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On the Gauss image of a spacelike hypersurface with constant mean curvature in Minkowski space

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I. Introduction

To generalize Bernstein's theorem on minimal surfaces Chern [5] proposed to study the distribution of normals to complete constant mean curvature hypersurface in Euclidean space. In this direction there is a remarkable theorem given by Hoffman, Osserman and Schoen [7]. The author also considered more general cases of this kind of problem in a previous work [2].

In [10] Palmer studied the analogous problem in the ambient Minkowski space.

Let M be an oriented spacelike hypersurface in a Minkowski space $\mathbb{R}^{n+1}_1$. Let $\mathscr{V}$ be the timelike unit normal vector field to M in $\mathbb{R}^{n+1}_1$. For any point $p \in M |\mathscr{V}(p)|^2 = -1$. By parallel translation to the origin in $\mathbb{R}^{n+1}_1$ we can regard $\mathscr{V}(p)$ as a point in the n-dimensional hyperbolic space $H^n(-1)$ which is canonically embedded in $\mathbb{R}^{n+1}_1$. In such a way we have the Gauss map $\gamma : M \to H^n(-1)$.

Palmer proved the following result:

THEOREM A [10]. *For $H \neq 0$ there exists a number $\tau(n, H) > 0$ with the following property: Let $M \to \mathbb{R}^{n+1}_1$ be a spacelike hypersurface with constant mean curvature H. If $\mathscr{V}(M)$ is contained in a geodesic ball of radius $\tau_1 < \tau$ in $H^n(-1)$ then M is not complete.*

We observe that in the case when M has constant mean curvature the Gauss map γ is a harmonic map into the hyperbolic space [8]. Then the Liouville theorem of harmonic maps is applicable provided one can show M has nonnegative Ricci curvature [3]. This can be done by using the maximum principle [2]. Therefore, by a totally different approach we generalize the above quoted Theorem A as follows:

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THEOREM B. *Suppose M is a complete spacelike hypersurface with constant mean curvature in Minkowski space $\mathbb{R}_1^{n+1}$. If the image under the Gauss map $\gamma : M \to H^n(-1)$ is bounded then M has to be a linear subspace.*

In view of the famous Calabi–Cheng–Yau result [1], [4] of non-existence of nontrivial complete maximal spacelike hypersurface in Minkowski space, any complete spacelike hypersurface with nonzero constant mean curvature in Minkowski space must have boundedless Gauss image.

It should be mentioned that Choi–Triebergs also study the Gauss maps of constant mean curvature graphs in Minkowski space [6].

In this note we will firstly give an estimate for the squared length of the second fundamental form in terms of mean curvature and Gauss image diameter and then prove Theorem B.

II. Preliminaries

Let N be an $(n+1)$-dimensional Lorentzian manifold with Lorentzian metric $\bar{g}$ of signature $(-, +, \ldots, +)$. Let $\{e_0, e_1, \ldots, e_n\}$ be a local Lorentzian orthonormal frame field in N. Let $\omega_0, \omega_1, \ldots, \omega_n$ be its dual frame field so that $\bar{g} = -\omega_0^2 + \Sigma_i \omega_i^2$. We agree the following range of indices:

$$1 \le i, j, \ldots \le n,$$

$$0 \le \alpha, \beta, \ldots \le n.$$

The Lorentzian connection forms $\omega_{\alpha\beta}$ of N are uniquely determined by the equations

$$\begin{aligned} d\omega_0 &= \sum \omega_{0i} \wedge \omega_i, \\ d\omega_i &= -\sum \omega_{i0} \wedge \omega_0 + \sum_i \omega_{ij} \wedge \omega_j, \qquad (1) \\ \omega_{\alpha\beta} &+ \omega_{\beta\alpha} = 0. \end{aligned}$$

The covariant derivatives are defined by the following equations

$$\begin{aligned} De_0 &= \sum \omega_{0i} e_i, \\ De_i &= \sum_j \omega_{ij} e_j - \omega_{i0} e_0. \end{aligned} \qquad (2)$$

The curvature forms $\bar{\Omega}_{\alpha\beta}$ of N are given by

$$\bar{\Omega}_{0i} = d\omega_{0i} - \sum_{i} \omega_{0k} \wedge \omega_{ki},$$
$$\bar{\Omega}_{ij} = d\omega_{ij} + \omega_{i0} \wedge \omega_{0j} - \sum_{j} \omega_{ik} \wedge \omega_{kj}, \tag{3}$$
$$\bar{\Omega}_{\alpha\beta} = -\frac{1}{2} \bar{R}_{\alpha\beta\gamma\delta} \omega_{\gamma} \wedge \omega_{\delta},$$

where $\bar{R}_{\alpha\beta\gamma\delta}$ are components of the curvature tensor $\bar{R}$ of N.

