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Duality and minimality in Riemannian foliations

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In this work we prove that a Riemannian foliation $\mathscr{F}$ defined on a smooth closed manifold M is *minimal*, in the sense that there exists a Riemannian metric on M for which all the leaves are minimal submanifolds, iff $\mathscr{F}$ is *unimodular*, that is, the basic cohomology of $\mathscr{F}$ in maximal dimension is nonzero. This result has been conjectured by Y. Carrière. We use the structure theorem for Riemannian foliations (Molino, [12]) to reduce the problem to transitive foliations, and a parametrix constructed by Sarkaria [15], that permits him to prove a finiteness theorem for transitive foliations. We also prove a duality theorem for the foliated cohomology conjectured in [8].

A good description of the theory of duality and minimality is given in Appendix B, by V. Sergiescu, in the book by P. Molino "Riemannian foliations" [12].

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1. The cohomology of transitive foliations

Let M be a smooth closed manifold of dimension $n + m$, which carries a smooth foliation $\mathscr{F}$ of dimension m. We denote by $\Omega(M)$ the algebra of all smooth differential forms on M. A smooth form of degree i is said to be of filtration $\geq p$ if it vanishes whenever $i - p + 1$ of the vectors are tangent to the foliation. We shall denote the subalgebra of all forms of filtration $\geq p$ by $F^p\Omega$. In this way, the de Rham complex of smooth forms becomes a filtered complex and we have the spectral sequence $E_r(\mathscr{F})$ which converges to the real cohomology of M. $E_1^{0,q}(\mathscr{F}) \cong H^q_{\mathscr{F}}$, the cohomology of M with coefficients in the sheaf of germs of locally constant differentiable functions along the leaves of the foliation.

We can define a differential operator

$$d_{\mathscr{F}} : \Omega^i(M) \to \Omega^{i+1}(M)$$

as follows: we consider a Riemannian metric on M and an orthogonal complement

$\nu(\mathscr{F})$ of $T\mathscr{F}$; we have

$$\Omega^i(M) \cong \sum_{r+s=i} \Gamma(\Lambda^r(\nu(\mathscr{F}))^* \otimes \Lambda^s(T\mathscr{F})^*).$$

We say that a differential form α is of "type" (r, s) if $\alpha \in \Omega^{r,s}(M) = \Gamma(\Lambda^r(\nu(\mathscr{F}))^* \otimes \Lambda^s(T\mathscr{F})^*)$, and the exterior differential decomposes as

$$d = d_{0,1} + d_{1,0} + d_{2,-1}.$$

$d_{\mathscr{F}}$ is $d_{0,1}$ and $d_{\mathscr{F}}^2 = 0$.

We define the *basic forms* $\Omega_B^p(M)$ by

$$\Omega_B^p(M) = (F^p\Omega^p) \cap \operatorname{Ker} d_{\mathscr{F}}.$$

$(\Omega_B^p(M), d)$ is a differential complex, and $E_2^{p,0}(\mathscr{F}) \cong H^p(\Omega_B^*(M))$ is called the *basic cohomology* of the foliation.

The terms $E_2^{*,m}(\mathscr{F})$ of the spectral sequence are isomorphic to the $\mathscr{F}$-relative cohomological groups introduced by Rummler (see [14] for the definition, [9] or [10] for the isomorphism).

Let $\chi(M)$ be the algebra of vector fields on M, $\Gamma(\mathscr{F})$ the Lie sub-algebra of vector fields tangent to the foliation. Let us denote by $\chi(M, \mathscr{F})$ the Lie algebra of the foliated vector fields, that is, the normalizer of $\Gamma(\mathscr{F})$ in $\chi(M)$. At each point $x \in M$, we get a subspace $\chi(M, \mathscr{F})(x)$ of the tangent space T_xM, by evaluating the vector fields at x. The foliation is called *transitive* if $\chi(M, \mathscr{F})(x) = T_xM$ for all x.

