to establish a very special case

LEMMA 5.2. *If G is a group of type $(FP)_m$ with centre Z then $\Sigma^m(G;\mathbb{Z})$ contains the complement of the subsphere $S(G, Z)$ in $S(G)$. In particular, if G is a finitely generated Abelian group then $\Sigma^m(G;\mathbb{Z}) = S(G)$ for all $m \geq 0$.*

Proof. Let $\mathbf{F} \twoheadrightarrow \mathbb{Z}$ be a G-free resolution with finite m-skeleton, and let $\chi : G \to \mathbb{R}$ be a character not in $S(G, Z)$. Then there is an element $z \in Z$ with $\chi(z) > 0$, and multiplication by z yields a chain-endomorphism $\varphi : \mathbf{F} \to \mathbf{F}$ as required in Theorem 4.1. This shows that $[\chi] \in \Sigma^m(G;\mathbb{Z})$.

Now we take the short exact sequence $N \rightarrowtail G \twoheadrightarrow Q$ and the G-module A as in Theorem 5.1 and assume that A is of type $(FP)_m$ over N. Let $\mathbf{F} \twoheadrightarrow A$ be an N-free resolution with finite m-skeleton, and let $\chi : G \to \mathbb{R}$ be a character with $\chi(N) = 0$. Then $N \subseteq G_\chi$ and $\mathbb{Z}G_\chi \otimes_N \mathbf{F}$ is a $\mathbb{Z}G_\chi$-free resolution with finite m-skeleton of the G_χ-module $\mathbb{Z}G_\chi \otimes_N A$. The G_χ-action on the tensor product is given by the action on the left hand factor. Since A is a G-module we have a G_χ-isomorphism

$$\mathbb{Z}G_\chi \otimes_N A \cong \mathbb{Z}Q_\chi \otimes A,$$

given by $g \otimes a \rightarrowtail \pi(g) \otimes ga$, $g \in G_\chi$, $a \in A$, where $\pi : G_\chi \twoheadrightarrow Q_\chi$ is the canonical projection and with the diagonal G_χ-action on the right hand side. By Lemma 5.2, there is a Q_χ-free resolution $\mathbf{E} \twoheadrightarrow \mathbb{Z}$ with finite skeleta, whence a resolution of the G_χ-module A by modules $E_i \otimes A \cong (\mathbb{Z}Q_\chi \otimes A)^m$. Each of these modules is of type $(FP)_m$ over G_χ, hence so is A by the usual mapping cone argument. The easy direction of Theorem 5.1 is thus established.

5.3. Before we can prove the more subtle direction of Theorem 5.1 we need some further notation. We identify the free Abelian group Q with the integral lattice of the Euclidean space $\mathbb{R}^n$, $n = rkQ$. We do this for two purposes. On the one hand, the inner product of $\mathbb{R}^n$ allows one to identify the sphere $S(G, N)$ with the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$ by assigning to each $u \in S^{n-1}$ the point $[\chi_u] \in S(G)$ represented by the character χ_u,

$$\chi_u(g) = \langle u, \pi(g) \rangle, \qquad g \in G.$$

We shall from now on identify u with $[\chi_u]$ and write $v_u : \mathbf{F} \to \mathbb{R}_\infty$ for the valuation extending χ_u on a free resolution as defined in Section 2.4 (or formula (2.5)). On the other hand, we can also consider the *norm* map $\|\cdots\| : G \to \mathbb{R}$, where the norm on a group element $g \in G$ is simply defined to be the norm of its image under the canonical projection $\pi : G \twoheadrightarrow Q \rightarrowtail \mathbb{R}^n$, $\|g\| = \|\pi(g)\|$.

The norm map $G \to \mathbb{R}$ is, of course, not a character, but it still satisfies $\|gh\| \leq \|g\| + \|h\|$ for all $g, h \in G$, so that one could call it a "semi-character".

Much of what we have been doing in Section 2 for characters and valuations can be extended to semi-characters and semi-valuations. For the sake of exposition, however, we prefer to treat the norm as an ad hoc notion.

