Equivariant Function Spaces and Stable Homotopy Theory I

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Let $F(S^n)$ denote the space of self-maps of the n-sphere with the compact-open topology and the identity as its basepoint. Results of Dold and Lashof [10] and Stasheff [27] show the importance of $F(S^n)$ in the classification of fiber spaces with fiber (homotopically equivalent to) S^n, and because of this the topological properties of $F(S^n)$ yield (or should yield, at least) considerable information about the topology of manifolds. Actually, for purposes of studying manifolds it is preferable to replace the spaces $F(S^n)$ by a so-called *stable version*. To construct this, we embed $F(S^n)$ in $F(S^{n+1})$ via the unreduced suspension functor and set

$$F = \operatorname{inj}\lim_k F(S^k).$$

(In the literature, this space is usually called G; however, we shall soon find it convenient to let G designate a compact Lie group).

If we are given an action of a compact Lie group G on S^n, we shall let $F_G(S^n)$ denote the subspace (submonoid, in fact) of all self-maps of S^n that are *equivariant* with respect to the given actions of G; we shall restrict our attention to group actions given by free orthogonal representations (see §3). In this paper we shall study the homotopy properties of these spaces $F_G(S^n)$ and their corresponding stable versions. Perhaps the most interesting consequence of our work is a relationship between the stable versions of the spaces $F_G(S^n)$ and stable homotopy theory that generalizes the fundamentally important natural isomorphism

$$\theta X: [X, F] \simeq \{X, S^0\}$$

essentially due to G. Whitehead [32], where $[\ ,\ ]$ and $\{\ ,\ \}$ denote homotopy classes of ordinary maps and S-maps respectively and X is a CW complex.

Just as the spaces $F(S^n)$ and F and the isomorphism θX are applicable to the topology of manifolds, the spaces $F_G(S^n)$, their stable analogs, and the results of this paper are applicable to the study of manifolds with G-actions. Applications of our results along these lines appear in [35] and [36].

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1. Introduction

We shall describe some of our results more precisely in this section. Let G be a compact Lie group and W a free G-module (see §3). Let $S(W)$ denote the underlying unit sphere of W. If V is a submodule of W, we denote by $F(V \mid W)$ the space of G-equivariant maps $S(V^*) \to S(W)$, where V^* is the orthogonal complement of V in W. If W' is another free G-module, then $S(W \oplus W')$ is equivariantly homeomorphic to the join of $S(W)$ and $S(W')$; furthermore, the orthogonal complement of V in $W \oplus W'$ is $V^* \oplus W'$. Hence the join functor induces an inclusion of $F(V \mid W)$ in $F(V \mid W \oplus W')$. We define

$$F(V) = \text{inj lim}_k F(V \mid kW),$$

where kW denotes the k-fold sum of W with itself and V is included in the first factor. If V is the trivial G-module $\{0\}$ we write F_G in place of $F(\{0\})$.

Our main result (Theorem (6.6)) gives a description of $F(V)$ as a space constructed from the classifying space of G in a natural way. For example, F_G is describable as follows: let B_G be a classifying space for G with total space E_G, let $\mathfrak{G}$ be the Lie algebra of G and G act on $\mathfrak{G}$ via the adjoint representation; the balanced product of E_G and $\mathfrak{G}$ is a vector bundle over BG which we shall call ζ and whose Thom space we shall call B_G^ζ. Then F_G is homotopy equivalent to $Q(B_G^\zeta)$, where $Q(Y)$ is defined for pointed spaces Y by

$$Q(Y) = \text{inj lim}_k \Omega^k S^k Y.$$

The homotopy equivalence is best understood using its alternate stable homotopy theoretic interpretation. Namely, under the canonical natural isomorphism

$$\theta X : [X, Q(Y)] \cong \{X, Y\}$$

it takes the form of a natural isomorphism

$$\varphi X : [X, F_G] \cong \{X, BG^\zeta\}.$$

If G is the trivial group, then φX is essentially the same as the previously mentioned θX.

There are many generalizations of the spaces F_G, and it is natural to ask whether they too are describable as $Q(Y)$ for suitable choices of Y. We mention two results in this direction:

(i) If G is finite and acts orthogonally on its real group algebra via the regular representation, the homotopy type of F_G is essentially given by results of Graeme Segal [25, Prop. 2 and Corollary to Prop. 7]. Using the techniques of [24] one can derive special cases of Segal's results from some of our results and vice versa.

(ii) Suppose G is finite and acts freely and topologically on S^n; results of R. Lee [17] and T. Petrie [22] show that some finite groups admit such actions (smooth actions, in fact) but not linear ones. In this case one can still define F_G and prove analogs of our results. Details will appear in Part II of this paper.

Sections 2 through 4 contain preliminary material on ex-spaces, vector bundles, and the transfer map for fiber bundles. Our main results are stated in Sections 5 and 6; some of the more technical arguments are postponed to Sections 7, 8, and 9. Finally, we consider the following problem: If H is a closed subgroup of G, there is an inclusion of F_G in F_H because every G-equivariant map is automatically H-equivariant; determine the image of $\pi_*(F_G)$ in $\pi_*(F_H)$. The last three sections (10–12) contain some quantitative results on this problem.

2. Sectioned Bundles

Let B denote a locally finite CW-complex. In the terminology of James [14], an *ex-space* of B is an object $\xi=(E_\xi, B, p_\xi, \Delta_\xi)$ consisting of maps $p_\xi : E_\xi \to B$ and $\Delta_\xi : B \to E_\xi$ such that $p_\xi \Delta_\xi$ is the identity. If ξ and ξ' are ex-spaces, we denote by $[\xi, \xi']$ the set of homotopy classes of fiber and cross section preserving maps $E_\xi \to E_{\xi'}$. Ex-spaces may be regarded as generalizations of pointed spaces and many of the standard constructions for pointed spaces, such as reduced join, wedge, etc., carry over to ex-spaces. This is usually done by performing the construction 'fiberwise'. For detailed accounts see [14], [15], [4].

An ex-space ξ will be called a *sectioned bundle* if it has the following local product structure. There is a pointed space F, with base point (say) x_0, a cover $\{U\}$ of B by open sets, and homeomorphisms $\psi_U : U \times F \to p_\xi^{-1}(U)$ such that the following diagrams are commutative.

$$\begin{array}{ccc} U \times F & \xrightarrow{\psi_U} & p_\xi^{-1}(U) \\ & {}_{p}\searrow \quad \swarrow_{p_\xi} & \\ & U & \end{array} \qquad \begin{array}{ccc} U \times F & \xrightarrow{\psi_U} & p_\xi^{-1}(U) \\ & {}_{\Delta}\nwarrow \quad \nearrow_{\Delta_\xi} & \\ & U & \end{array}$$

Here p is the projection and Δ is the cross section $b \to (b, x_0)$. We will also assume that F is a finite complex and $(E_\xi, \Delta_\xi(B), p_\xi)$ has the homotopy extension property [4; section 2].

The fiberwise reduced join of ξ and α will be denoted by $\xi \wedge \alpha$. There is a suspension map

$$\sigma : [\xi, \xi'] \to [\xi \wedge \alpha, \xi' \wedge \alpha] \tag{2.1}$$

defined by $f \to f \wedge 1$, and the following suspension theorem is proved in [15] (see also [14]).

(2.2) THEOREM. *Suppose that* α *is a sphere bundle and the fiber of* ξ' *is* $(n-1)$-*connected. Then* σ *is injective if* E_ξ *is* $(2n-1)$-*connected and surjective if* E_ξ *is* $2n$-*co-connected.*

If Y and $\hat{Y}$ are homeomorphic pointed spaces let $H(Y, \hat{Y})$ denote the space of base point preserving homeomorphisms from Y to $\hat{Y}$. If $\xi=(E, B, p, \Delta)$ and $\hat{\xi}=(\hat{E}, \hat{B}, \hat{p}, \hat{\Delta})$ are sectioned bundles with the same fiber F, let

$$H(E, \hat{E}) = \bigcup_{(b, \hat{b}) \in B \times \hat{B}} H(p^{-1}(b), \hat{p}^{-1}(\hat{b}))$$

and let $q: H(E, \hat{E}) \to B \times \hat{B}$ denote the obvious projection. For each pair of coordinate maps

$$\psi_U: U \times F \to p^{-1}(U), \qquad \psi_V: V \times \hat{F} \to \hat{p}^{-1}(V)$$

we obtain

$$\psi_{U \times V}: (U \times V) \times H(F, \hat{F}) \to q^{-1}(U \times V)$$

by $(b, \hat{b}, \varphi) \to \psi_b \varphi \psi_{b^{-1}}$. Let $H(E, \hat{E})$ have the smallest topology such that each $\psi_{U \times V}$ is continuous. Then, with this topology, it is easy to check that $(H(E, \hat{E}), B \times \hat{B}, q)$ is a fiber bundle which we denote by $H(\xi, \hat{\xi})$. Now the following bundle covering homotopy property is an immediate consequence of the covering homotopy property for $H(\xi, \hat{\xi})$.

(2.3) THEOREM. *Let* $H: B \times I \to \hat{B}$ *and* $k: E \to \hat{E}$ *be such that* $\hat{p}k = H_0$, k *is cross section preserving, and* k *is a homeomorphism on each fiber. Then there is* $K: E \times I \to \hat{E}$ *such that* $pK = H$, $K_0 = k$, K_t *is cross section preserving, and* K_t *is a homeomorphism on each fiber.*

We conclude this section with some notation and remarks. If X is a pointed space with base point x_0, let $\hat{X}$ denote the sectioned bundle $(B \times X, B, p, \Delta)$ where $p(b, x) = b$ and $\Delta(b) = (b, x_0)$. If α is a vector bundle over B, define $\bar{\alpha}$ to be the sectioned bundle obtained by taking the fiberwise one point compactification of E_α and letting $\Delta_{\bar{\alpha}}$ be the cross section at infinity. Observe that $\overline{\alpha \oplus \beta}$ is canonically equivalent to $\bar{\alpha} \wedge \bar{\beta}$.

There is a functor T from sectioned bundles to pointed spaces defined by $T(\xi) = E_\xi / \Delta_\xi(B)$. If α is a vector bundle, $T(\bar{\alpha})$ is simply the Thom space of α which we will alternately denote by $T(\alpha)$ or B^α. More generally, if $A \subset B$ let

$$(B, A)^\alpha = E_{\bar{\alpha}} / \Delta_{\bar{\alpha}}(B) \cup p_{\bar{\alpha}}^{-1}(A).$$

If X is a pointed space we have $T(X \wedge \xi) = X \wedge T(\xi)$. Note also that projection onto the second factor induces a bijection

$$[\xi, \dot{X}] \to [T(\xi), X]. \tag{2.4}$$

3. Vector Bundles

Suppose that M is a compact differentiable manifold without boundary and G is a compact Lie group acting freely and differentiably on M. By a result of Gleason [11] the orbit map $p: M \to M/G$ has the structure of a principal G-bundle (in fact, a smooth bundle, compare [34]). The tangent bundles of M and M/G are related as follows. Let $\mathrm{Ad}(G)$ denote the G-module determined by the adjoint representation of G. The vector bundle with fiber $\mathrm{Ad}(G)$ associated with $p: M \to M/G$ will be denoted by ζ. One then has an identification

$$\tau(M)/G \simeq \zeta \oplus \tau(M/G), \tag{3.1}$$

and this identification is natural with respect to smooth G-maps [28].

A G-module V will always be assumed to be real, finite dimensional, and equipped with a G-invariant metric. The unit sphere of V will be denoted by $S(V)$ and the quotient space $S(V)/G$ by $M(V)$. We say that V is *free* if G acts freely on $S(V)$. In this case $M(V)$ is a smooth manifold and $p: S(V) \to M(V)$ is a principal G-bundle.