Let M be a spacelike hypersurface in a Lorentzian $(n+1)$-manifold N. We choose a local Lorentzian orthonormal frame field $e_0, e_1, \ldots, e_n$ in N such that, restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M. When we restrict their dual forms to M, then

$$\omega_0 = 0$$

and the induced Riemannian metric g of M is written as $g = \Sigma_i \omega_i^2$ and the induced structure equations of M are

$$d\omega_i = \omega_{ik} \wedge \omega_k, \qquad \omega_{ij} + \omega_{ji} = 0,$$
$$d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} - \omega_{i0} \wedge \omega_{0j} + \bar{\Omega}_{ij}, \tag{4}$$
$$\Omega_{ij} = d\omega_{ij} - \sum_{k} \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l,$$

where Ω_{ij} and R_{ijkl} denote the curvature forms and the components of curvature tensor of M, respectively.

By Cartan's lemma, we have

$$\omega_{i0} = h_{ij} \omega_j, \tag{5}$$

where h_{ij} are components of the second fundamental form of M in N. From (3), (4) and (5) we obtain the Gauss formula

$$R_{ijkl} = \bar{R}_{ijkl} - (h_{ik} h_{jl} - h_{il} h_{jk}). \tag{6}$$

The Ricci tensor is

$$R_{ik} = \bar{R}_{ik} + \sum_{j} h_{ij} h_{kj} - nH h_{ik},$$

where $H = (1/n)\,\Sigma_i h_{ii}$ is the mean curvature of M in N. If N has Ricci curvature bounded below by C_N then M has Ricci curvature bounded below as follows:

$$\text{Ricc} \geq C_N - \frac{1}{4} m^2 H^2. \tag{7}$$

Let h_{ijk} denote the covariant derivative of h_{ij} so that

$$\sum h_{ijk}\omega_k = dh_{ij} + \sum_k h_{ik}\omega_{kj} + \sum h_{kj}\omega_{ki}. \tag{8}$$

Then by exterior differentiating (5) and using (4) we obtain the Codazzi equation

$$h_{ijk} - h_{ikj} = \bar{R}_{0ijk}. \tag{9}$$

Define the covariant derivative of h_{ijk} by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_l h_{ljk}\omega_{li} + \sum_l h_{ilk}\omega_{li} + \sum_l \omega_{ijl}\omega_{lk}. \tag{10}$$

Then by exterior differentiating (9), one obtains the Ricci formula

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{11}$$

From (9) and (11) it follows that the Laplacian satisfies

$$\Delta h_{ij} = \sum_k h_{kkij} + \sum_{k,m} h_{mk}R_{mijk} + \sum_{k,m} h_{im}R_{mkjk} + \sum_k \bar{R}_{0ijkk} + \sum_k \bar{R}_{0kikj}, \tag{12}$$

where

$$\sum_l \bar{R}_{0ijkl}\omega_l = d\bar{R}_{0ijk} - \sum_l \bar{R}_{0ljk}\omega_{il} - \sum_l \bar{R}_{0ilk}\omega_{lj} - \sum_l \bar{R}_{0ijl}\omega_{lk}.$$

Let $S = \Sigma_{i,j} h_{ij}^2$ be the squared length of the second fundamental form of M in N. Then (12) gives

$$\frac{1}{2}\Delta S = \sum_{i,j,k} h_{ijk}^2 + n\sum_{i,j} h_{ij}H_{ij} + S^2 - nH\sum_{i,j,k} h_{ij}h_{jk}h_{ki} + \sum_{i,j,k,m} h_{ij}h_{km}\bar{R}_{mijk}$$
$$+ \sum_{i,j,k,m} h_{ij}h_{im}\bar{R}_{mkjk} + \sum_{i,j,k} h_{ij}\bar{R}_{0ijkk} + \sum_{i,j,k} h_{ij}\bar{R}_{0kikj}. \tag{13}$$

If M has constant mean curvature in Minkowski space N then

$$\frac{1}{2}\Delta S \geq \sum_{i,j,k} h_{ijk}^2 + S^2 - n|H|S^{3/2}. \tag{14}$$

III. A proof of the main result

Let r, $\tilde{r}$ be the respective distance functions on M and $H^n(-1)$ relative to fixed points $x_0 \in M$, $\tilde{x}_0 \in H^n(-1)$. Let $B(a)$ and $\tilde{B}(a)$ be closed balls of radius a around x_0 and $\tilde{x}_0$ respectively. Define the maximum modulus of Gauss map $\gamma : M \to H^n(-1)$ on $B(a)$ by

$$\mu(\gamma, a) \overset{\text{def}}{=} \max\{\tilde{r}(\gamma(x)); x \in B(a) \subset M\}. \tag{15}$$

For a fixed positive number a choose $b > ch(\mu(\gamma, a))$. Define $f : B(a) \to \mathbb{R}$ by

$$f = \frac{(a^2 - r^2)^2 S}{(b - h \circ \gamma)^2}, \tag{16}$$

where S is the squared length of the second fundamental form of M in $\mathbb{R}_1^{n+1}$, $h = ch\tilde{r}$.