In [15] Sarkaria constructs a 2-parametrix for a transitive foliation. We shall topologise $\Omega(M)$ with the usual C^∞ topology. In this way it becomes a Hausdorff locally convex topological vector space. If E and F are Hausdorff LCTVSes a linear map $s : E \to F$ is called *compact* if it maps some neighbourhood U of 0 to a set $s(U)$ with compact closure. A 2-parametrix will be a pair of linear maps $s, h : \Omega(M) \to \Omega(M)$ satisfying

$$\begin{cases} \text{(a)} \ s \text{ is compact,} \\ \text{(b)} \ 1 - s = dh + hd, \\ \text{(c)} \ s(F^p\Omega) \subseteq F^p\Omega \text{ for all } p, \\ \text{(d)} \ h(F^p\Omega) \subseteq F^{p-1}\Omega \text{ for all } p. \end{cases} \tag{1}$$

In order to define s and h, let us fix a finite dimensional vector space $V \subseteq \chi(M, \mathscr{F})$ such that $V(x) = T_xM$ for all $x \in M$. Since M is compact and $\mathscr{F}$ is transitive, we can always extract such a finite dimensional vector subspace. Furthermore, one chooses a Riemannian metric g on V, of volume element $|g|$, and a smooth function f on V supported in a compact neighbourhood of 0. One defines a map $s : \Omega^i(M) \to \Omega^i(M)$ by

$$(s\alpha)(x) = \int_V (\phi_X^*\alpha)(x) \cdot f(X) \cdot |g|,$$

where $\phi_{tX} : M \to M$ denotes the flow of the vector field $X \in V$, and ϕ_X is the diffeomorphism corresponding to $t = 1$. Sarkaria constructs the smooth kernel $K(x, y)$ of s, such that

$$(s\alpha)(x) = \int_M K(x, y)\alpha(y),$$

and so s is a compact operator of trace class.

One normalises the function by $\int_V f(X)|g| = 1$ and one defines h by

$$(h\alpha)(x) = \int_V \int_0^1 (i_X\phi_{tX}^*\alpha)(x) \cdot f(X) \cdot dt \cdot |g|.$$

Now we can use the Riesz theory of compact operators ([6], [13]). Let

$$K = \bigcup_r (1 - s)^{-r}(0), \qquad I = \bigcap_r (1 - s)^r(\Omega(M)).$$

Then K and I are topological supplements stable under s and $1 - s$, K finite dimensional. Moreover, $1 - s$ induces a nilpotent operator in K and a *TVS* automorphism in I. In fact, the sequences $(1 - s)^{-r}(0)$ and $(1 - s)^r(\Omega(M))$ are stationary starting from the same rank v. In this case, we say that $(1 - s)$ has finite ascent (and finite descent) v [13].

$F^p\Omega$ is a closed subspace of $\Omega(M)$, and s defines a compact operator $s : F^p\Omega \to F^p\Omega$ for each $p, 0 \le p \le n$. Let k be the maximum of the ascents of each $(1 - s)|_{F^p\Omega}, 0 \le p \le n$. K and I are filtered differential algebras: we define F^pK as $K \cap (F^p\Omega)$, and F^pI as $\bigcap_r (1 - s)^r F^p\Omega$.

Let $u = \{(1 - s)^k|_I\}^{-1}$. We have a split exact sequence

$$0 \to K \to \Omega(M) \to I \to 0$$

of filtered differential complexes, where $\Omega(M) \to I$ is $u \circ (1 - s)^k$. In this way, we have

$$F^p\Omega \cong F^pK \oplus F^pI$$

as a topological differential complex. We can define two spectral sequences $E_2(K)$ and $E_2(I)$, and we obtain

$$E_2(\mathscr{F}) \cong E_2(K) \oplus E_2(I).$$

But now the inclusion $E_2(I) \to E_2(\mathscr{F})$ is zero, because the homotopy h satisfies the condition (d) in (1).