Suppose now that W is a free G-module and $V \subset W$ is a submodule. Let U denote the orthogonal complement of V in W and let η denote the balanced product vector bundle

$$\eta = (S(U) \times V/G, M(U), p) \tag{3.2}$$

Let ξ be the sectioned bundle

$$\xi = (S(U) \times S(W)/G, M(U), p, \Delta) \tag{3.3}$$

where $\Delta[u] = [u, u]$. We have an identification

$$\xi \simeq \overline{\eta \oplus \tau(S(U))/G} \tag{3.4}$$

given by

$$[u, v \oplus u'] \to \left[u, \frac{v}{1 - u \cdot u'}\right] \oplus \left[u, \frac{u' - (u \cdot u')\, u}{1 - u \cdot u'}\right]$$

Combining this with (3.1) we have

$$\xi \simeq \overline{\eta \oplus \zeta \oplus \tau}, \tag{3.5}$$

where τ is the tangent bundle of $M(U)$.

We will also need a description of the Thom space of $\eta \oplus \zeta$ along the lines of [2, Proposition (4.3)]. The map

$$S(U) \times (V \oplus \mathrm{Ad}(G)) \to S(W) \times \mathrm{Ad}(G)$$

by

$$(u, v, y) \to (\sqrt{1 - (|v|/(1 + |v|^2))}\, u \oplus (1/(1 + |v|\, v), y))$$

is equivariant and its quotient extends to an identification

$$M(U)^{\eta\oplus\zeta} \simeq (M(W), M(V))^{\zeta}. \tag{3.6}$$

4. The Transfer

In this section we will give a brief description of the transfer or 'umkehr' map associated with a differentiable fiber bundle. Our account follows that of Boardman [6]. By a manifold we mean a compact differentiable manifold without boundary. Let N be a manifold and M a submanifold of N with normal bundle ω. Choose an embedding $E_\omega \subset N$ of E_ω as a tubular neighborhood of M. Let α be a sectioned bundle over N and consider the maps

$$\begin{array}{ccc} E_\alpha \mid E_\omega & \xrightarrow{k_t} & E_\alpha \mid E_\omega \\ \downarrow p_\alpha & & \downarrow p_\alpha \\ E_\omega & \xrightarrow{j_t} & E_\omega, \end{array} \qquad 0 \leqslant t \leqslant 1. \tag{4.1}$$

where j_t is the canonical homotopy given by $j_t(x) = (1-t)\,x$, and k_t is a sectioned bundle morphism covering j_t such that k_0 is the identity, k_t is the identity on $E_\alpha \mid M$ (where $M \subset E_\omega$ is the 0-section), and k_t is a homeomorphism on each fiber. Such a homotopy exists by (2.3). Define

$$h_\alpha : \alpha \mid E_\omega \to p_\omega^*(\alpha \mid M) \tag{4.2}$$

by $h_\alpha(a) = (p_\alpha(a), k_1(a))$ and let

$$\bar{h}_\alpha : \alpha \mid E_\omega \to \alpha \mid M \tag{4.3}$$

denote the map k_1. The Pontrjagin-Thom map

$$c : T(\alpha) \to T(\bar{\omega} \wedge \alpha \mid M) \tag{4.4}$$

is then given by

$$c(a_x) = \begin{cases} x \wedge \bar{h}_\alpha(a_x), & x \in E_\omega \\ \infty, \text{ if } & x \notin E_\omega. \end{cases}$$

It follows by a standard argument that the homotopy class of c does not depend on the particular choice of covering homotopy.

Let $p : M \to N$ be a differentiable fiber bundle. Choose an embedding $\hat{p} : M \to N \times \times R^s$ homotopic to p and let ω denote the normal bundle. If α is a sectioned bundle over N there is the product bundle $\alpha \times 0$ over $N \times R^s$ and $\alpha \times 0 \mid M \simeq p^*(\alpha)$. Since $T(\alpha \times 0) = T(\alpha) \times R^s/R^s$, the Pontrjagin-Thom map has the form

$$c : T(\alpha) \times R^s/R^s \to T(p^*(\alpha) \oplus \omega).$$

Representing S^s as the one point compactification of R^s, c may be extended to a map

$$t: T(\alpha) \wedge S^s \to T(p^*(\alpha) \oplus \omega). \tag{4.5}$$

In particular, if G is a compact Lie group acting freely on a manifold M and H is a closed subgroup, we have the fiber bundle $p: M/H \to M/G$. Let ζ_G (respectively, ζ_H) denote the bundle over M/G (respectively, M/H) having fiber $\mathrm{Ad}(G)$ (respectively, $\mathrm{Ad}(H)$). Now

$$\tau(M/H) \oplus \omega \simeq p^*(\tau(M/G) \oplus R^s).$$

Adding $\zeta_H \oplus p^*(\zeta_G)$ to both sides and using (3.1) we have

$$\tau(M)/H \oplus p^*(\zeta_G) \oplus \omega \simeq \tau(M)/H \oplus \zeta_H \oplus R^s.$$

For sufficiently large s we may cancel $\tau(M)/H$ obtaining an equivalence

$$p^*(\zeta_G) \oplus \omega \simeq \zeta_H \oplus R^s. \tag{4.6}$$

Thus, the map t of (4.5) yields

$$t: T(\alpha \wedge \zeta_G) \wedge S^s \to T(p^*(\alpha) \wedge \zeta_H) \wedge S^s. \tag{4.7}$$

The stable homotopy class of this map does not depend on the particular choice of embedding because of the following: (a) isotopic embeddings determine homotopic maps. (b) the effect of replacing $\hat{p}: M/G \to M/H \times R^s$ by $i\hat{p}: M/G \to M/H \times R^{s+1}$, where i is the usual inclusion, is to replace t by its suspension. (c) for sufficiently large s, any two embeddings homotopic to p are isotopic.

We shall call t in (4.7) the *transfer* associated with the bundle $p: M/H \to M/G$. It is easily seen that t is functorial with respect to smooth G-maps. Moreover, if H has finite index in G (so that p is a finite covering map) t agrees with the transfer defined and axiomatized by Roush [23]. A proof of this fact will be given in the appendix.

Consider now the situation of the previous section. If V is a G-module write $V = V_G$ and let V_H denote its underlying H-module. Suppose that $V_G \oplus U_G = W_G$. Let η_G and η_H be as in (3.2). We have the fiber bundle $p: M(U_H) \to M(U_G)$ and since $p^*(\eta_G) = \eta_H$ we obtain a transfer map

$$T(\eta_G \oplus \zeta_G) \wedge S^s \to T(\eta_H \oplus \zeta_H) \wedge S^s.$$

Making the identification (3.6) we have

$$t: (M(W_G), M(V_G))^{\zeta_G} \wedge S^s \to (M(W_H), M(V_H))^{\zeta_H} \wedge S^s. \tag{4.8}$$

5. The Spaces $F(V \mid W)$

If α and β are sectioned bundles, let $\mathscr{M}(\alpha, \beta)$ denote the space of fiber and cross section preserving maps $E_\alpha \to E_\beta$, with the compact-open topology. Recall that if Y is a pointed space

$$Q(Y) = \text{inj lim}_k \mathcal{M}(S^k, Y \wedge S^k).$$

Let V and W be free G-modules such that $V \subset W$ and $V \neq W$. Let V^* denote the orthogonal complement of V in W. We define $F(V \mid W)$ to be the pointed space of G-equivariant maps $S(V^*) \to S(W)$, the inclusion map being the base point. Our objective is to construct a map

$$\lambda: F(V \mid W) \to Q\big((M(W), M(V))^\zeta\big). \tag{5.1}$$

Let

$$\xi = \big(S(V^*) \times S(W)/G, M(V^*), p, \Delta\big) \tag{5.2}$$

where p and Δ are induced by the projection and diagonal respectively. From (3.4) we have an identification

$$\xi \simeq \overline{\eta \oplus \zeta \oplus \tau}, \tag{5.3}$$

where τ is the tangent bundle of $M(V^*)$ and $\eta = \big(S(V^*) \times V/G, M(V^*), p\big)$. The function

$$\theta: F(V \mid W) \to \mathcal{M}(\dot{S}^0, \xi) \tag{5.4}$$

defined by sending $f: S(V^*) \to S(W)$ to $f': S^0 \times M(V^*) \to S(V^*) \times S(W)/G$, where $f'(0, [y]) = [y, f(y)]$ and $f'(\infty, [y]) = [y, y]$ is easily seen to be a homeomorphism of function spaces.[3] Making the identification (3.4), θ becomes

$$\theta: F(V \mid W) \to \mathcal{M}(\dot{S}^0, \overline{\eta \oplus \zeta \oplus \tau}). \tag{5.5}$$

Choose an embedding $M(V^*) \subset R^s$ and let ν denote the normal bundle. Let $\psi: \tau \oplus \nu \to R^s$ denote the associated trivialization and $c: S^s \to T(\nu)$ the Pontrjagin-Thom map. The map λ is to be the following composition.

$$\begin{aligned} &F(V \mid W) \xrightarrow{\theta} \mathcal{M}(\dot{S}^0, \overline{\eta \oplus \zeta \oplus \tau}) \xrightarrow{\sigma} \mathcal{M}(\bar{\nu}, \overline{\eta \oplus \zeta \oplus \tau \oplus \nu}) \\ &\xrightarrow{\mathcal{M}(1 \oplus \psi)} \mathcal{M}(\bar{\nu}, \overline{\eta \oplus \zeta \oplus R^s}) \xrightarrow{T} \mathcal{M}(T(\nu), T(\eta \oplus \zeta) \wedge S^s) \\ &\xrightarrow{\mathcal{M}(c)} \mathcal{M}(S^s, T(\eta \oplus \zeta) \wedge S^s) \to Q(T(\eta \oplus \zeta)) \\ &\longrightarrow Q\big((M(W), M(V))^\zeta\big). \end{aligned} \tag{5.6}$$

Here σ is suspension and the last map is given by the identification (3.6). It is easy to check that the homotopy class of λ does not depend on the choice of embedding.

[3]) We use S^n to denote the one point compactification of R^n. The sphere of unit vectors in R^{n+1} will be denoted by $S(R^{n+1})$.

(5.7) THEOREM. *λ is an n-equivalence where* $n = \dim(V) + \dim(W) + \dim(G) - 2$.

Proof. It follows from the suspension theorem (2.2) that σ is an n-equivalence. It remains to show that $\mathcal{M}(c)\,T$ is an n-equivalence for large s. Let $\alpha = \eta \oplus \zeta \oplus R^s$. Choose a complementary bundle β and let $\varphi: \beta \oplus \alpha \to R^t$ be a trivialization. We then have a duality map

$$\mu: S^{s+t} \to T(\nu \oplus \beta) \wedge T(\alpha)$$

given by the composite

$$S^{s+t} \xrightarrow{c \wedge 1} T(\nu) \wedge S^t \xrightarrow{T(1 \oplus \varphi^{-1})} T(\nu \oplus \beta \oplus \alpha) \xrightarrow{\Delta} T(\nu \oplus \beta) \wedge T(\alpha),$$

where Δ is the diagonal map. Let X be a finite complex such that $\dim(X) \leqslant n$. The associated correspondence

$$D_\mu: [X \wedge T(\nu \oplus \beta), S^t] \to [X \wedge S^{s+t}, T(\alpha) \wedge S^t]$$

defined by sending $f: X \wedge T(\nu \oplus \beta) \to S^t$ to the map

$$X \wedge S^{s+t} \xrightarrow{1 \wedge \mu} X \wedge T(\nu \oplus \beta) \wedge T(\alpha) \xrightarrow{f \wedge 1} S^t \wedge T(\alpha) \to T(\alpha) \wedge S^t$$

is bijective, provided we are in the stable range. Let us take t to be large enough so that this is the case.