Since $f|_{\partial B(a)} \equiv 0$, f achieves an absolute maximum in the interior of $B(a)$, say $f \leq f(z)$, for some z inside $B(a)$. By using the technique of support functions we may assume that f is c^2 near z. We may also assume $S(z) \neq 0$. Then from

$$\nabla f(z) = 0,$$

$$\Delta f(z) \leq 0$$

we obtain at the point z the following:

$$-\frac{2\nabla r^2}{a^2 - r^2} + \frac{\nabla S}{S} + \frac{2\nabla(h \circ \gamma)}{b - h \circ \gamma} = 0, \tag{17}$$

$$\frac{-2|\nabla r^2|^2}{(a^2 - r^2)^2} - \frac{2\Delta r^2}{a^2 - r^2} + \frac{\Delta S}{S} - \frac{|\nabla S|^2}{S^2} + \frac{2\Delta(h \circ \gamma)}{b - h \circ \gamma} + \frac{2|\nabla(h \circ \gamma)|^2}{(b - h \circ \gamma)^2} \leq 0. \tag{18}$$

The Schwarz inequality implies that

$$\frac{|\nabla S|^2}{S} \leq 4 \sum_{i,j,k} h_{ijk}^2. \tag{18}$$

Hence (14) and (19) give

$$\Delta S \geq \frac{|\nabla S|^2}{2S} + 2S^{3/2}(S^{1/2} - n|H|) \tag{20}$$

so that

$$\frac{\Delta S}{S} - \frac{|\nabla S|^2}{S^2} \geq \frac{-2|\nabla(h \circ \gamma)|^2}{(b - h \circ \gamma)^2} - \frac{4|\nabla(h \circ \gamma)||\nabla r^2|}{(b - h \circ \gamma)(a^2 - r^2)}$$
$$- \frac{2|\nabla r^2|^2}{(a^2 - r^2)^2} + 2S^{1/2}(S^{1/2} - n|H|). \tag{21}$$

Substituting (21) into (18) gives

$$\frac{-2\Delta r^2}{a^2 - r^2} - \frac{4|\nabla r^2|^2}{(a^2 - r^2)^2} - \frac{4|\nabla(h \circ \gamma)||\nabla r^2|}{(b - h \circ \gamma)(a^2 - r^2)} + \frac{2\Delta(h \circ \gamma)}{b - h \circ \gamma} + 2S^{1/2}(S^{1/2} - n|H|) \leq 0. \tag{22}$$

It is easily seen that

$$\gamma_* e_i = h_{ij} e_j,$$
$$|\nabla(h \circ \gamma)|^2 = \langle \operatorname{grad} h, \gamma_* e_i \rangle \langle \operatorname{grad} h, \gamma_* e_i \rangle \leq (sh^2 \tilde{r}) S. \tag{23}$$

Since

$$\operatorname{Hess} \tilde{r} = \coth \tilde{r} (\tilde{g} - d\tilde{r} \otimes d\tilde{r}),$$

we have

$$\operatorname{Hess} h = (ch\tilde{r})\tilde{g}, \tag{24}$$

where $\tilde{g}$ is the metric tensor of $H^n(-1)$. It follows that

$$\Delta(h \circ \gamma) = \operatorname{Hess}(h)(\gamma_* e_i, \gamma_* e_i) + \langle \operatorname{grad} h, \nabla_{e_i} \gamma_* e_i \rangle$$
$$= (ch\tilde{r})S + \langle \operatorname{grad} h, h_{iji} e_j \rangle$$
$$= (ch\tilde{r})S. \tag{25}$$

The last equality follows from the Codazzi equation (9) and the assumption of constant mean curvature.