In fact,

$$E_1^{p,q}(\mathscr{F}) \cong H^q\left(\frac{F^p\Omega^{p+q}}{F^{p+1}\Omega^{p+q}}\right),$$

and

$$d_1 : E_1^{p,q}(\mathscr{F}) \to E_1^{p+1,q}(\mathscr{F})$$

is induced by the connecting of the exact sequence

$$0 \to \frac{F^{p+1}\Omega^{p+q}}{F^{p+2}\Omega^{p+q}} \to \frac{F^p\Omega^{p+q}}{F^{p+2}\Omega^{p+q}} \to \frac{F^p\Omega^{p+q}}{F^{p+1}\Omega^{p+q}} \to 0.$$

We shall see that if $[\eta] \in E_2^{p,q}(\mathscr{F})$, then $\eta - s(\eta)$ is a coboundary in $E_1^{p,q}(\mathscr{F})$: $\eta \in F^p\Omega$ represents a class in $E_2^{p,q}(\mathscr{F})$ if $d\eta = \alpha + d\beta$, with $\alpha \in F^{p+2}\Omega$ and $\beta \in F^{p+1}\Omega$. So, $h\alpha \in F^{p+1}\Omega$ and $h\,d\beta = d(-h\beta) + (\beta - s\beta)$, with $(\beta - s\beta) \in F^{p+1}\Omega$. Hence,

$$\eta - s(\eta) = dh(\eta - \beta) \bmod F^{p+1}\Omega.$$

Since $(1 - s)_2 : E_2(I) \to E_2(I)$ is an isomorphism, we have proved

$$E_2(\mathscr{F}) \cong E_2(K).$$

With the C^∞ topology the quotients $E_1(\mathscr{F})$ are not always Hausdorff. As the exterior differential d is continuous, we can define a new differential complex $\mathbb{E}_1(\mathscr{F})$ with

$$\mathbb{E}_1(\mathscr{F}) = E_1(\mathscr{F})/\bar{O}_{\mathscr{F}},$$

where $\bar{O}_{\mathscr{F}}$ is the closure of $\{0\}$ in $E_1(\mathscr{F})$, and $\mathbb{E}_2(\mathscr{F}) = H(\mathbb{E}_1(\mathscr{F}))$.

We have $E_1(\mathscr{F}) \cong E_1(K) \oplus E_1(I)$, and $E_1(K)$ is Hausdorff. Then,

$$\mathbb{E}_1(\mathscr{F}) \cong E_1(K) \oplus \mathbb{E}_1(I),$$

where $\mathbb{E}_1(I) \cong E_1(I)/\bar{O}_I$, and $\bar{O}_I$ is the closure of $\{0\}$ in $E_1(I)$. Finally,

$$\mathbb{E}_2(\mathscr{F}) \cong E_2(K) \oplus \mathbb{E}_2(I),$$

and $\mathbb{E}_2(I) = H(\mathbb{E}_1(I)) = 0$ for the same reason that $E_2(I) = 0$. So, we have proved $E_2(\mathscr{F}) \cong \mathbb{E}_2(\mathscr{F})$.

2. The cohomology of Riemannian foliations

(A reference for this section, with very detailed calculus, is [2]. For the cohomology of operations and related spectral sequences, see [5]).

Let us consider now a Riemannian foliation $\mathscr{F}$ on M. If $\mathscr{F}$ or M are not orientable we can work in a convenient covering space. We shall use throughout a bundle-like metric on M. Let (P, π, M) be the principal $SO(n)$-bundle of oriented orthonormal transverse frames, associated to $\mathscr{F}$ and the metric. Let $\tilde{\mathscr{F}}$ be the lifted foliation in P, which is transitive (it is transversally parallelizable [12]). $\tilde{\mathscr{F}}$ is invariant by the right action of $SO(n)$ on P.