We have the following commutative diagram.

$$\begin{array}{ccccc}
[\dot{X} \wedge \bar{\nu}, \bar{\alpha}] & \xrightarrow{T} & [X \wedge T(\nu), T(\alpha)] & \xrightarrow{(1 \wedge c)^\#} & [X \wedge S^s, T(\alpha)] \\
\downarrow \sigma & & & & \downarrow \sigma \\
[\dot{X} \wedge \overline{\nu \oplus R^t}, \overline{\alpha \oplus R^t}] & \xrightarrow{T} & [X \wedge T(\nu) \wedge S^t, T(\alpha) \wedge S^t] & \xrightarrow{(1 \wedge c \wedge 1)^\#} & [X \wedge S^{s+t}, T(\alpha) \wedge S^t] \\
\uparrow 1 \wedge (1 \oplus \varphi^{-1})^\# & & & & \\
[\dot{X} \wedge \overline{\nu \oplus \beta \oplus \alpha}, \overline{\alpha \oplus R^t}] & & & & \uparrow D_\mu \\
\uparrow & & & & \\
[\dot{X} \wedge \overline{\nu \oplus \beta \oplus \alpha}, \overline{R^t \oplus \alpha}] & & & & \\
\uparrow \sigma & & & & \\
[\dot{X} \wedge \nu \oplus \beta, R^t] & & \xrightarrow{\qquad\qquad T \qquad\qquad} & & [X \wedge T(\nu \oplus \beta), S^t]
\end{array}$$

For sufficiently large s the suspension maps in the above diagram are bijective and therefore $(1 \wedge c)^\# T$ is bijective as desired.

We will now consider the functorial properties of the map λ. We identify the unreduced join $S(V) * S(W)$ with $S(V \oplus W)$ by the map $[v, w, t] \to tv \oplus \sqrt{1-t^2}\,w$. If

$V \subset U \subset W$ there is an inclusion map

$$j: F(V \mid U) \to F(V \mid W) \tag{5.8}$$

induced by the join operation as follows. Let V^* denote the orthogonal complement of V in U and U^* the orthogonal complement of U in W. Then j is defined by sending $f: S(V^*) \to S(U)$ to $f * 1: S(V^*) * S(U^*) \to S(U) * S(U^*)$. Let

$$i: (M(U), M(V))^{\zeta} \to (M(W), M(V))^{\zeta} \tag{5.9}$$

denote the inclusion, and let X denote a finite complex.

(5.10) *The following diagram is commutative.*

$$\begin{array}{ccc} [X, F(V \mid U)] & \xrightarrow{\lambda^{\#}} & [X, Q((M(U), M(V))^{\zeta})] \\ \downarrow{\scriptstyle j_{\#}} & & \downarrow{\scriptstyle Q(i)_{\#}} \\ [X, F(V \mid W)] & \xrightarrow{\lambda^{\#}} & [X, Q((M(W), M(V))^{\zeta})]. \end{array}$$

Let

$$r: F(V \mid W) \to F(U \mid W) \tag{5.11}$$

denote the map defined by restricting $f: S(V^*) \to S(W)$ to $S(U^*)$, and let

$$c: (M(W), M(V))^{\zeta} \to (M(W), M(U))^{\zeta} \tag{5.12}$$

be the collapsing map.

(5.13) *The following diagram is commutative*

$$\begin{array}{ccc} [X, F(V \mid W)] & \xrightarrow{\lambda^{\#}} & [X, Q((M(W), M(V))^{\zeta})] \\ \downarrow{\scriptstyle r_{\#}} & & \downarrow{\scriptstyle Q(c)_{\#}} \\ [X, F(U \mid W)] & \xrightarrow{\lambda^{\#}} & [X, Q((M(W), M(U))^{\zeta})]. \end{array}$$

Finally, if H is a closed subgroup of G there is the natural forgetful map

$$\varphi: F(V_G \mid W_G) \to F(V_H \mid W_H), \tag{5.14}$$

and for sufficiently large s, there is a transfer map

$$t: (M(W_G), M(V_G))^{\zeta_G} \wedge S^s \to (M(W_H), M(V_H))^{\zeta_H} \wedge S^s \tag{5.15}$$

as in (4.8).

(5.16) *The following diagram is commutative*

$$\begin{array}{ccc} [X, F(V_G \mid W_G)] & \xrightarrow{\lambda_\#} & [X, Q((M(W_G), M(V_G))^{\zeta_G})] \\ \downarrow{\scriptstyle \varphi_\#} & & \downarrow{\scriptstyle Q(t)_\#} \\ [X, F(V_H \mid W_H)] & \xrightarrow{\lambda_\#} & [X, Q((M(W_H), M(V_H))^{\zeta_H})]. \end{array}$$

Proofs for (5.10), (5.13) and (5.16) are given in section 8.

6. The Spaces $F(V)$.

Given a free G-module V, choose a free G-module W such that $V \subset W$ and $V \neq W$. Let kW denote the k-fold direct sum of W and define

$$F(V) = \text{inj lim}_k F(V \mid kW) \tag{6.1}$$

and

$$B(V)^\zeta = \text{inj lim}_k (M(kW), M(V))^\zeta \tag{6.2}$$

If X is a pointed finite CW-complex the map

$$\lambda_\#: [X; F(V \mid kW)] \to [X, Q((M(kW), M(V))^\zeta)]$$

is, by (5.10), compatible with the above inclusions. Hence we obtain

$$\lambda(V): [X; F(V)] \to [X; Q(B(V)^\zeta)] \tag{6.3}$$

as the injective limit of the $\lambda_\#$. As a result of theorem (4.5) we have

(6.4) THEOREM. *$\lambda(V)$ is a natural equivalence of homotopy functors on the category of finite* CW-*complexes.*

We next show that $F(V)$ has the homotopy type of a CW-complex. To do this it is sufficient to show that the spaces $F(V \mid W)$ have the homotopy type of a CW-complex. Since $F(V \mid W)$ is homeomorphic to the space of cross sections to the bundle $S(V^*) \times S(W)/G \to M(V^*)$, the result for $F(V \mid W)$ is a consequence of the following.

(6.5) LEMMA. *Let $p: E \to B$ be a Hurewicz fibration with fiber F. Suppose that B is compact and both B and F have the homotopy type of a* CW-*complex. Then the space of cross sections to p has the homotopy type of a* CW-*complex.*

Proof. Let $\mathscr{CH}$ denote the category of spaces having the homotopy type of a CW-complex. First, suppose that $p: E \to B$ is a fibration such that E and B are in $\mathscr{CH}$. We will show that the fiber F is in $\mathscr{CH}$. If we replace the inclusion $i: F \to E$ by a fibration $i':F'\to E$ in the usual way, the fiber over e has the homotopy type of $\Omega(B, p(e))$ [21]. By a result of Milnor [19], $\Omega(B, p(e))$ is in $\mathscr{CH}$. Hence by a theorem of Stasheff [27], F' is in $\mathscr{CH}$. Therefore F is in $\mathscr{CH}$.

Now let $p:E\to B$ be as in the statement of the lemma. By the exponential law, $p':E^B\to B^B$ is also a Hurewicz fibration and since both E^B and B^B are in $\mathscr{CH}$ [19], the fiber over the identity is in $\mathscr{CH}$. This is just the space of cross sections to p.

As a consequence of (6.4), we have proved the following:

(6.6) THEOREM. *The space $F(V)$ is homotopy equivalent to $Q(B(V)^\zeta)$.*

Since the homotopy type of $B(V)^\zeta$ clearly does not depend on the choice of ambient G-module W, Theorem (6.6) has an obvious consequence.

(6.7) COROLLARY. *The homotopy type of $F(V)$ depends only on the representation V.*

There are two functorial properties of the transformation $\lambda(V)$. Firstly, if V is a submodule of U we obtain from (5.13) the following commutative diagram

$$\begin{array}{ccc} [X: F(V)] & \xrightarrow{\lambda(V)} & [X; Q(B(V)^\zeta)] \\ \downarrow r_\# & & \downarrow Q(c)_\# \\ [X; F(U)] & \xrightarrow{\lambda(U)} & [X; Q(B(U)^\zeta)]. \end{array} \tag{6.8}$$

Secondly, if H is a closed subgroup of G, we have a *transfer*

$$t_\#:[X; Q(B(V_H)^{\zeta_H})]\to[X; Q(B(V_G)^{\zeta_G})] \tag{6.9}$$

defined to be the injective limit of the maps $Q(t)_\#$, where $Q(t)$ is the map appearing in (5.16). Then by (5.16) we have a commutative diagram

$$\begin{array}{ccc} [X; F(V_G)] & \xrightarrow{\lambda(V_G)} & [X; Q(B(V_G)^{\zeta_G})] \\ \downarrow \varphi_\# & & \downarrow t_\# \\ [X; F(V_H)] & \xrightarrow{\lambda(V_H)} & [X; Q(B(V_H)^{\zeta_H})]. \end{array} \tag{6.10}$$

Actually, by the methods of [6], one can construct in a natural way, a map $t:Q(B(V_G)^{\zeta_G})\to Q(B(V_H)^{\zeta_H})$ which realizes the transfer $t_\#$. Since we will not need such a map, we do not carry out the construction here.

If V is the trivial G-module $\{0\}$ we shall write F_G in place of $F(V)$ and B_G^ζ in place of $B(V)^\zeta$. Thus, F_G is the injective limit of the space of G-equivariant self maps of $S(kW)$ and B_G^ζ is the Thom space of the bundle with fiber $\mathrm{Ad}(G)$ associated to the universal principal G-bundle.

We shall now examine some special cases of the preceding results. First, if G is the trivial group we write F in place of F_G. In this case $B_G^\zeta=S^{\infty+}$ may be identified with S^0 by collapsing S^∞ to a point. Let us write $Q^{(0)}(S^0)$ (respectively, $Q^{(1)}(S^0)$) to denote $Q(S^0)$ with the constant map (respectively, the identity map) as base point. We will relate

$$\lambda : [X; F] \to [X; Q^{(0)}(S^0)] \tag{6.11}$$

to a more familiar map. Let

$$T: [X; Q^{(1)}(S^0)] \to [X; Q^{(0)}(S^0)] \tag{6.12}$$

be defined as follows. First let $T' : \Omega^k(S^k) \to \Omega^k(S^k)$ send f to the composite

$$S^k \xrightarrow{h} S^k \vee S^k \xrightarrow{1 \vee Rf} S^k \vee S^k \xrightarrow{g} S^k,$$

where h is the pinching map, R is the reflection $(x_1, x_2, \ldots, x_k) \to (-x_1, x_2, \ldots, x_k)$, and g is the folding map. (With respect to loop addition T' sends f to $1-f$). Let $H: S^k \times I \to S^k$ denote the canonical homotopy from $T'(1)$ to the constant map. Then T is to be the injective limit of

$$[X' \Omega^k(S^k)] \xrightarrow{T'_\#} [X; \Omega^k(S^k)] \xrightarrow{H_\#} [X; \Omega^k(S^k)]$$

There is also a natural inclusion $\iota: F \to Q^{(1)}(S^0)$ defined by sending $f: S(R^k) \to \to S(R^k)$ to its radial extension $\bar{f}: S^k \to S^k$ given by $\bar{f}(tv) = tf(v)$, $t \geqslant 0$, $|v| \geqslant 1$.

(6.13) THEOREM. *The triangle*

$$\begin{array}{ccc} [X, F] & \xrightarrow{\quad\lambda\quad} & [X; Q^{(0)}(S^0)] \\ & {\scriptstyle \iota_\#}\searrow \quad \nearrow{\scriptstyle T} & \\ & [X; Q^{(1)}(S^0)] & \end{array}$$

is commutative.

A proof of (6.13) will be given in Section 9.

Now let K denote one of the fields R, C or H, the real, complex, or quaternionic numbers and let d denote the dimension of K over R. Let $G = S^{d-1}$ and let V denote the standard representation of G on R^d given by scalar multiplication. Then the space $F(kV)$, which we shall now denote by L_k, is the injective limit over n of the spaces L_k^n, where L_k^n is the space of S^{d-1}-equivariant maps $S^{d(n-k)-1} \to S^{dn-1}$.