For every admissible free resolution $\mathbf{F} \twoheadrightarrow A$ we define the *"norm map"* $\|\cdots\| : \mathbf{F} \to \mathbb{R} \cup \{-\infty\}$ by putting, for each $c \in \mathbf{F}$,

$$\|c\| = \begin{cases} -\infty, & \text{if } c = 0, \\ \max \|\mathrm{supp}_X c\|, & \text{if } c \neq 0. \end{cases} \tag{5.1}$$

If we wish to emphasize the basis $X \subseteq \mathbf{F}$, we write $\|c\|_X$ for $\|c\|$. The formal properties of the norm map are highly analogous to those of the valuations in Section 2.4.

$$\|c + c'\| \leq \max \{\|c\|, \|c'\|\}, \quad \text{all} \quad c, c' \in \mathbf{F} \tag{5.2}$$

$$\|gc\| \leq \|g\| + \|c\|, \quad \text{all} \quad g \in G, c \in \mathbf{F} \tag{5.3}$$

$$\|-c\| = \|c\|, \quad \text{all} \quad c \in \mathbf{F} \tag{5.4}$$

$$\|\partial c\| \leq \|c\|, \quad \text{all} \quad c \in \mathbf{F} \tag{5.5}$$

Remark. 1) (5.2) is an equality if $c \in \mathbb{Z}TX$, $c' \in \mathbb{Z}T'X$, where T and T' are *disjoint* subsets of G.

2) The definition (5.1) applies also when $\mathbf{F}$ is the group ring $\mathbb{Z}G$ concentrated in dimension 0. In this sense (5.3) can be generalized to

$$\|\lambda c\| \leq \|\lambda\| + \|c\|, \quad \text{all } \lambda \in \mathbb{Z}G, \qquad c \in \mathbf{F}. \tag{5.6}$$

For $0 \neq \lambda \in \mathbb{Z}N$, (5.3)′ is an equality.

5.4. Let $\mathbf{F} \twoheadrightarrow A$ be an admissible free resolution with finitely generated m-skeleton. We consider for each real number $r \geq 0$,

$$\mathbf{F}_r = \{c \in \mathbf{F} \mid \|c\| \leq r\}. \tag{5.7}$$

By (5.2)–(5.5), $\mathbf{F}_r$ is an N-subcomplex of $\mathbf{F}$. We claim that its m-skeleton, $\mathbf{F}_r^{(m)} = \mathbf{F}_r \cap \mathbf{F}^{(m)}$, is a free N-module of finite rank.

Observe first that $\mathbf{F}_r$ is free Abelian on the set $\{y \in GX \mid \|y\| \leq r\}$. Since N acts freely on GX we find that $\mathbf{F}_r$ is a free N-module on $B = \{y \in TX \mid \|y\| \leq r\}$, where T stands for a transversal modulo N. We have to show that $B^{(m)} = B \cap \mathbf{F}^{(m)}$ is finite. So let $y = tx \in B^{(m)}$. Then $\|tg\| \leq r$ for every $g \in \mathrm{supp}_X x$. Hence

$$\|t\| \leq \|tg\| + \|g^{-1}\| \leq r + \max_X \|x\|,$$

which shows that $\|t\|$ is bounded. Since the canonical projection $\pi: G \twoheadrightarrow Q$ maps T bijectively onto the discrete subset $\mathbb{Z}^n$ of $\mathbb{R}^n$, it follows that $B^{(m)}$ is finite.

So we have shown that (5.7) defines a free N-subcomplex of $\mathbf{F}$ which is finitely generated in each dimension $\leq m$. We shall establish Theorem 5.1 by showing that if we choose $\mathbf{F}$ carefully and r sufficiently large then $\mathbf{F}_r \twoheadrightarrow A$ is, in fact, exact in all dimension $<m$.

5.5. Now we choose an admissible free resolution $\mathbf{F} \twoheadrightarrow A$ with finite m-skeleton satisfying condition (iii) of Theorem 4.7 for the compact subset $\Gamma = S(G, N) \subseteq S(G)$. Thus we are given a finite set ϕ of chain endomorphisms $\varphi: \mathbf{F} \to \mathbf{F}$, with the property that for each $u \in S^{n-1} = S(G, N)$ there is $\varphi \in \phi$ and a chain homotopy $\sigma: \varphi \simeq \mathrm{Id}_{\mathbf{F}}$ such that we have

$$v_u(\varphi(x)) - v_u(x) > 0, \quad \text{for all } x \in X^{(m)}, \tag{5.8}$$

and

$$\sigma(X_i) \subseteq X_{i+1} \cup \{0\} \quad \text{for all } 0 \leq i \leq m. \tag{5.9}$$

We shall need the following two real parameters. On the one hand, we consider, for each $u \in S(G, N)$, the positive real number

$$\rho(u) = \max_{\varphi \in \phi} \min_{x \in X^{(m)}} (v_u(\varphi(x)) - v_u(x)).$$

This defines a continuous and positive real function $S(G, N) \to \mathbb{R}$; since $S(G, N)$ is compact it attains a positive infimum

$$r = \inf \{\rho(u) \mid u \in S(G, N)\} > 0. \tag{5.10}$$

On the other hand we put

$$s = \max \{\|\sigma(y)\| \mid y \in GX^{(m)},\ 1 \in \mathrm{supp}_X y\} > 0. \tag{5.11}$$

Note that, since G operates freely on GX, only finitely many translates of the finite set $\mathrm{supp}_X x$, $x \in X$, can contain the unit element $1 \in G$. Hence s is well defined.