If $\xi \in \mathbf{so}(n)$, the Lie algebra of $SO(n)$, we denote by $\theta(\xi)$ the Lie derivative with respect to the fundamental vector field associated to ξ. Let $\Omega(P)_{\theta=0}$ be the subalgebra of $\Omega(P)$ of the differential forms satisfying

$$\theta(\xi)\eta = 0, \qquad \xi \in \mathbf{so}(n).$$

Analogously, we denote by $i(\xi)$ the interior product with respect to fundamental field associated to ξ, and $\Omega(P)_{i=0}$ the subalgebra of the differential forms satisfying

$$i(\xi)\eta = 0, \qquad \xi \in \mathbf{so}(n).$$

The existence of a connection on P permits to set up an isomorphism

$$\Omega(P) \cong \Omega(P)_{i=0} \otimes \Lambda\, \mathbf{so}(n)^*.$$

The induced filtration on $\Omega(P)_{i=0}$ defines a spectral sequence $E_r(\Omega(P)_{i=0})$, and we have

$$E_1(\tilde{\mathscr{F}}) \cong E_1(\Omega(P)_{i=0}) \otimes \Lambda\, \mathbf{so}(n)^*$$

because the d_0-differential is induced by $d_{\tilde{\mathscr{F}}}$, and the forms in $\Lambda\,\mathbf{so}(n)^*$ are $d_{\tilde{\mathscr{F}}}$-closed. As a consequence, we have $E_1(\tilde{\mathscr{F}})_{i=0} \cong E_1(\Omega(P)_{i=0})$. Let $j : \Omega(P)_{\theta=0} \to \Omega(P)$

be the inclusion. Since $SO(n)$ is compact and connected, there exist linear maps

$$\rho : \Omega^i(P) \to \Omega^i(P)_{\theta=0}; \qquad h : \Omega^i(P) \to \Omega^{i-1}(P)$$

compatible with the action of $SO(n)$, and satisfying

$$\rho \circ j = id; \qquad id - j \circ \rho = dh + hd.$$

The homotopy h has the property that $h \mid \Omega(P)_{i=0} = 0$. Moreover, as $\tilde{\mathscr{F}}$ is invariant by the right action of $SO(n)$ on P,

$$h(F^p\Omega(P)) \subseteq F^{p-1}\Omega(P); \qquad \rho(F^p\Omega(P)) \subseteq F^p\Omega(P),$$
$$\rho \circ d_{\tilde{\mathscr{F}}} = d_{\tilde{\mathscr{F}}} \circ \rho; \qquad h \circ d_{\tilde{\mathscr{F}}} = d_{\tilde{\mathscr{F}}} \circ h.$$

So, the action of $\mathbf{so}(n)$ on $\Omega(P)$ defines also an action on $E_1(\tilde{\mathscr{F}})$, and we have:

LEMMA. $E_1(\tilde{\mathscr{F}})_{i=0,\theta=0} \cong E_1(\mathscr{F})$.

In fact, let $[\alpha] \in E_1(\tilde{\mathscr{F}})_{i=0,\theta=0}$. We can write

$$d\alpha = d_{\tilde{\mathscr{F}}}\alpha + d_{1,0}\alpha + d_{2,-1}\alpha.$$

But $d_{\tilde{\mathscr{F}}}\alpha = 0$ and $d_{2,-1}\alpha \in F^{p+2}\Omega(P)$, $hd_{2,-1}\alpha \in F^{p+1}\Omega(P)$ and it is zero in E_1. Finally, $d_{1,0}\alpha$ has two parts. One of them belongs to $\Omega(P)_{i=0}$, and their image by h is zero. The other, $d_\pi\alpha$, the derivative along the fibres of π, is as follows: if $\{\xi_i\}$ is a basis of the fundamental vector fields and $\{\xi_i^*\}$ is the dual basis of differential forms, we have

$$d_\pi\alpha = \sum_i \xi_i^* \wedge \theta(\xi_i)\alpha.$$

But $\theta(\xi)[\alpha] = 0$, i.e., $\theta(\xi)\alpha = d_{\tilde{\mathscr{F}}}\beta$, $\beta \in F^p\Omega(P)_{i=0}$, and $d_{\tilde{\mathscr{F}}}\xi_i^* = 0$, then, if we put $\theta(\xi_i)\alpha = d_{\tilde{\mathscr{F}}}\beta_i$, then

$$hd_\pi\alpha = hd_{\tilde{\mathscr{F}}}\sum_i \xi_i^* \wedge \beta_i = d_{\tilde{\mathscr{F}}}h\sum_i \xi_i^* \wedge \beta_i,$$

with $h(\xi_i^* \wedge \beta_i) \in \Omega(P)_{i=0}$ and

$$\alpha - j \circ \rho(\alpha) = d_{\tilde{\mathscr{F}}}h\sum_i \xi_i^* \wedge \beta_i.$$