Let S^{d-1} act on $S^{dn-1} \times S^{d-1}$ by $(x, y) \to (gx, gyg^{-1})$, $g \in S^{d-1}$. The *quasi-projective* space $\tilde{P}_n$ defined by James [12] is the space obtained from $S^{dn-1} \times S^{d-1}/S^{d-1}$ by identifying the section $S^{dn-1} \times \{1\}/S^{d-1}$ to a point (see [2; section 5]). It is easy to see that $\tilde{P}_n$ is the Thom space of the bundle with fiber $\mathrm{Ad}(S^{d-1})$ associated with the bundle $S^{dn-1} \to P_n$, where P_n is the projective space S^{dn-1}/S^{d-1} [2]. Let $\tilde{P}_\infty = \mathrm{inj\,lim}_n \tilde{P}_n$ and let $\tilde{P}_0$ be the base point of $\tilde{P}_\infty$. Then with these changes in notation we obtain from (6.6) a homotopy equivalence

$$L_k \simeq Q(\tilde{P}_\infty/\tilde{P}_k). \tag{6.14}$$

In particular,

$$F_{S^{d-1}} \simeq Q(\tilde{P}_\infty). \tag{6.15}$$

Note that $R\tilde{P}_\infty = RP^{\infty+}$ and $C\tilde{P}^\infty = (CP^{\infty+}) \wedge S^1$.

7. Morphisms of Sectioned Bundles

In this section we take up some properties of the mapping set $[\alpha, \beta]$ which will be needed to establish the functorial properties of the transformation λ.

Suppose that N is a manifold and $M \subset N$ is a submanifold with normal bundle ω. Let $E_\omega \subset N$ as a tubular neighborhood. Then if α is a sectioned bundle over N we have

$$h_\alpha : \alpha \mid E_\omega \to p_\omega^*(\alpha \mid M), \qquad \tilde{h}_\alpha : \alpha \mid E_\omega \to \alpha \mid M$$

as in (4.2) and (4.3). Let β denote another sectioned bundle over N and define

$$e : [\bar{\omega} \wedge \alpha \mid M, \beta \mid M] \to [\alpha, \beta] \tag{7.1}$$

by

$$e(f)(a_x) = \begin{cases} h_\beta^{-1}(x, f(x \wedge \tilde{h}_\alpha(a_x))), & x \notin E_\omega \\ \Delta_\beta(x), & x \notin E_\omega . \end{cases}$$

The map e is easily seen to be natural with respect to suspension. That is, if γ is another sectioned bundle over N, the following diagram is commutative.

$$\begin{array}{ccc} [\bar{\omega} \wedge \alpha \mid M, \beta \mid M] & \xrightarrow{\sigma} & [\bar{\omega} \wedge (\alpha \wedge \gamma) \mid M, (\beta \wedge \gamma) \mid M] \\ \downarrow e & & \downarrow e \\ [\alpha, \beta] & \xrightarrow{\sigma} & [\alpha \wedge \gamma, \beta \wedge \gamma]. \end{array} \tag{7.2}$$

The relation between e and the Pontrjagin-Thom map $c : T(\alpha) \to T(\bar{\omega} \wedge \alpha \mid M)$ is given by the following commutative diagram.

$$\begin{array}{ccc} [\bar{\omega} \wedge \alpha \mid M, \beta \mid M] & \xrightarrow{T} & [T(\bar{\omega} \wedge \alpha \mid M), T(\beta \mid M)] \\ & & \downarrow c^\# \\ \downarrow e & & [T(\alpha), T(\beta \mid M)] \\ & & \downarrow i_\# \\ [\alpha, \beta] & \xrightarrow{T} & [T(\alpha), T(\beta)]. \end{array} \tag{7.3}$$

Here $i: T(\beta \mid M) \to T(\beta)$ denotes the inclusion. To prove (7.3) we have

$$T\,e(f)\,(a_x) = \begin{cases} h_\beta^{-1}(x, f(x \wedge \tilde{h}_\alpha(a_x))), & x \in E_\omega \\ \infty, & x \notin E_\omega, \end{cases}$$

and

$$i: T(f)\,c\,(a_x) = \begin{cases} f(x \wedge \tilde{h}_\alpha(a_x), & x \in E_\omega \\ \infty, & x \notin E_\omega. \end{cases}$$

A connecting homotopy H is given by

$$H(a_x, t) = \begin{cases} h_\beta^{-1}(tx, f(x \wedge \tilde{h}_\alpha(a_x))), & x \in E_\omega \\ \infty, & x \notin E_\omega. \end{cases}$$

Now consider the restriction map

$$r: [\alpha, \beta] \to [\alpha \mid M, \beta \mid M]. \tag{7.4}$$

Note that for $f: \alpha \to \beta$ we have $\tilde{h}_\beta f \simeq f \mid M \tilde{h}_\alpha$ since both are the end of a homotopy from $\alpha \mid E_\omega$ to $\beta \mid E_\omega$ which begins at f and covers the homotopy j_t of (4.1). From this observation and a straightforward calculation we obtain the following commutative diagram.

$$\begin{array}{ccc} [\alpha, \beta] & \xrightarrow{\quad T \quad} & [T(\alpha), T(\beta)] \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle c_\#} \\ [\alpha \mid M, \beta \mid M] & & [T(\alpha), T(\beta \mid M \wedge \bar{\omega})] \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle c^\#} \\ [\alpha \mid M \wedge \bar{\omega}, \beta \mid M \wedge \bar{\omega}] & \xrightarrow{T} & [T(\alpha \mid M \wedge \bar{\omega}), T(\beta \mid M \wedge \bar{\omega})]. \end{array} \tag{7.5}$$

Suppose now that $p: M \to N$ is a map and α, β are sectioned bundles over N. There is then the induced map

$$p^*: [\alpha, \beta] \to [p^*(\alpha), p^*(\beta)] \tag{7.6}$$

defined by $p^*(f)(m, a) = (m, f(a))$, $m \in M$, $a \in E_\alpha$. Suppose further that $p: M \to N$ is a differentiable fiber bundle. Let $\hat{p}: M \to N \times R^s$ be an embedding homotopic to p, let ω denote the normal bundle, and let $\pi: N \times R^s \to N$ denote the projection. Let us also choose $\hat{p}$ so that $\pi\hat{p} = p$. Then $p^*(\alpha) = \pi^*(\alpha) \mid M$ and under this identification the map p^* of (7.6) corresponds to

$$[\alpha, \beta] \xrightarrow{\pi^*} [\pi^*(\alpha), \pi^*(\beta)] \xrightarrow{r} [p^*(\alpha), p^*(\beta)].$$

where r is the restriction map. Hence by the commutativity of (7.5) and the definition of the transfer t, we have the following commutative diagram

$$\begin{array}{ccc} [\alpha, \beta] \xrightarrow{T} [T(\alpha), T(\beta)] & \xrightarrow{\sigma} & [T(\alpha) \wedge S^s, T(\beta) \wedge S^s] \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle t_\#} \\ [p^*(\alpha), p^*(\beta)] & & [T(\alpha) \wedge S^s, T(p^*(\beta) \wedge \bar{\omega})] \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle t^\#} \\ [p^*(\alpha) \wedge \bar{\omega}, p^*(\beta) \wedge \bar{\omega}] & \xrightarrow{T} & [T(p^*(\alpha) \wedge \bar{\omega}), T(p^*(\beta) \wedge \bar{\omega})]. \end{array} \tag{7.7}$$

8. The Functorial Properties of λ

We will first establish property (5.10). Let U, V, and W be free G-modules such that $V \subset U \subset W$. Let V^* denote the orthogonal complement of V in W and V^{**} the orthogonal complement of V in U. We then have $M(V^{**}) \subset M(V^*)$. We let η, ζ, τ denote the bundles over $M(V^*)$ which appear in the definition of $\lambda(V \mid W)$ and η_0, ζ_0, τ_0 those over $M(V^{**})$ which appear in the definition of $\lambda(V \mid U)$. Let ω denote the normal bundle of $M(V^{**})$ in $M(V^*)$.

Let X be a finite complex. Since the restriction of τ to $M(V^{**})$ is $\tau_0 \oplus \omega$, we have

$$[\dot{X}, \overline{\eta_0 \oplus \zeta_0 \oplus \tau_0}] \xrightarrow{\sigma} [\dot{X} \wedge \bar{\omega}, \overline{\eta_0 \oplus \zeta_0 \oplus \tau_0 \oplus \omega}] \xrightarrow{e} [\dot{X}, \overline{\eta \oplus \zeta \oplus \tau}].$$

and we denote this composite by $\hat{e}$. A lengthy but straightforward calculation shows that the following diagram is commutative.

$$\begin{array}{ccc} [X, F(V \mid U)] & \xrightarrow{\theta_\#} & [\dot{X}, \overline{\eta_0 \oplus \zeta_0 \oplus \tau_0}] \\ \downarrow{\scriptstyle j_\#} & & \downarrow{\scriptstyle \hat{e}} \\ [X, F(V \mid W)] & \xrightarrow{\theta_\#} & [\dot{X}, \overline{\eta \oplus \zeta \oplus \tau}] \end{array} \tag{8.1}$$

Now let $M(V^*) \subset R^s$ with normal bundle ν and let ν_0 denote normal bundle of the composite embedding $M(V^{**}) \subset R^s$. Then $\nu_0 \simeq \omega \oplus (\nu \mid M(V^{**}))$ so that, by (7.2) and the definition of $\hat{e}$ we obtain a commutative diagram

$$\begin{array}{ccc} [\dot{X}, \overline{\eta_0 \oplus \zeta_0 \oplus \tau_0}] & \xrightarrow{\sigma} & [\dot{X} \wedge \bar{\nu}_0, \overline{\eta_0 \oplus \zeta_0 \oplus \tau_0 \oplus \nu_0}] \\ \downarrow{\scriptstyle \hat{e}} & & \downarrow{\scriptstyle e} \\ [\dot{X}, \eta \oplus \zeta \oplus \tau] & \xrightarrow{\sigma} & [\dot{X} \wedge \bar{\nu}, \overline{\eta \oplus \zeta \oplus \tau \oplus \nu}]. \end{array} \tag{8.2}$$

Let $\psi : \tau \oplus \nu \to R^s$ and $\psi_0 : \tau_0 \oplus \nu_0 \to R^s$ denote the trivializations associated with the embeddings. Since ψ_0 is the restriction of ψ we have the commutativity relation.

$$\begin{array}{ccc} [\dot{X} \wedge \bar{\nu}_0, \overline{\eta_0 \oplus \zeta_0 \oplus \tau_0 \oplus \nu_0}] & \xrightarrow{(1 \oplus \psi_0)_\#} & [\dot{X} \wedge \bar{\nu}_0, \overline{\eta_0 \oplus \zeta_0 \oplus R^s}] \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\ [\dot{X} \wedge \bar{\nu}, \overline{\eta \oplus \zeta \oplus \tau \oplus \nu}] & \xrightarrow{(1 \oplus \psi)_\#} & [\dot{X} \wedge \nu, \overline{\eta \oplus \zeta \oplus R^s}]. \end{array} \tag{8.3}$$

Now, by (7.3) we have a commutative diagram

$$\begin{array}{ccc} [\dot{X} \wedge \bar{\nu}_0, \overline{\eta_0 \oplus \zeta_0 \oplus R^s}] & \xrightarrow{T} & [X \wedge T(\nu_0), T(\eta_0 \oplus \zeta_0) \wedge S^s] \\ \Big\downarrow e & & \Big\downarrow c^{\#} \\ & & [X \wedge T(\nu), T(\eta_0 \oplus \zeta_0) \wedge S^s] \\ & & \Big\downarrow i_{\#} \\ [\dot{X} \wedge \bar{\nu}, \overline{\eta \oplus \zeta \oplus R^s}] & \xrightarrow{T} & [X \wedge T(\nu), T(\eta \oplus \zeta) \wedge S^s]. \end{array} \tag{8.4}$$

Property (5.10) now follows easily from the commutativity of the diagrams (8.1) through (8.4), together with the relation

$$\begin{array}{ccc} & & T(\nu) \\ & \nearrow^{c} & \\ S^s & & \Big\downarrow c \\ & \searrow_{c} & \\ & & T(\nu_0). \end{array} \tag{8.5}$$

We turn now to the proof of (5.16). Let V_G and W_G be free G-modules such that $V_G \subset W_G$ and let V_H and W_H denote their underlying H-modules. Let $p: M(V_G^*) \to \to M(V_G^*)$ denote the projection and choose an embedding $\hat{p}: M(V_H^*) \to M(V_G^*) \times R^{s_1}$ such that $\pi\hat{p} = p$, where $\pi: M(V_G^*) \times R^{s_1} \to M(V_G^*)$ is the projection. Let ω denote the normal bundle to this embedding. The bundles over $M(V_G^*)$ which appear in the definition of λ will be denoted by a subscript G and those over $M(V_H^*)$ by a subscript H. We then have $p^*(\eta_G) = \eta_H$ and $p^*(\zeta_G \oplus \tau_G) = \zeta_H \oplus \tau_H$.