The crucial technical lemma is

LEMMA 5.3. *For every cycle $z \in F_{j-1}$ with $0 \leq j \leq m$ there is a chain $c \in F_j$ with $\partial c = z$ and $\|c\| \leq \max(\|z\|, s^2/2r)$.*

The lemma shows that if t is a real number $> s^2/2r$ then every $(j-1)$-cycle z of $\mathbf{F}_t$ is the boundary of a j-chain of $\mathbf{F}$ with $\|c\| \leq t$; that is, c is, in fact a j-chain in $\mathbf{F}_t$. Hence $\mathbf{F}_t \to A \to 0$ is exact in all dimensions $<m$, whence the required result that A is of type $(FP)_m$ over N.

5.6. *Proof* (of Lemma 5.3). We take any $c \in F_j$ with $\partial c = z$ and assume $\|c\| = a > \max(\|z\|, s^2/2r)$. We shall show that c can be replaced by a chain of smaller norm. The idea to prove this is the following. We consider the support $\operatorname{supp}_X c \subseteq G$ and pick $g \in G$ with $\|g\| = a$. Then we modify c so as to remove this element from the support, at the expense of introducing new elements h with $\|h\| < a$. This reduces the number of elements with maximum norm in $\operatorname{supp}_X c$ by one – thus repeating the argument will eventually yield a chain of smaller norm. Since the values of norms attain only square roots of integers, 0 is the only cluster point of norm values. Hence the procedure yields eventually a chain c with $\|c\| \leq s^2/2r$.

So let $g \in \operatorname{supp}_X c$ with $\|c\| = \|g\| = a$. We consider the expansion $c = \Sigma n_y y$, $y \in GX_j$, with $0 \neq n_y \in \mathbb{Z}$, and decompose $c = c' + c''$, where c' collects all terms $n_y y$ with the property that $hg \in \operatorname{supp}_X y$, for some $h \in N$. Now let

$$u = -\frac{\pi(g)}{\|g\|} \in S(G, N).$$

The corresponding character χ_u, when restricted to $\operatorname{supp}_X c$, takes its minimum value at elements of the form hg, $h \in N$, and this minimum value is equal to $-\|g\| = -a$. Since both $\operatorname{supp}_X z$ and $\operatorname{supp}_X c''$ are contained in $\operatorname{supp}_X c$ but do not contain such elements, we have $v_u(z) > -a$ and $v_u(c'') > -a$. Let $\varphi \in \phi$ and $\sigma : \varphi \cong \mathrm{Id}_{\mathbf{F}}$ as in (5.8) and (5.9). In view of the definition (5.10) we may assume that $v_u(\varphi(x)) - v_u(x) \geq r$ for every $x \in X^{(m)}$. Our aim is to replace c by

$$\begin{aligned} \tilde{c} &= c + \partial\sigma(c') \\ &= \varphi(c') - \sigma(\partial c') + c''. \end{aligned}$$

We have to show that $\|\tilde{c}\| \leq \|c\|$ and that the number of elements of maximum norm in the support has decreased.

Note first, that by (5.10) and Lemma 2.1,

$$v_u(\varphi(c')) \geq v_u(c') + r \geq -a + r.$$

Also,

$$\begin{aligned} v_u(\sigma(\partial c')) &\geq v_u(\partial c'), \quad \text{by Lemma 4.3,} \\ &= v_u(z - \partial c'') \\ &\geq \min\{v_u(z), v_u(c'')\} \\ &> -a. \end{aligned}$$

Hence $v_u(\tilde{c}) \geq \min\{v_u(\varphi(c')), v_u(\sigma(\partial c')), v_u(c'')\} > -a$, which shows that g is certainly not contained in $\operatorname{supp}_X \tilde{c}$.