Since the Ricci curvature of M is bounded from below by $-n^2H^2/4$ we can use the Laplacian comparison theorem and obtain

$$\Delta r^2 \leq 2 + 2(n-1)cr(\coth cr) \leq 2n + 2(n-1)cr, \tag{26}$$

where $c = (n/2)|H|$. Substituting (23), (25) and (26) into (22) we have

$$\left(\frac{ch\tilde{r}}{b - ch\tilde{r}} + 1\right)S - \left(\frac{4(sh\tilde{r})r}{(b - ch\tilde{r})(a^2 - r^2)} + n|H|\right)\sqrt{S} \\ - \left(\frac{2(n + (n-1)cr)}{a^2 - r^2} + \frac{8r^2}{(a^2 - r^2)}\right) \leq 0. \tag{27}$$

It is easily seen that if $ax^2 - bx - c \leq 0$ with a, b, c all positive, then

$$x^2 \leq k\left(\frac{b^2}{a^2} + \frac{c}{a}\right),$$

where k is an absolute constant and in what follows k may be different in different inequalities. Thus, we obtain at the point z,

$$S \leq k\left[\frac{\left(\frac{4(sh\tilde{r})r}{(b - chr^2)(a^2 - r^2)} + n|H|\right)^2}{\left(\frac{ch\tilde{r}}{b - ch\tilde{r}} + 1\right)^2} + \frac{2(n + (n-1)cr)(a^2 - r^2) + 8r^2}{\left(\frac{ch\tilde{r}}{b - ch\tilde{r}} + 1\right)(a^2 - r^2)^2}\right]$$

and

$$f(z) \leq k\left[\frac{\left(\frac{4(sh\mu)a}{b - ch\,\mu} + na^2|H|\right)^2}{\left(\frac{1}{b} + 1\right)^2(b - ch\mu)} + \frac{2(na^2 + (n-1)ca^3) + 8a}{\left(\frac{1}{b} + 1\right)(b - ch\mu)}\right].$$

Choosing $b = 2ch\mu$ we have

$$f(z) \leq k\left[\frac{(4(sh\mu)a + n(ch\mu)a^2|H|)^2}{(1 + 2ch\mu)^2(b - ch\mu)} + \frac{(n+4)a^2 + (n-1)ca^3}{1 + 2ch\mu}\right] \tag{28}$$

and

$$S(x) = \frac{(b - ch\tilde{r})f(x)}{(a^2 - r^2)^2} \leq \frac{(b - ch\tilde{r})f(z)}{(a^2 - r^2)^2} \leq \frac{2ch\mu}{(a^2 - r^2)^2} f(z)$$

$$\leq k\left(\frac{(4(sh\mu)a + n(ch\mu)a^2|H|)^2}{(1 + 2ch\mu)^2(a^2 - r^2)^2} + \frac{(n + 4)(ch\mu)a^2 + (n - 1)ca^3}{(1 + 2ch\mu)(a^2 - r^2)^2}\right), \tag{29}$$

where $c = (n/2)|H|$ and μ is defined by (15). We state the above estimate in the following theorem.

THEOREM B′. *Let M be a spacelike hypersurface of constant mean curvature H in Minkowski space $\mathbb{R}_1^{n+1}$ such that for a certain $x_0 \in M$, the geodesic ball of radius a centered at x_0 is compact. Let $S = \Sigma_{i,j} h_{ij}^2$ be the squared length of the second fundamental form M in $\mathbb{R}_1^{n+1}$. Then we have the estimate* (29).

Now we are in a position to prove the main result stated in the introduction.

A proof of Theorem B. If the image under the Gauss map is bounded, then the maximum modulus $\mu(\gamma, a)$ is bounded. We also have bounded smooth function $h = ch\tilde{r}(\gamma(x))$ on the complete manifold M of Ricci curvature bounded below by $-n^2H^2/4$. Thus the Omori–Yau [9], [11] maximum principle is applicable to h. For any $\varepsilon > 0$ and $p_0 \in M$ there exists a point p such that

$$h(p) \geq h(p_0), \qquad |\text{grad } h|_p < \varepsilon \quad \text{and} \quad \Delta h|_p < \varepsilon. \tag{30}$$

By (25)

$$\Delta h = (ch\tilde{r})S,$$

which means

$$\inf S = 0.$$

On the other hand

$$H^2 \leq \frac{S}{n}$$

and H is constant. This forces $H \equiv 0$. From (29) it follows

$$S(x) \leq k\left[\frac{16(sh^2\mu)a^2}{(1+2ch\mu)^2(a^2-r^2)^2}+\frac{(n+4)(ch\mu)a^2}{(1+2ch\mu)^2(a^2-r^2)^2}\right]. \tag{31}$$

Hence we may fix x and let a tend to infinity in (31). Then we obtain $S(x) = 0$ for all $x \in M$. This completes the proof of Theorem B.

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