We have the following commutative diagram

$$\begin{array}{ccc} [X, F(V_G \mid W_G)] & \xrightarrow{\theta} & [\dot{X}, \overline{\eta_G \oplus \zeta_G \oplus \tau_G}] \\ \Big\downarrow \varphi & & \Big\downarrow p^* \\ [X, F(V_H \mid W_H)] & \xrightarrow{\theta} & [\dot{X}, \overline{\eta_H \oplus \zeta_H \oplus \tau_H}] \end{array} \tag{8.6}$$

Now choose an embedding $M(V_G^*) \subset R^{s_1}$ with normal bundle ν_G and let ν_H denote the normal bundle of the composite embedding

$$M(V_H^*) \xrightarrow{\hat{p}} M(V_G^*) \times R^{s_1} \to R^{s_1 + s_2}.$$

We then have the relation

$$\nu_H \simeq p^*(\nu_G) \oplus \omega, \tag{8.7}$$

and from (4.6),

$$\zeta_H \oplus R^{s_1} \simeq p^*(\zeta_G) \oplus \omega. \tag{8.8}$$

Let $\psi_G:\tau_G \oplus \nu_G \to R^{s_2}$ and $\psi_H:\tau_H \subset \nu_H \to R^{s_1+s_2}$ denote the trivializations associated with the embeddings. Making use of the identifications (8.7) and (8.8) we have the following commutative diagrams

$$\begin{array}{ccc}
[\dot{X}, \overline{\eta_G \oplus \zeta_G \oplus \tau_G}] & \xrightarrow{(1\oplus\psi_G)_{\#}\sigma} & [\dot{X} \wedge \bar{\nu}_G, \overline{\eta_G \oplus \zeta_G \oplus R^{s_2}}] \\
 & & \downarrow p^* \\
\downarrow p^* & & [\dot{X} \wedge p^*(\bar{\nu}_G), \overline{\eta_H \oplus p^*(\zeta_G) \oplus R^{s_2}}] \\
 & & \downarrow \sigma \\
[\dot{X}, \overline{\eta_H \oplus \zeta_H \oplus \tau_H}] & \xrightarrow{(1\oplus\psi_H)_{\#}\sigma} & [\dot{X} \wedge \bar{\nu}_H, \overline{\eta_H \oplus \zeta_H \oplus R^{s_1+s_2}}]
\end{array} \tag{8.9}$$

Next, by the commutativity of (7.8) we have (see (4.7))

$$\begin{array}{ccc}
[\dot{X} \wedge \bar{\nu}_G, \eta_G \oplus \zeta_G \oplus R^{s_2}] & \xrightarrow{T} & [X \wedge T(\nu_G), T(\eta_G \oplus \zeta_G) \wedge S^{s_2}] \\
\downarrow p^* & & \downarrow \sigma \\
 & & [X \wedge T(\nu_G) \wedge S^{s_1}, T(\eta_G \oplus \zeta_G) \wedge S^{s_1+s_2}] \\
[\dot{X} \wedge p^*(\bar{\nu}_G), \eta_H \oplus p^*(\zeta_G) \oplus R^{s_2}] & & \downarrow t_{\#} \\
 & & [X \wedge T(\nu_G) \wedge S^{s_1}, T(\eta_H \oplus \zeta_H) \wedge S^{s_1+s_2}] \\
\downarrow \sigma & & \uparrow (1\wedge t)^{\#} \\
[\dot{X} \wedge \nu_H, \eta_H \oplus \zeta_H \oplus R^{s_1+s_2}] & \xrightarrow{T} & [X \wedge T(\nu_H), T(\eta_H \oplus \zeta_H) \wedge S^{s_1+s_2}]
\end{array} \tag{8.10}$$

Finally, from the relation

$$\begin{array}{ccc}
 & \overset{c}{\swarrow} & T(\nu_G) \wedge S^{s_1} \\
S^{s_1+s_2} & & \downarrow t \\
 & \underset{c}{\searrow} & T(\nu_H)
\end{array}$$

we obtain the following commutative diagram.

$$\begin{array}{ccc}
[X \wedge T(\nu_G), T(\eta_G \oplus \zeta_G) \wedge S^{s_2}] & \xrightarrow{c^{\#}} & [X \wedge S^{s_2}, T(\eta_G \oplus \zeta_G^G) \wedge S^{s_2}] \\
\downarrow t_{\#}\sigma & & \\
[X \wedge T(\nu_G) \wedge S^{s_1}, T(\eta_H \oplus \zeta_H) \wedge S^{s_1+s_2}] & \overset{(1\wedge c)^{\#}}{\searrow} & \downarrow t_{\#}\sigma \\
\uparrow (1\wedge t)^{\#} & & \\
[X \wedge T(\nu_H), T(\eta_H \oplus \zeta_H) \wedge S^{s_1+s_2}] & \xrightarrow{(1\wedge t)^{\#}} & [X \wedge S^{s_1+s_2}, T(\eta_H \oplus \zeta_H) \wedge S^{s_1+s_2}]
\end{array} \tag{8.11}$$

Property (5.16) now follows from the commutativity of the diagrams (8.6) through (8.11).

Property (5.13) requires a similar analysis but we will leave the details to the reader. The key relation needed here is given in (7.5).

9. Proof of (6.13)

Let X be a finite complex such that $\dim(X)<n-1$ and let p_2 denote the projection $X\times S^n\to S^n$. The proof of (6.13) is based on the following commutative diagram

$$\begin{array}{ccccc} [X\times S(R^n), S(R^n)] & \xrightarrow{s} & [X\times S^n, S^n] & \xrightarrow{T'} & [X\times S^n, S^n] \\ \downarrow & & \downarrow & & \downarrow \\ [X, F] & \xrightarrow{\iota} & [X, Q^{(1)}(S^0)] & \xrightarrow{T} & [X, Q^{(0)}(S^0)] \end{array}$$

Here s is defined by $s(f)(x, tv)=tf(x, v)$, $t\geqslant 0$, $|v|=1$. The vertical maps are given by the obvious exponential correspondence and T' is the map $T'(u)=[p_2]-u$. Since we are in the stable range $[X\times S^n, S^n]$ has a natural abelian group structure.

Let $f: X\times S(R^n)\to S(R^n)$ represent an element of $[X, F]$ and let

$$\lambda(f): X\times S^n\to S^n \tag{9.2}$$

represent its image under the equivalence $\lambda: [X, F]\to[X, Q^{(0)}(S^0)]$. From the commutativity of the above diagram it is sufficient to show that

$$[\lambda(f)]=[p_2]-[s(f)]. \tag{9.3}$$

To do this we will give an explicit description of $\lambda(f)$. The standard embedding $S(R^n)\subset R^n$ has a trivial normal bundle and a tubular neighborhood map $S(R^n)\times\times R\to R^n$ is given by $(v, t)\to e^t v$. Hence, the associated Pontrjagin-Thom map

$$c: S^n\to S^1\times S(R^n)/S(R^n)$$

is given by $c(v)=(\log|v|, v/|v|)$. (It will be understood throughout this section that a point for which a formula is not defined is to be mapped to the base point.)

Let $\psi: \gamma\oplus\dot{R}\to R^n$ denote the standard trivialization $\psi((v, w)\oplus t)=tv+w$. If v is a non-zero vector let $\hat{v}=v/|v|$. Using this data to construct λ, we have

$$\lambda(f)(x, v)=\frac{f(x, \hat{v})-(\hat{v}\cdot f(x, \hat{v}))\,\hat{v}}{1-\hat{v}\cdot f(x, \hat{v})}+\log|v|\,\hat{v}.$$

Let

$$h: S^n\times S^n\to S^n \tag{9.4}$$

be defined by

$$h(v, w)=\frac{\hat{w}-(\hat{v}\cdot\hat{w})\,\hat{v}}{1-\hat{v}\cdot\hat{w}}+\log|v|\,\hat{v}.$$

Let $a\in\pi_n(S^n)$ denote a generator and let $a_1, a_2\in\pi_n(S^n\times S^n)$ denote the image of a under inclusion onto the first and second factor respectively.

(9.5) LEMMA. *Suppose that n is odd. Then* $h_{\#}(a_1)=a$ *and* $h_{\#}(a_2)=-a$.

Proof. Since h maps the diagonal to the base point we have $h_{\#}(a_1+a_2)=0$. Now let $d: S^n \to S^n \times S^n$ send v to $(v, -v)$ and consider the composite $hd: S^n \to S^n$. Its adjoint $(hd)': S(R^n) \to \Omega(S^n)$ is given by

$$(hd)'(v)(t) = \begin{cases} \log(t)\, v, & t > 0 \\ -\log(t)\, v, & t < 0. \end{cases}$$

Let $i: S(R^n) \to \Omega(S^n)$ denote the adjoint of the identity. Evidently, $(hd)'$ represents $[i]-[iA]$, where $A: S(R^n) \to S(R^n)$ is the antipodal map. If n is odd $[iA] = -[i]$ and $(hd)'$ represents $2[i]$. Therefore hd has degree 2. Since $d_{\#}(a) = a_1 - a_2$ we have $h_{\#}(a_1-a_2)=2a$. The lemma follows now from this and the relation $h_{\#}(a_1+a_2)=0$.

We suppose now that n is odd. The map $\lambda(f)$ admits a factorization

$$X \times S^n \xrightarrow{\tilde{f}} S^n \times S^n \xrightarrow{h} S^n$$

where $\tilde{f}(x, v) = (v, s(f)(x, v))$. Because of the dimensional restriction on X we may deform $\tilde{f}$ into $S^n \vee S^n$. That is, there exists a homotopy commutative diagram of the form

$$\begin{array}{ccccc} X \times S^n & \xrightarrow{\tilde{f}} & S^n \times S^n & \xrightarrow{h} & S^n \\ & \searrow^{\tilde{f}} & \uparrow & & \\ & & S^n \vee S^n & & \end{array}$$

It now follows from the lemma and an elementary diagram chase that $h\tilde{f} = \lambda(f)$ represents $[p_2]-[s(f)]$.

10. The Image of $\pi_*(F_G)$ in $\pi_*(F)$, $G = Z_p$.

The stable homotopy theoretic interpretation of the forgetful homomorphism from F_G to F_H yields considerable information on the image of $\pi_*(F_G)$ in $\pi_*(F_H)$. There is a natural division into two cases depending on whether G is finite or infinite; we defer the infinite case to the next two sections.

We begin with an easy observation.

(10.1) PROPOSITION. *Suppose G is finite and admits a free linear representation. Then the induced homomorphism*

$$\pi_*(F_G) \otimes Z[|G|^{-1}] \to \pi_*(F) \otimes Z[|G|^{-1}]$$

is an isomorphism.

Proof. According to (6.10), the above map is equivalent to the transfer homomorphism

$$\tau_*: \mathbf{S}_*(B_G^+) \to \mathbf{S}_*(S^{\infty +})$$

tensored with $Z[|G|^{-1}]$. However, if $p: S^\infty \to B_G$ is projection, the composite $(p^+)_* \circ \tau_*$ is an isomorphism when tensored with $Z[|G|^{-1}]$ (see [23]). By a spectral sequence argument, $(p^+)_*$ is an isomorphism when tensored with $Z[|G|^{-1}]$. Hence the same is true of τ_*.

As one might expect, considerably stronger results hold for suitable choices of G. We limit our discussion to the following

(10.2) THEOREM. *Let $G=Z_p$, where p is a prime. Then the forgetful map from $\pi_*(F_G)$ to $\pi_*(F)$ is surjective in positive dimensions.*

Proof. By (10.1) the image of the forgetful map contains all torsion in $\pi_*(F)$ of order prime to p. Since $\pi_*(F)$ is finite in positive dimensions, it suffices to prove that the p-primary component of $\pi_*(F_G)$ maps onto the p-primary component of $\pi_*(F)$ in positive dimensions. We shall establish this using results of D. S. Kahn and S. B. Priddy [16]; the cases $p=2$ and $p \neq 2$ require separate treatment.