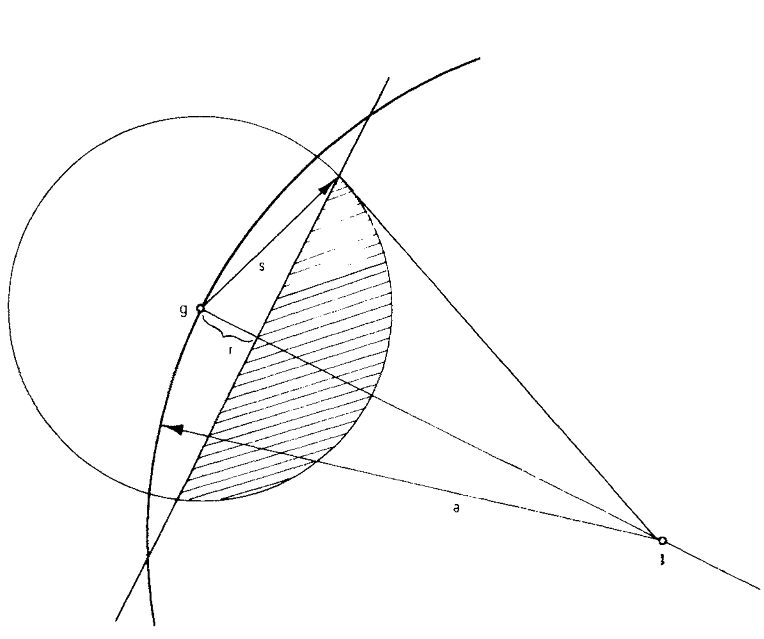
As $\operatorname{supp}_X \partial\sigma(c') \subseteq \operatorname{supp}_X \sigma(c')$ we have, by Lemma 4.3,

$$\operatorname{supp}_X \tilde{c} \subseteq \operatorname{supp}_X c \cup G_{-a+r}. \tag{5.12}$$

Also, for each $y \in GX_j$ occurring in the expansion $c' = \Sigma n_y y$, $g^{-1} \operatorname{supp}_X y$ contains an element of N. Hence, by the definition (5.11)

$$\|g^{-1} \operatorname{supp}_X \partial\sigma(c')\| \leq \|g^{-1} \operatorname{supp}_X \sigma(c')\| \leq s. \tag{5.13}$$

The conjunction of (5.12) and (5.13) then shows that the new elements of $\operatorname{supp}_X \tilde{c}$, that is, $\operatorname{supp}_X \tilde{c} \setminus \operatorname{supp}_X c$, is contained in the range exhibited in the following figure.



By Pythagoras' Theorem the maximum norm of points in the exhibited range is

$$((a-r)^2+s^2-r^2)^{1/2}=(a^2-2ar+s^2)^{1/2}<a.$$

Thus we have shown that $\|\bar{c}\| \leq \|c\| = a$ and that the elements of norm a in $\mathrm{supp}_X\, \bar{c}$ are already in $\mathrm{supp}_X\, c$. Since $g \notin \mathrm{supp}_X\, \bar{c}$, the number of these elements is reduced by at least 1. This completes the proof of Lemma 5.3 and hence also of Theorem 5.1.

6. Topological interpretation and connection with $\Sigma_{G'}$ of [5]

6.1. Let us assume that the G-module A is a *permutation module*. Then the definition of $\Sigma^m(G;A)$ can easily be translated into more topological language by interpreting the admissible free resolution $\mathbf{F}$ as the cellular chain complex of an acyclic CW-complex $\mathfrak{X}$ with a free cellular G-action, such that $\mathfrak{X}/G$ has finite m-skeleton and $H_0(\mathfrak{X})$ is isomorphic to A. The reader might find this translation suggestive – it was certainly invaluable in the process of finding both the results and the proofs in the previous sections. It will also lead to a convenient connection to the invariant Σ_M of [5].

6.2. For simplicity we assume that $\mathfrak{X}$ is, in fact, given with a *simplicial* structure, $\mathfrak{X}=|K|$, and a simplicial free G-action. Then, given a character $\chi: G \to \mathbb{R}$ we find a continuous map $f:\mathfrak{X}\to\mathbb{R}$ as follows. We choose a set X_0 of representatives of the G-orbits on the 0-skeleton $\mathfrak{X}^0$ of $\mathfrak{X}$, and we put, for each $x\in\mathfrak{X}^0$, $f(x)=\chi(g)$, where $g\in G$ is the unique element with $x\in gX_0$. Then we extend the map $f:\mathfrak{X}^0\to\mathbb{R}$ linearly to the higher skeleta. The resulting map $f:\mathfrak{X}\to\mathbb{R}$ is continuous, piecewise linear, and satisfies

$$f(gx)=\chi(g)+f(x), \quad \text{all } g\in G, \qquad x\in\mathfrak{X}.$$

Let $\mathbf{F}=\mathbf{C}(K)$ be the simplicial chain complex of K. $\mathbf{F}$ has the set K of all simplices as a canonical $\mathbb{Z}$-basis acted on by G. Any choice of representatives X of the G-orbits is a G-basis of $\mathbf{F}$, and $\mathbf{F}$ is admissible in the sense of Section 2.4, with respect to this basis. Let us choose X such that X_0 coincides with the previously chosen representatives in $K^0=\mathfrak{X}^0$. Then we find that for every simplex $\sigma\in K$ the support $\mathrm{supp}_X\,\sigma$ and the valuation $v(\sigma)$, as defined in Section 2, are given by