Case 1. $p=2$. In this case $B_G = RP^\infty$. Embed RP^∞ in the infinite special orthogonal group via the reflection construction; since SO is contained in F_G (linear maps are Z_2-equivariant) and F_{Z_2} is homotopy equivalent to $Q(RP^{\infty +})$, this yields a map from RP^∞ to $Q(RP^{\infty +})$. The results of [18] imply the existence of a unique map

$$h: Q(RP^\infty) \to Q(RP^{\infty +}) \tag{10.3}$$

which is a map of infinite loop spaces and makes the following diagram commute:

$$\begin{array}{ccccc}
\pi_*(RP^\infty) & & \longrightarrow & & \pi_*(Q(RP^\infty)) \\
\downarrow{\scriptstyle p^*} & & & & \downarrow{\scriptstyle h^*} \\
\pi_*(SO) & \to & \pi_*(F_{Z_2}) & \xrightarrow{\lambda_{2*}} & \pi_*(Q(RP^{\infty +})) \\
& {\scriptstyle J^*}\searrow & \downarrow & & \downarrow{\scriptstyle t^*} \\
& & \pi_*(F) & \xrightarrow{\lambda_{1*}} & \pi_*(Q(S^0))
\end{array} \tag{10.4}$$

It is well-known that $\lambda_1 Jp$ induces an isomorphism of fundamental groups. Thus by [16, Theorem 4.1] its adjoint induces a surjection of 2-primary components in positive-dimensional homotopy. But this adjoint induces $t_* h_*$ in homotopy by standard adjoint functor formulas, and hence t_* must also induce a surjection of 2-primary components in positive-dimensional homotopy.

Case 2. $p \neq 2$. Suppose $f: X \to QY$ is continuous where X and Y are pointed CW-complexes. Then there is an essentially unique factorization of f through Y *as an S-map* (i.e., in the category of CW-spectra). Hence for any cohomology theory h^* there is a canonical induced homomorphism

$$f^{\#}: h^*(Y) \to h^*(X) \tag{10.6}$$

making the following diagram commutative

$$\begin{array}{ccc} h^*(Q(Y)) & \xrightarrow{f^*} & h^*(X) \\ \downarrow{\scriptstyle i^*} & \nearrow{\scriptstyle f^\#} & \\ h^*(Y) & & \end{array}$$

Furthermore the correspondence $f \to f^*$ is functorial. Let $L = B_{Z_p}$ and let $t: Q(L^+) \to \to Q(S^0)$ denote some map which realizes the transformation

$$t_\# : [\ ; Q(L^+)] \to [\ ; Q(S^0)].$$

For any such choice of t we have the following commutative diagram (where H^* denotes singular cohomology with Z_p coefficients).

$$\begin{array}{ccccc} & & H^*(F) & \xleftarrow{\lambda^*} & H^*(Q(S^0)) \\ & & \downarrow & & \downarrow{\scriptstyle t^*} \\ H^*(U) & \leftarrow & H^*(F_{Z_p}) & \xleftarrow{\lambda^*} & H^*(Q(L^+)) \\ & & & \searrow{\scriptstyle \lambda^\#} & \downarrow{\scriptstyle i^*} \\ & & & & H^*(L^+) \end{array} \tag{10.7}$$

Let $\sigma(q_i) \in H^{2i(p-1)-1}(F)$ represent the loop-suspension of the i-th Wu class

$$q_i \in H^{2i(p-1)}(BF) \tag{10.8}$$

and let $r_i = \lambda^{*-1}(\sigma(q_i))$. By the results of Kahn and Priddy [16, Remark 4.3] together with a lemma of Tsuchiya [30, Lemma 6.3], in order to show that the adjoint of the composite

$$L^+ \xrightarrow{i} Q(L^+) \xrightarrow{t} Q(S^0)$$

induces an epimorphism of p-primary components in stable homotopy (in positive dimensions) it is sufficient to show that the images of the r_i in $H^{2i(p-1)-1}(L^+)$ are non zero. From the diagram (10.7) this will follow by showing that the classes $\sigma(q_i)$ map non-trivially into $H^*(U)$. Now the image of $\sigma(q_i)$ in $H^*(U)$ is the loop-suspension of the i-th Wu class in $H^*(BU)$ which is a non zero multiple of the Chern class of dimension $(p-1)\,i$ modulo decomposables (see [33] or [30, p. 120]). Hence it is non zero in $H^*(U)$.

11. The Image of $\pi_*(F_G)$ in $\pi_*(F)$, G Infinite

In contrast to the above results for $G = Z_p$ the image of $\pi_k(F_G)$ in $\pi_k(F)$ is always a proper subgroup if G is infinite and $k \equiv \pm 1 \bmod 8$ with the exception of $k = 1$ if

$G \neq S^3$ (since F_{S^3} is 2-connected by (6.6), clearly the generator of $\pi_1(F) = Z_2$ does not come from $\pi_*(F_{S^3})$). The proof has two basic ingredients – an investigation of the image of $\pi_*(U)$ in $\pi_*(F)$ and a computation of the Adams e-invariant of elements in $\pi_*(F)$ which come from torsion in $\pi_*(F_{S^1})$.

In [8] Browder essentially proved that $\pi_*(U) \to \pi_*(F_{S^1})$ is monic. Using his methods one can prove a much stronger result.

(11.1) THEOREM. *The map from $\pi_*(U)$ to $\pi_*(F_{S^1})$ is an injection onto a direct summand, and the complementary summand of the latter group is finite.*

We shall need the notion of G-equivariant fiber bundle as defined by Tom Dieck [29]; however, all of our equivariant bundles will be over trivial G-spaces, and hence the formulation of equivariant local triviality is easily understandable. In particular, if $\mathrm{Top}(X, \varphi)$ is the group of G-equivariant homeomorphisms of the G-space X with action $\varphi: G \times X \to X$, then equivariant (X, φ) bundles over a trivial base are classified by maps from the base into $B\,\mathrm{Top}(X, \varphi)$.

The Dold-Lashof classification of ordinary fiber bundles up to fiber homotopy type [10, Theorem 7.5, p. 303] generalizes to equivariant fiber bundles over trivial G-spaces with only minor changes.

(11.2) PROPOSITION. *Let (X, φ) be as above, and let $F(X, \varphi)$ be its space of equivariant self-maps. Two equivariant fiber bundles over a CW complex with fiber (X, φ) are equivariantly fiber homotopy equivalent if and only if the composites of their classifying maps with the induced function*

$$B\,\mathrm{Top}(X, \varphi) \to BF(X, \varphi).$$

are homotopic.

The following result generalizes the main step in Browder's argument. It is apparently well-known but (to our knowledge) unpublished.

(11.3) LEMMA. (i) *Let ξ be an n-dimensional complex vector bundle over a finite complex, and assume that its unit sphere bundle is equivariantly fiber homotopically trivial (with the obvious free S^1 action). Then the complex K-theoretic Chern classes of ξ are trivial.* (ii) *Let ξ be an n-dimensional quaternionic vector bundle over a finite complex, and assume that the unit sphere bundle of ξ is equivariantly fiber homotopically trivial (with the obvious free S^3 action). Then the real K-theoretic symplectic Pontrjagin classes of ξ are trivial.*

The characteristic classes mentioned above are defined in [9].

Proof. (i) Let $S(\xi)$ be the associated S^{2n-1} bundle of ξ and let $P(\xi)$ be the associated CP^{n-1} bundle. Then $S(\xi) \to P(\xi)$ is a principal S^1 bundle projection we shall call the *canonical line bundle* of ξ. An equivariant fiber homotopy equivalence from $S(\xi)$ to $B\times S^{2n-1}$ induces a fiber homotopy equivalence from $P(\xi)$ to $B\times CP^{n-1}$ under which the canonical line bundle over $B\times CP^{n-1}$ (namely, id $\times p: B\times S^{2n-1}\to \to B\times(P^{n-1})$ pulls back to the canonical line bundle on ξ. Since K-theoretic Chern classes satisfy an analog of the Grothendieck relation for ordinary Chern classes (compare [9, pp. 45–48] or [3, pp. 84, 109], Browder's argument [8, p. 33] works for complex K-theory as well as singular cohomology.

(ii) This follows from a virtually identical argument with canonical quaternionic line bundles replacing complex line bundles and KO-theoretic symplectic Pontrjagin classes [9, pp. 45–48, 52–58] replacing K-theoretic Chern classes.

(11.4) COROLLARY. *If ξ satisfies the hypotheses of Proposition* 8.3, *it is stably trivial.*

Proof. The results of [9, Section 9] show that the first K-theoretic Chern or symplectic Pontrjagin class of ξ is its stable equivalence class in $K^2(X)\cong\tilde{K}(X)$ or $KO^4(X)\cong\widetilde{KSp}(X)$.

Proof of Theorem (11.1). Since U and F_{S^1} are both arcwise connected, the result is trivial for π_0. We shall first prove the result for π_1 and use the low-dimensional cases in providing the higher-dimensional ones.

Let $F(CP^{n-1})$ be the space of self maps of CP^{n-1}. Regarding C^n as an S^1 module we have the space $F_{S^1}(C^n)$. A result of James [13] states that the 'passage to orbit space' homomorphism

$$F_{S^1}(C^n)\to F(CP^{n-1}) \tag{11.5}$$

is a fibration whose fiber is homeomorphic to the space of functions from CP^{n-1} to S^1. It is easy to show that the latter is a $K(Z, 1)$ and the inclusion of S^1 as the set of diagonal matrices is an explicit homotopy equivalence. Thus we have the following commutative diagram whose rows represent fibrations and whose left-hand vertical map is a homotopy equivalence;

$$\begin{array}{ccccc} S^1 & \to & U_n & \longrightarrow & PSU_n \\ \downarrow\cong & & \downarrow & & \downarrow \\ X & \to & F_{S^1}(C^n) & \to & F(CP^{n-1}) \end{array} \tag{11.6}$$

as usual, PSU_n denotes the projective group. Consider the induced mappings of fundamental groups; in the first row one obtains the short exact sequence $0\to Z\to \to Z\to Z_n\to 0$. By (11.4), the induced map from $\pi_1(U_n)=Z$ to $\pi_1(F_{S^1}(C^n))$ is monic. Thus the induced map from $\pi_1(X)$ to $\pi_1(F_{S^1}(C^n))$ is also monic; notice that $\pi_1(F_{S^1}(C^n))=Z$ holds if $n\geqslant 2$ by Theorem (5.7). An application of [26, Theorem 4.11, p. 452] shows that $\pi_1(F(CP^{n-1}))\cong Z_n$, and it follows that the bottom row of the above diagram also yields the short exact sequence $0\to Z\to Z\to Z_n\to 0$ in fundamental groups. But this forces the map from $\pi_1(U_n)$ to $\pi_1(F_{S^1}(C^n))$ to be an isomorphism. Since $\pi_1(U_n)\cong\pi_1(U)$ and $\pi_1(F_{S^1}(C^n))\cong\pi_1(F_{S^1})$ if n is large, the proof of the theorem in dimension 1 is complete.

Consider the following extended fibration sequence

$$U_n \to F_{S^1}(C^n) \xrightarrow{g} Y_n \xrightarrow{f} BU_n \xrightarrow{h} BF_{S^1}(C^n). \tag{11.7}$$

By the results of the previous paragraphs, Y_n is 1-connected. Thus Lemma (6.5) and results of Stasheff [27] and Milnor [19] imply Y_n has the homotopy type of a CW complex with finitely many cells in each dimension.