$$\mathrm{supp}_X\,\sigma=\{g\in G \mid \bar{\sigma}\cap gX_0\neq\emptyset\}$$
$$v(\sigma)=\min f(\bar{\sigma}),$$

where we write $\bar{\sigma} \subset \mathfrak{X}$ for the cell corresponding to σ. This shows that the valuation subcomplex $\mathbf{F}_v$ of Section 3.1 coincides with the simplicial complex $\mathbf{C}(K_v)$, where K_v stands for the full subcomplex of K generated by all 0-simplices $x \in K^0 = \mathfrak{X}^0$ with $f(x) \geq 0$. Instead of using the corresponding subspace $|K_v|$ of $|K| = \mathfrak{X}$ it seems more natural, in the present circumstance, to use the subspace $\mathfrak{X}_\chi = \mathfrak{X}_{\chi,0}$, where

$$\mathfrak{X}_{\chi,r} = \{x \in \mathfrak{X} \mid f(x) \geq -r\}, \qquad r \in \mathbb{R}.$$

PROPOSITION 6.1. $[\chi] \in \Sigma^m(G; A)$ *if and only if* (*the permutation module*) *A is finitely generated over G_χ and there is a real number $r \geq 0$ with the property that the homomorphism*

$$\tilde{H}_i(\mathfrak{X}_\chi) \to \tilde{H}_i(\mathfrak{X}_{\chi,r}),$$

induced by inclusion, is the zero map for all $i < m$, $\tilde{H}_i$ is the reduced homology, i.e., $\tilde{H}_i = H_i$ for $i > 0$ and $\tilde{H}_0 = \ker(H_0 \to A)$.

Proof. Show that $|K_v|$ is a deformation retract of $\mathfrak{X}_\chi$ and apply Theorem 3.2.

6.3. Let G be a finitely generated group and $T \subseteq G$ a finite set of generators of G, $1 \notin T$. Then Proposition 6.1 applies, for $A = \mathbb{Z}$ and $m = 1$, if we take for $\mathfrak{X}^1$ the Cayley graph $\Gamma(G, T)$ (in dimension 0 there is no need to pass to a simplicial subdivision). Recall that $\Gamma(G, T)$ is the graph with vertices G and edges $G \times T$, where g is the origin and gt the terminus of the edge $(g, t) \in G \times T$. The condition in Proposition 6.1 for $i = 0$ asserts, that $\mathfrak{X}_\chi$ is "essentially connected" in the sense that each pair of points in $\mathfrak{X}_\chi^0$ can be connected by an edge path of $\mathfrak{X}_{\chi,r}$ for some fixed $r \in \mathbb{R}$. Hence

COROLLARY 6.2. $[\chi] \in \Sigma^1(G; \mathbb{Z})$ *if and only if there is a real number $r \geq 0$ with the property that every element $g \in G_\chi$ can be written as a product $g = t_1 t_2 t_3 \cdots t_s$, with $t_i \in T^{\pm 1}$ and $\chi(t_1 t_2 \cdots t_i) \geq -r$ for every $1 \leq i \leq s$.*

At the expense of adjoining to T an additional generator one can strengthen the condition in Corollary 6.2. Let $t \in G$ with $\chi(t) > r$. If $g \in G_\chi$ then $t^{-1}gt \in G_\chi$, whence $t^{-1}gt = t_1 t_2 \cdots t_s$ as in the corollary. Hence $g = t t_1 t_2 \cdots t_s t^{-1}$ is a product with all its initial segments $t t_1 \cdots t_i \in G_\chi$. That is, we have

COROLLARY 6.3. $[\chi] \in \Sigma^1(G; \mathbb{Z})$ *if and only if G has a finite set of generators T with the property that each $g \in G_\chi$ can be written as a product $g = t_1 t_2 \cdots t_s$ such that all initial segments $t_1 t_2 \cdots t_i$, $1 \leq i \leq s$, are in G_χ.*

6.4. We are now in a position to prove

PROPOSITION 6.4. *If G is a finitely generated group, then $\Sigma^1(G;\mathbb{Z})$ coincides with $-\Sigma_{G'}$, the antipodal set of the invariant of* [5], *where G' is the commutator subgroup of G acted on by conjugation from the right.*

Remark. The slightly unpleasant sign in Proposition 6.4 arises because the groups in [5] are acted on from the right, whereas in the present paper we use left modules. The sign disappears if one considers G' with the left action by conjugation or, alternatively, if one considers $\mathbb{Z}$ as the trivial right G-module.