Let W_n be the $2n$-skeleton of such a complex homotopically equivalent to Y_n, and let $j: W_n \to Y_n$ be the 'inclusion' map. Then hfj is homotopically trivial, so that the composite of f_i with the canonical map from BU_n to BU is homotopically trivial by Corollary (11.4). Since (BU, BU_n) is $(2n+1)$-connected and dim $W_n \leqslant 2n$, it follows that fj is homotopically trivial. Since f is a fibration, this means that j factors through g up to homotopy. Since g is a fibration, this means that the induced fibration

$$U_n \to j^*F_{S^1}(C^n) \to W_n$$

has a cross section. Therefore

$$\pi_*(j^*F_{S^1}(C^n)) \cong \pi_*(W_n) \oplus \pi_*(U_n).$$

However, the pair $(F_{S^1}(C^n), j^*F_{S^1}(C^n))$ is $2n$-connected, and hence it is immediate that $\pi_i(U_n) \to \pi_i(F_{S^1}(C^n))$ is an injection onto a direct summand if $i<2n$. Since (U, U_n) is $2n$-connected and $(F_{S^1}, F_{S^1}(C^n))$ is $(2n-2)$-connected by 5.5 and 6.6, an obvious diagram chase shows that $\pi_*(U) \to \pi_*(F_{S^1})$ is also an injection onto a direct summand. The finiteness of the complementary summand follows because rank $\pi_i(F_{S^1})$ is 1 if i is odd and 0 if i is even, the same as the corresponding rank of $\pi_i(U)$.

(11.8) *Addendum to* 11.1. A completely analogous argument shows that $\pi_*(Sp) \to \pi_*(F_{S^3})$ is an injection onto a direct summand with finite complementary summand; we shall omit the details.

(11.9) THEOREM. *Let n be odd, and let $u \in \pi_n(F_{S^1})$ have finite order. Then the image of u in $\pi_n(F)$ has trivial complex e-invariant.*

See [1, §3] for the definition and properties of the complex Adams e-invariant.

Proof. Let $T: S^{2m+1}(CP^{r+}) \to S^{2m}(S^{2r+1+})$ be the transfer, where $r \gg n$ and $2m \gg r$. Let $u': S^{2m+n} \to S^{2m+1}(CP^{r+})$ correspond to u. The image v of u in $\pi_n(F)$ corresponds to cTu', where $c: S^{2m}(S^{2r+1+}) \to S^{2m}$ collapses the $S^{2m+2r+1}$ wedge factor.

To show $e_{\mathbf{C}}(\text{image } u)=0$, it suffices to prove that $\tilde{K}(C(v)) \cong \tilde{K}(S^{2m}) \oplus \tilde{K}(S^{2m+n+1})$

as modules over the Adams ψ operations (compare [1, §6]). Consider the following diagram

$$\begin{array}{ccccccccc} S^{2m+n} & \xrightarrow{v} & S^{2m} & \to & C(v) & \to & S^{2m+n+1} & \to & S^{2m+1} \\ \downarrow u' & & \downarrow = & & \downarrow & & \downarrow Su' & & \downarrow = \\ S^{2m+1}(CP^{r+}) & \xrightarrow{cT} & S^{2m} & \to & Y & \to & S^{2m+2}(CP^{r+}) & \to & S^{2m+1} \end{array} \tag{11.9}$$

Apply $\tilde{K}$ to this diagram; since $\tilde{K}(X)=0$ if X is a finite complex with cells of only odd dimensions, we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 \leftarrow \tilde{K}(S^{2m}) & \leftarrow & \tilde{K}(Y) & \longleftarrow & \tilde{K}(S^{2m+2}CP^{r+}) & \leftarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \leftarrow \tilde{K}(S^{2m}) & \leftarrow & \tilde{K}(C(v)) & \leftarrow & \tilde{K}(S^{2m+n+1}) & \longleftarrow & 0 \end{array} \tag{11.10}$$

Let α generate $\tilde{K}(S^{2m})=Z$, let $\xi'\in\tilde{K}(Y)$ map to α, let $\xi\in\tilde{K}(C(v))$ denote the image of ξ'.

It suffices to show that $\psi^k(\xi)=k^m\xi$. By naturality,

$$\psi^k(\xi) = k^m\xi + \tau, \tag{11.11}$$

where $\tau\in\text{Image}(u')^*$. But the order of $(u')^*$ is finite since the order of u' is; since $\tilde{K}(S^{2m+n+1})=Z$, this means $(u')^*$ must vanish. Therefore $\tilde{K}(C(v))$ splits as a ψ-module.

Theorems (11.1) and (11.9) readily yield the following result:

(11.12) THEOREM. (i) *Let $\mu_k (k\geqslant 1)$ denote the Adams-Barratt element in $\pi_{8k+1}(F)$. Then μ_k is not in the image of $\pi_{8k+1}(F_{S^1})$.*

(ii) *Let $\sigma_k (k\geqslant 1)$ denote the generator of the image of J in dimension* $8k-1$. *Then σ_k is not in the image of $\pi_{8k-1}(F_{S^1})$.*

(iii) *In the notation of* (ii), *twice σ_k is not in the image of $\pi_{8k-1}(F_{S^3})$.*

Proof. The results of Adams show that μ_k and $2\sigma_k$ have nontrivial e-invariant [1, pp. 68 and 44–45]. Thus they can only come from elements of $\pi_*(F_{S^3})$ or $\pi_*(F_{S^1})$ having infinite order. An easy application of Theorem (11.1) and its addendum shows that if they come from $\pi_*(F_{S^3})$ or $\pi_*(F_{S^1})$, they also come from $\pi_*(Sp)$ or $\pi_*(U)$ respectively. Since μ_k is not in the image of J, conclusion (i) follows. On the other hand, the Bott periodicity theorems imply that $\pi_{8k-1}(G)=Z$ if $G=0$, U, or Sp and the canonical maps

$$\pi_{8k-1}(U) \to \pi_{8k-1}(0)$$
$$\pi_{8k-1}(Sp) \to \pi_{8k-1}(0)$$

are multiplication by 2 and 4 respectively (for example, see [7]). This shows that σ_k and $2\sigma_k$ do not come from $\pi_{8k-1}(F_{S^1})$ and $\pi_{8k-1}(F_{S^3})$ respectively, proving (ii) and (iii).

12. The Image of $\pi_*(F_{S^3})$ in $\pi_*(F_{S^1})$

The pathologies discussed in Section 11 are definitely 2-primary in nature. For example, if p is odd the generators of the p-primary components of the image of J always come from $\pi_*(F_{S^3})$; in fact, they come from $\pi_*(Sp)$ because the canonical map from $\pi_*(Sp)$ to $\pi_*(0)$ is an isomorphism mod (graded) finite 2-groups. Thus one is led to ask whether the induced map from $\pi_*(F_{S^3})\otimes Z[\frac{1}{2}]$ to $\pi_*(F)\otimes Z[\frac{1}{2}]$ is surjective in positive dimensions. Although we cannot prove this, we can prove that the images of $\pi_*(F_{S^3})\otimes Z[\frac{1}{2}]$ and $\pi_*(F_{S^1})\otimes Z[\frac{1}{2}]$ in $\pi_*(F)\otimes Z[\frac{1}{2}]$ are the same.

By Theorem (5.15) the above statement is equivalent to saying that the images of the transfer homomorphisms

$$\mathbf{S}_*((H\tilde{P})^\infty)\otimes Z[\tfrac{1}{2}] \to \mathbf{S}_*(S^0)\otimes Z[\tfrac{1}{2}]$$
$$\mathbf{S}_*(S(CP^{\infty +}))\otimes Z[\tfrac{1}{2}] \to \mathbf{S}_*(S^0)\otimes Z[\tfrac{1}{2}]$$

are equal. We shall deduce this using the following result.

(12.1) THEOREM. *Let k be the involution of CP^∞ given by conjugation. Then the transfer from $\mathbf{S}_*(H\tilde{P}^\infty)\otimes Z[\frac{1}{2}]$ to $\mathbf{S}_*(S(CP^\infty))\otimes Z[\frac{1}{2}]$ is surjective, and its image is the subgroup left fixed by $S(k^+)_*$.*

Assuming this, we state and prove the fact mentioned above.

(12.2) THEOREM. *The images of $\mathbf{S}_*(H\tilde{P}^\infty)\otimes Z[\frac{1}{2}]$ and $\mathbf{S}_*(S(CP^{\infty +}))\otimes Z[\frac{1}{2}]$ in $\mathbf{S}_*(S^0)\otimes Z[\frac{1}{2}]$ are equal.*

Proof. Let S^∞ be the total space of the universal S^1 bundle over CP^∞. Then k lifts to an involution l of S^∞, and by the naturality of the transfer we have the following commutative diagram:

$$\begin{array}{ccccc} S(CP^{\infty +}) & \to & S^{\infty +} & \simeq & S^0 \\ {\scriptstyle S(k^+)}\downarrow & & {\scriptstyle l^+}\downarrow & & {\scriptstyle id}\downarrow \\ S(CP^{\infty +}) & \to & S^{\infty +} & \simeq & S^0 \end{array}$$

It follows that if $y \in \mathbf{S}_*(S(CP^{\infty +}))$, then y and $S(k^+)_* y$ have the same image in $\mathbf{S}_*(S^0)$. Clearly this remains true after tensoring with $Z[\frac{1}{2}]$.

Consider the element $\frac{1}{2}(y+S(k^+)_* y)$ in $\mathbf{S}_*(S(CP^{\infty +}))\otimes Z[\frac{1}{2}]$. By the discussion of the preceding paragraph its image in $\mathbf{S}_*(S^0)\otimes Z[\frac{1}{2}]$ is the same as the image of y. On the other hand, it is clearly left fixed by $S(k^+)_*$, so that it lies in the image of $\mathbf{S}_*(H\tilde{P}^\infty)\otimes Z[\frac{1}{2}]$ by Theorem (12.1).

Let N be the normalizer of S^1 in S^3; then the transfer from $H\tilde{P}^\infty$ to $S(CP^{\infty +})$ factors through BN^ζ. The proof of Theorem (12.1) has two parts – an examination of the image of $\mathbf{S}_*(H\tilde{P}^\infty)\otimes Z[\frac{1}{2}]$ in $\mathbf{S}_*(BN^\zeta)\otimes Z[\frac{1}{2}]$ and an examination of the image of $\mathbf{S}_*(BN^\zeta)\otimes Z[\frac{1}{2}]$ in $\mathbf{S}_*((CP^{\infty +}))\otimes Z[\frac{1}{2}]$.

(12.3) PROPOSITION. *The induced homomorphism from* $\mathbf{S}_*(H\tilde{P}^\infty)\otimes Z[\frac{1}{2}]$ *to* $\mathbf{S}_*(BN^\zeta)\otimes Z[\frac{1}{2}]$ *is an isomorphism.*

Proof. Let $k \geqslant 0$ be given, and let n be large with respect to k. It suffices to prove that

$$t_*\mathbf{S}_k((HP^{n-1})^{\zeta(S^3)}) \to \mathbf{S}_k((S^{4n-1}/N)^{\zeta(N)})$$

is an isomorphism when tensored with $Z[\frac{1}{2}]$.

The Atiyah-Hirzebruch spectral sequence for stable homotopy theory yields a spectral sequence map converging to the homomorphism under consideration. On the E_2 level it takes the form

$$t_*: H_p((HP^{n-1})^\zeta; \mathbf{S}_q)\otimes Z[\tfrac{1}{2}] \to H_p((S^{4n-1}/N)^\zeta; \mathbf{S}_q)\otimes Z[\tfrac{1}{2}].$$

The homology groups of X^ζ are isomorphic to unreduced cohomology groups of X (where $X = HP^{n-1}$ or S^{4n-1}/N) by the Thom isomorphism and Poincaré duality. Techniques of Boardman [6, §6] show that under these isomorphisms t_* corresponds to the cohomology map induced by the projection

$$p: S^{4n-1}/N \to HP^{n-1}.$$

Therefore it suffices to know that p^* is an isomorphism in $Z[\frac{1}{2}]$-module coefficients. This follows from the Serre spectral sequence; for p is an orientable fiber bundle projection whose fiber is RP^2, a $Z[\frac{1}{2}]$-acyclic space.

We shall need a slight generalization of a familiar result on the transfer in singular cohomology.

(12.4) PROPOSITION. *Suppose* $p: X \to Y$ *is a regular n-sheeted covering (Y is a CW complex) and G is the full group of covering transformations. Let* ξ *be a k-plane bundle over Y whose pullback to X is trivial, and let* $p^\xi: S^kX^+ \to Y^\xi$ *denote the induced map of Thom spaces.*

(i) *If* $t: Y^\xi \to S^kX^+$ *is the transfer, then* $p^\xi t$ *is an isomorphism in any homology theory taking values in the category of* $Z[1/n]$*-modules.*

(ii) *Let* h_* *be a homology theory taking values in the category of* $Z[1/n]$*-modules. Then* t_* *is injective and its image is the stationary set of* $h_*(S^kX^+)$ *under the action of G induced by covering transformations.*

The proof of the first part is an exercise in the techniques of [6, §6] and [23]. The proof of the second part is an elementary algebraic exercise based on the canonical isomorphism from $h_*(S^kX^+)/G$ to $h_*(Y^\xi)$ induced by p^ξ.

The following result and Proposition (12.3) imply Theorem (12.1).

(12.5) PROPOSITION. *The transfer map from* $\mathbf{S}_*(BN^\zeta)\otimes Z[\frac{1}{2}]$ *to* $\mathbf{S}_*(S(CP^{\infty +}))\otimes Z[\frac{1}{2}]$ *is injective and its image is the subgroup left fixed under* $\mathbf{S}(k)_*$.

Proof. If ζ is the line bundle over BN given by the adjoint representation, then the pullback of ζ to CP^∞ is trivial. On the other hand, CP^∞ is a double covering of BN, and an elementary argument shows that the covering involution of CP^∞ is homotopic to k. Thus the proposition follows from Proposition (12.4).

APPENDIX

13. The Transfer

Let $p:M\to N$ be a finite covering space where M and N are compact smooth manifolds without boundary. In section 4, we described a well known method of associating with a sectioned bundle α over N an S-map.

$$t:T(\alpha)\wedge S^s\to T(p^*(\alpha))\wedge S^s.$$

For the purposes of this section we refer to t as the 'umkehr' map. On the other hand, there are general constructions of Roush [23] and of Kahn and Priddy [16] which associate with a finite covering pair a wrong way map called the 'transfer'. In particular, for the covering pair $(E_{p^*(\alpha)}, M)\to(E_\alpha, N)$ there is a transfer

$$\tau:T(\alpha)\wedge S^s\to T(p^*(\alpha))\wedge S^s.$$

The object of this appendix is to give a direct proof that the umkehr map agrees with the transfer. In this direction Roush has shown that their induced homomorphisms agree for any (*co*) homology theory h for which N is h-orientable (taking $\alpha=0$).

We begin by describing the transfer for finite coverings. Let $\mathscr{C}$ denote the subcategory of the stable homotopy category of CW-spectra [6,31] having pointed CW-complexes as objects. Let G be a finite group and H a subgroup. Let $\mathscr{P}_G$ denote the category whose objects are CW-pairs (X, A) with a free and cellulair action of G on X which leaves A invariant. The morphisms in $\mathscr{P}_G$ are to be equivariant maps of pairs. We will call (X, A) a free G-pair. There is the forgetful functor $\mathscr{R}:\mathscr{P}_G\to\mathscr{P}_H$ obtained by restricting the action of G to H. There is also the quotient functor $\mathscr{Q}_G:\mathscr{P}_G\to\mathscr{C}$ defined by sending (X, A) to $X/A/G$. As usual, we write X^+ for $X/\Phi = X\cup +\}$ and, in general, $+$ will denote the base point of a pointed space. If $f:(X, A)\to(X', A')$ is a G-map, we also let f denote the quotient map $f:X/A/G\to X'/A'/G$.

There is a 'suspension' functor $\mathscr{P}_G\to\mathscr{P}_G$ defined by sending (X, A) to the pair $(X, A)\times(S^1, +)$ with G acting on the first factor. Note that the quotient of $(X, A)\times(S^1, +)$ is equal to $X/A/G\wedge S^1$.

Suppose that $\Delta:X/G\to X/H$ is a cross section to the covering $p:X/H\to X/G$.

There is then a retraction $q: X/A/H \to X/A/G$ by

$$q(y) = \begin{cases} p(y), & \text{if } y = \Delta(p(y)). \\ +, & \text{otherwise.} \end{cases}$$

(13.1) DEFINITION. *An $H-G$ transfer is a natural transformation $\tau: \mathcal{Q}_G \to \mathcal{Q}_H \mathcal{R}$ having the following properties*:

(a) $\tau(X, A) \times (S^1, +) = \tau(X, A) \wedge 1$.

(b) *If $\Delta: X/G \to X/H$ is a cross section.*

the composite

$$X/A/G \xrightarrow{\tau} X/A/H \xrightarrow{q} X/A/G$$

is the identity.

Although our formulation of the transfer is slightly different than Roush's his results are easily translated. Hence we have

(13.2) THEOREM. (Roush [23]). *There exists a unique $H-G$ transfer.*

The construction of τ that follows is equivalent to that of Roush and also of Kahn and Priddy. If Y is a pointed space let $P(Y)$ denote the space of functions $\sigma: G/H \to Y$, where G/H denotes the set of left cosets of H in G. Let G act on $P(Y)$ by $g\sigma(wH) = \sigma(g^{-1}wH)$, $g, w \in G$. We have an equivariant embedding

$$(G/H)^+ \wedge Y \to P(Y)$$

by $wH \wedge y \to \sigma$, where $\sigma(wH) = y$ and $\sigma(w'H) = +$ if $w'H \neq wH$. Topologically, the pair $(P(Y), (G/H)^+ \wedge Y)$ is simply the n-fold product of Y modulo the n-fold wedge, where n is the index of H in G. Hence it is a $(2s-1)$-connected pair if Y is $(s-1)$-connected.

Now we may write

$$P(Q(Y) = \text{inj lim}_k \Omega^i(P(Y \wedge S^k))$$

and

$$Q((G/H)^+ \wedge Y) = \text{inj lim}_p \Omega^k((G/H)^+ \wedge (Y \wedge S^k)).$$

Moreover, the embedding (13.3) is compatible with the injective limit maps and so we obtain

$$i: Y((G/H)^+ \wedge Y) \to P(Q(Y)). \tag{13.4}$$

By the remarks of the preceding paragraph, the relative homotopy groups of the pair $(P(Q(Y)), Q((G/H)^+ \wedge Y))$ are trivial.

Now let (X, A) be a free G-pair and set $Y=X/A/H$. Define

$$\varphi:(X, A) \to (P(Y), +) \tag{13.5}$$

by $\varphi(x)(wH)=[w^{-1}x]$. Then φ is a G-map. We will also let φ denote the map $(X, A)\to(P(Q(Y)), +)$ obtained by composing with the canonical inclusion $P(Y)\subset P(Q(Y))$. Consider the diagram

$$\begin{array}{ccl} (X, A) & \xrightarrow{\varphi} & (P(Q(Y)), +) \\ & \searrow^{\varphi'} & \uparrow i \\ & & (Q((G/H)^+ \wedge Y), +) \xrightarrow{Q(\lambda)} Q(Y), \end{array}$$

where λ is the 'folding map' $(G/H)^+ \wedge Y\to Y$ defined by $\lambda(wH\wedge y)=y$. There are no obstructions to equivariantly deforming φ relative to A into $Q\big((G/H)^+ \wedge Y\big)$. The end of such a homotopy is denoted by φ' in the diagram. Upon taking quotients $Q(\lambda)$ yields a map

$$\tau':X/A/G \to Q(Y) = Q(X/A/H). \tag{13.7}$$

Now the transfer τ is the map in the stable homotopy category which is the adjoint of τ'. It is easy to check that τ is well defined and meets the requirements of definition (13.1).

To obtain a transfer on the category of n-fold coverings let $G=\mathscr{S}_n$, the symmetric group on n letters, and let $H=\mathscr{S}_{n-1}$. If $p:(E, E')\to(B, B')$ is an n-fold covering pair let X denote the total space of the associated principal G-bundle. Precisely, X is the space of maps $\sigma:\{1,\dots, n\}\to E$ such that σ is fiber preserving and one-one. Let A be the subspace of maps whose image lies in E'. If G acts on X by $\sigma\to\sigma\psi^{-1}$, $\psi\in G$, we have a free G-pair (X, A) and the assignment which sends the covering pair to (X, A) is clearly functorial. Moreover $p:(X/H, A/H)\to(X/G, A/G)$ is naturally equivalent to the original covering pair. The identifications $X/H\to E$ and $X/G\to B$ are given by $\sigma\to\sigma(n)$ and $\sigma\to p\sigma(n)$ respectively. Hence the $H-G$ transfer yields a transfer for n-fold coverings.

Now let $p:M\to N$ be a finite covering of index n where M and N are smooth manifolds. By the preceding remarks, we may write it in the form $p:X/H\to X/G$ where $G=\mathscr{S}_n$, $H=\mathscr{S}_{n-1}$, and X is a smooth manifold. To define the umkehr map we will construct a particular embedding

$$\hat{p}:X/H \to X/G \times R^s \tag{13.8}$$

Let V denote the G-module consisting of R^n plus the action of $G=\mathscr{S}_n$ on R^n through permutations. There is an embedding $X/H\to X\times V/G$ by $[x]_H\to[x, e_n]_G$. Now for the vector bundle $\pi:X\times V/G\to X/G$ there is, for large, s, a map

$$\sigma:X \times V/G \to R^s \tag{13.9}$$

which is a monomorphism on each fiber. Let $\hat{p}$ be the composite embedding

$$X/H \to X \times V/G \xrightarrow{(\pi,\sigma)} X/G \times R^s .$$

Explicitly, $\hat{p}([x]) = ([x], \sigma([x, e_n]))$. The embedding $\hat{p}$ has trivial normal bundle and for ε sufficiently small we have a tubular neighborhood map

$$\hat{p}: X/H \times R^s \to X/G \times R^s \tag{13.10}$$

by $\hat{p}([x], v) = ([x], \varrho(x, v))$, where

$$\varrho: X \times R^s \to R^s \tag{13.11}$$

is defined by $\varrho(x, v) = \sigma([x, e_m]) + \varepsilon v/1 + |v|$.

Let β be a sectioned bundle over X/G and α its pullback over X. Then $\beta = \alpha/G$ and $p^*(\beta) = \alpha/H$. Using the above tubular neighborhood embedding, the umkehr map

$$t: T(\alpha/G) \wedge S^s \to T(\alpha/H) \wedge S^s \tag{13.12}$$

is given by

$$t([a] \wedge v) = \begin{cases} [g^{-1}a] \wedge v', & \text{if } \quad v = \varrho(g^{-1}p_\alpha(a), v') \\ +, & \text{otherwise} \end{cases}$$

On the other hand there is the transfer

$$\tau: T(\alpha/G) \wedge S^s \to T(\alpha/H) \wedge S^s \tag{13.13}$$

associated with the free G-pair (E_α, X).

We will show now that $t = \tau$. To this end let $Y = T(\alpha/H)$ and define

$$\theta: (E_\alpha, X) \times (S^s, +) \to (G/H)^+ \wedge Y \wedge S^s$$

by

$$\theta(a, v) = \begin{cases} gH \wedge [g^{-1}a] \wedge v', \; v = \varrho(g^{-1}p_\alpha(a), v') \\ +, \qquad \text{otherwise} \end{cases}$$

Consider the following diagram

$$\begin{array}{ccc} (E_\alpha, X) \xrightarrow{\varphi} (P(Y \wedge S^s), +) & & \\ \quad \searrow^{\theta} \qquad\quad \uparrow{\scriptstyle i} & & \\ \qquad ((G/H)^+ \wedge Y \wedge S^s, +) & \xrightarrow{\lambda} & (Y \wedge S^s, +) \end{array}$$

Since the umkehr map t is the quotient of $\lambda\theta$, we will have $\pi = t$ provided $i\theta$ is equivariantly homotopic to φ. The required homotopy F_t is given by

$$F_t(a, v)(gH) = \begin{cases} [g^{-1}a] \wedge v', \\ \text{if} \quad v = t\varrho(g^{-1}p_\alpha(a), v') + (1-t)\, v'. \\ +, \qquad \text{otherwise.} \end{cases}$$

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