Proof. If one is to prove that $[\chi] \in \Sigma^1(G;\mathbb{Z})$ by Corollary 6.2 or 6.3 it suffices to verify the corresponding conditions for $g \in G'$ (since there are no problems to verify them modulo G'). Hence the "equational condition" (ii) of [5], Proposition 2.1, shows that if $[\chi] \in -\Sigma_{G'}$ then $[\chi] \in \Sigma^1(G;\mathbb{Z})$. Conversely, assume that the condition in Corollary 6.3 holds, and pick $a \in G'$. Then $\chi(a) = 0$ and so $a = t_1 t_2 \cdots t_s$ as in the corollary. But then

$$a^t = [t, t_1^{-1}][t, t_2^{-1}]^{t_1^{-1}} \cdots [t, t_s^{-1}]^{(t_1 \cdots t_{s-1})^{-1}} a,$$

for all $a \in \mathfrak{R} = \{[u, v] \mid u, v \in T^{\pm 1}\}$ and all $t \in T$, shows that Condition (iv) of [5], Proposition 2.1, is satisfied for $-\chi$.

6.5. *Remark.* Proposition 6.1 suggests that there is a homotopy version of the invariant $\Sigma^m(G;\mathbb{Z})$, which is defined by replacing reduced homology, in the statement of Proposition 6.1, by homotopy. Let us write $^*\Sigma^m(G)$ for these new invariants of the group G; clearly $^*\Sigma^1(G) = \Sigma^1(G;\mathbb{Z})$. For $m \geq 2$ the invariants $^*\Sigma^m(G)$ have been investigated by the second author. This will appear in a separate publication. It turns out that

(a) $^*\Sigma^m(G)$ is an open subset of $S(G)$,
(b) $^*\Sigma^m(G) = {}^*\Sigma^2(G) \cap \Sigma^m(G;\mathbb{Z})$
(c) if $N \lhd G$ is a normal subgroup with Abelian quotient G/N, then N is finitely presented, if and only if $S(G, N) \subseteq {}^*\Sigma^2(G)$.

Whether $^*\Sigma^2(G) = \Sigma^2(G;\mathbb{Z})$ is open and related to the open problem as to whether every group of type $(FP)_2$ is finitely presented.

7. One relator groups

7.1. Throughout this section we write $\Sigma^m(G)$ for the invariant $\Sigma^m(G;\mathbb{Z})$.

Let F be a free group on a basis $X \subseteq F$, $R = gp_F(w) \lhd F$ the normal closure, in

F, of a single word $w \in F$, and G the one relator group F/R. The invariant $\Sigma^1(G)$ has been determined by K. S. Brown [8]; and Walter D. Neumann showed us recently a (topological) argument, based on Brown's computation of $\Sigma^1(G)$ and the invariant $^*\Sigma^2$, which proves $\Sigma^m(G) = \Sigma^1(G)$ for all $m \geq 2$. Thus $\Sigma^m(G)$ is known for all m.

In this section we show how all this can rather nicely be obtained by our techniques of Section 4.

7.2. We have to start with the preliminary

LEMMA 7.1. *If G is a one relator group with $\Sigma^1(G;\mathbb{Z}) \neq \emptyset$ then G is an ascending HNN-extension over a free base group.*

Proof. The set of all points of $S(G)$ represented by an integral character is dense and hence intersects every open set non-trivially. So let $\chi: G \to \mathbb{Z}$, with $[\chi] \in \Sigma^1(G)$, and consider the composite $F \to G \to \mathbb{Z}$. Since the automorphism group of F acts transitively on the equivalence classes in Hom $(F, \mathbb{Z})$ we can apply a free automorphism to the generators X so as to achieve that there is one basis element $t \in X$ with $\chi(t) > 0$ whereas $\chi(x) = 0$ for all remaining elements of X. This implies that the exponent sum of t in w is zero. Hence G admits the usual *HNN*-decomposition with stable letter t over a base group G_1, which is again a one relator group (see [11]). Moreover, the associated subgroups are Magnus subgroups of G_1 and hence are free. Since $[\chi] \in \Sigma^1(G)$ this *HNN*-extension must be ascending ([5], Proposition 4.4). Hence the base group G_1 coincides with one of the (free) associated subgroups.

7.3. Another immediate consequence of $\Sigma^1(G) \neq \emptyset$ is that G is of type $(FP)_1$ and hence finitely generated. One could have a slightly closer look at the *HNN*-extension used above and deduce that G is, in fact, generated by 2 elements (this argument is used by Ken Brown [8]). But the 2-generation will later drop out, essentially at no extra cost.

Now we consider the Lyndon resolution of G,

$$\mathbb{Z}Ge_w \xrightarrow{\partial_2} \bigoplus_{x \in X} \mathbb{Z}Ge_x \xrightarrow{\partial_1} \mathbb{Z}G \to \mathbb{Z} \to 0. \tag{7.1}$$

The differentials ∂_1, ∂_2 are given by

$$\partial_1 e_x = x - 1, \quad \text{all} \quad x \in X$$

$$\partial_2 e_w = \sum_{x \in X} \frac{\partial w}{\partial x} e_x,$$

where $\partial w/\partial x \in \mathbb{Z}G$ stands for the image of the partial Fox derivative under the canonical epimorphism $\mathbb{Z}F \twoheadrightarrow \mathbb{Z}G$. The resolution (7.1) is admissible if $w \neq 1$, and has finitely generated 2-skeleton.

Let $\chi : G \to \mathbb{R}$ be a character. By replacing the basis elements $x \in X$ by x^{-1}, if necessary, we may assume that $\chi(x) \geq 0$ for all $x \in X$, and $\chi(t) > 0$ for one specific $t \in X$. Let v be the valuation on the resolution (7.1) extending χ as defined in Lemma 4.2. Since $\partial e_x = x - 1$, with $v(x) \geq 0$, we find, by definition of v, that $v(e_x) = v(x-1) = 0$ for every $x \in X$.

We define $\sigma_0 : \mathbb{Z}G \to \bigoplus \mathbb{Z}Ge_x$, by putting $\sigma_0(1) = e_t$. Then $v(1 + \partial_1 \sigma_0(1)) = v(t) = \chi(t) > 0$, so that Proposition 4.5 applies for $m = 1$. It follows that $[\chi] \in \Sigma^1(G)$, if and only if we are able to find, for each $x \in X$, an element $\mu_x \in \mathbb{Z}G$ (which we need to define $\sigma_1(e_x) = \mu_x e_w$) such that

$$v\left(e_x + \mu_x \sum_{y \in X} \frac{\partial w}{\partial y} e_y + (x-1)e_t\right) > 0. \tag{7.2}$$

This is certainly very easy for $x = t$, where it suffices to choose $\mu_t = 0$. We can thus rephrase (7.2) by saying that $[\chi] \in \Sigma^1(G)$, if and only if we are able to find, for each $x \in X$, $x \neq t$, an element $\mu_x \in \mathbb{Z}G$ such that the following three inequalities hold

$$v\left(1 + \mu_x \frac{\partial w}{\partial x}\right) > 0 \tag{7.3}$$

$$v\left(x - 1 + \mu_x \frac{\partial w}{\partial t}\right) > 0 \tag{7.4}$$

$$v\left(\mu_x \frac{\partial w}{\partial y}\right) > 0 \quad \text{for all} \quad y \in X - \{x, t\}. \tag{7.5}$$

7.4. Now we infer from Lemma 7.1 that G is (locally free) -by-$\mathbb{Z}$ and hence the group ring $\mathbb{Z}G$ has no zero divisors. The analysis of (7.3)–(7.5) is then greatly simplified by the observation

LEMMA 7.2. *If the group ring $\mathbb{Z}G$ has no zero divisors, then the valuation $v : \mathbb{Z}G \to \mathbb{R}_\infty$ extending χ with respect to the basis $\{1\}$ is multiplicative; that is,*

$$v(\lambda\mu) = v(\lambda) + v(\mu) \quad \textit{for all} \quad \lambda, \mu \in \mathbb{Z}G. \tag{7.6}$$

Proof. For each $\lambda \in \mathbb{Z}G$ we define $\lambda_\chi \in \mathbb{Z}G$ to be the first term in the unique decomposition $\lambda = \lambda_\chi + \lambda_+$ with $\chi(\operatorname{supp} \lambda_\chi) = v(\lambda)$ and $v(\lambda_+) > v(\lambda)$. Then $\lambda\mu =$

$\lambda_\chi\mu_\chi + \lambda_+\mu + \lambda_\chi\mu_+$. Now

$$v(\lambda_+\mu + \lambda_\chi\mu_+) > v(\lambda) + v(\mu)$$

and

$$\operatorname{supp}(\lambda_\chi\mu_\chi) \subseteq (\operatorname{supp}\lambda_\chi)(\operatorname{supp}\mu_\chi).$$

Moreover, $\lambda_\chi\mu_\chi \neq 0$ since $\mathbb{Z}G$ has no zero divisors, whence $\chi(\operatorname{supp}(\lambda_\chi\mu_\chi)) = v(\lambda) + v(\mu)$. This shows that one has

$$(\lambda\mu)_\chi = \lambda_\chi\mu_\chi, \quad \text{all} \quad \lambda, \mu \in \mathbb{Z}G, \tag{7.7}$$

and this implies (7.6).

7.5. The analysis of (7.3)–(7.5) is now easily completed. First we apply Remark 1 of Section 2.1 to the sums in (7.3) and (7.4). We obtain

$$v\left(\mu_x \frac{\partial w}{\partial x}\right) = v(\mu_x) + v\left(\frac{\partial w}{\partial x}\right) = 0,$$

and

$$v\left(\mu_x \frac{\partial w}{\partial t}\right) = v(\mu_x) + v\left(\frac{\partial w}{\partial t}\right) = 0,$$

showing that $v(\mu_x) = -v(\partial w/\partial x) = a$ is the same constant value for each $x \in X$, $x \neq t$. This contradicts (7.5), unless (7.5) is empty, i.e., $X = \{t, x\}$. Thus we have shown that $\Sigma^1(G) \neq \emptyset$ implies G to be a 2-generator group.

It remains to analyse the two inequalities (7.3) and (7.4) with the single parameter $\mu = \mu_x \in \mathbb{Z}G$. By the definition of λ_x in the proof of Lemma 7.2 above and formula (7.7), these inequalities are equivalent to

$$\mu_\chi\left(\frac{\partial w}{\partial x}\right)_\chi = -1, \tag{7.8}$$

and

$$\mu_\chi\left(\frac{\partial w}{\partial t}\right)_\chi = (1-x)_\chi, \quad \text{respectively.} \tag{7.9}$$

But since we have always the equality

$$\frac{\partial w}{\partial x}(x-1) + \frac{\partial w}{\partial t}(t-1) = w - 1 = 0 \quad (\text{in } \mathbb{Z}G),$$

which implies

$$\left(\frac{\partial w}{\partial x}\right)_\chi (x-1)_\chi = \left(\frac{\partial w}{\partial t}\right)_\chi,$$

one can see that (7.9) is a consequence of (7.8). Hence $[\chi] \in \Sigma^1(G)$, if and only if there is $\mu \in \mathbb{Z}G$ satisfying (7.8).

Now (7.8) asserts, in particular, that $(\partial w/\partial x)_\chi$ is a unit in $\mathbb{Z}G$. The structure of G, exhibited in Lemma 7.1, makes it obvious that G is locally indicable (i.e., every finitely generated subgroup admits an infinite cyclic image), hence, by a result of G. Higman, $\mathbb{Z}G$ has only the trivial units. Hence $(\partial w/\partial x)_\chi \in \pm G$. But if so, it is certainly very easy to choose $\mu = \mu_\chi \in \pm G$ so that (7.8) holds. Thus we have the final result

THEOREM 7.3. *Let* $G = \langle t, x = x_1, x_2, \ldots, x_n \mid w \rangle$ *be a one relator group and* $\chi: G \to \mathbb{R}$ *a character with* $\chi(t) > 0$ *and* $\chi(x_i) \geq 0$ *for* $1 \leq i \leq n$. *Then* $[\chi] \in \Sigma^1(G)$, *if and only if* $n = 1$ *and*

$$\left(\frac{\partial w}{\partial x}\right)_\chi \in \pm G.$$

It is, of course, an easy matter to verify whether the condition of Theorem 7.3 holds in a specific situation. If one writes down the Fox derivatives for a general word one recovers Brown's explicit description of $\Sigma(G)$ in [8].

7.6. Let us now assume that $[\chi] \in \Sigma^1(G)$ and choose the element $\mu = \mu_\chi \in \pm G$ such that (7.8) holds. Then we define $\sigma_1: \oplus \mathbb{Z}Ge_x \to \mathbb{Z}Ge_w$ by putting

$$\sigma_1(e_t) = 0,$$
$$\sigma_1(e_x) = \mu e_w.$$

By using (7.3) and (7.4) one finds that $v(\mu e_w) \geq 0$, so that the assumptions of Proposition 4.5 are fulfilled. Hence $[\chi] \in \Sigma^2(G)$, if and only if

$$v(e_w + \sigma_1(\partial_2 e_w)) > v(e_w)$$

(there is no choice for σ_2 left). This inequality is equivalent to

$$v\left(1 + \frac{\partial w}{\partial x}\mu\right) > 0,$$

hence to

$$\left(\frac{\partial w}{\partial x}\right)_\chi \mu_\chi = -1. \tag{7.10}$$

It is now obvious that (7.8) implies (7.10) since $(\partial w/\partial x)_\chi$ and μ_χ are group elements inverse to one another up to a sign. This proves Walter Neumann's result that $\Sigma^2(G) = \Sigma^1(G)$.

To complete the picture, we note that Corollary 3.4 applies to G, whence $\Sigma^m(G) = \Sigma^2(G)$ for every $m \geq 2$.

We summarize

THEOREM 7.4. *If G is a one relator group, then $\Sigma^m(G) = \Sigma^1(G)$ for every $m \geq 1$.*

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