Now, to compute $E_2^{*,q}(\tilde{\mathscr{F}})$ we construct the spectral sequence $E(q)$ associated to the action of $\mathbf{so}(n)$ on $E_1(\tilde{\mathscr{F}})$,

$$E_2^{r,s}(q) = E_2^{s,q}(\mathscr{F}) \otimes H^r(\mathbf{so}(n), \mathbb{R}) \Rightarrow H(E_1^{*,q}(\tilde{\mathscr{F}})_{\theta=0}) \cong E_2^{*,q}(\tilde{\mathscr{F}}).$$

All the maps that we have used are continuous, then we also have a spectral sequence

$$\mathbb{E}_2^{r,s}(q) = \mathbb{E}_2^{s,q}(\mathscr{F}) \otimes H^r(\mathbf{so}(n), \mathbb{R}) \Rightarrow \mathbb{E}_2^{*,q}(\tilde{\mathscr{F}}).$$

Finally, the Zeeman's comparison theorem permits us to conclude:

THEOREM. *Let $\mathscr{F}$ be a Riemannian foliation. Then $E_2^{p,q}(\mathscr{F}) \cong \mathbb{E}_2^{p,q}(\mathscr{F})$.*

We can also conclude that $E_2^{p,q}(\mathscr{F})$ is finite dimensional, which is the principal theorem in [2].

3. A criterion for minimality

For a Riemannian foliation of codimension n on a compact manifold, we have $E_2^{n,0}(\mathscr{F}) = 0$ or $E_2^{n,0}(\mathscr{F}) = \mathbb{R}$ [4]. It is a well known fact that $E_2^{n,0}(\mathscr{F}) \neq 0$ is a necessary condition for minimality (vid., for instance, [12, Appendix B]). This is a consequence of the following Rummler–Sullivan criterion [14], [16], [7]:

"*Let g_0 be a smooth scalar product on $T\mathscr{F}$. It is induced by a Riemannian metric g on M for which the leaves are minimal submanifolds iff the volume m-form χ_0 on the leaves defined by g_0 (and the orientation) is the restriction to the leaves of an m-form χ on M which is relatively closed, namely, $d\chi(X_1, \ldots, X_{m+1}) = 0$ if the first m vector fields X_i are tangent to the leaves.*"

So, we shall assume that $E_2^{n,0}(\mathscr{F}) = \mathbb{R}$.

Now, we consider the star operator $*$ associated to the bundle-like metric on M and the scalar product

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta.$$

The star operator takes forms of type (p, q) into forms of type $(n-p, m-q)$. If we denote $\delta_{\mathscr{F}} = *d_{\mathscr{F}}*$, we have

$$\overline{\operatorname{Im} d_{\mathscr{F}}} = (\operatorname{Ker} \delta_{\mathscr{F}})^{\perp}, \tag{2}$$

where $\perp$ denotes the orthogonal complement with respect to the scalar product. Obviously, $\overline{\operatorname{Im} d_{\mathscr{F}}} \subseteq (\operatorname{Ker} \delta_{\mathscr{F}})^{\perp}$ and $(\overline{\operatorname{Im} d_{\mathscr{F}}})^{\perp} \subseteq \operatorname{Ker} \delta_{\mathscr{F}}$. We have also $\Omega = \overline{(\operatorname{Ker} \delta_{\mathscr{F}})^{\perp} \oplus \operatorname{Ker} \delta_{\mathscr{F}}} = \overline{\operatorname{Im} \delta_{\mathscr{F}}} \oplus \operatorname{Ker} \delta_{\mathscr{F}}$. In fact, if $\alpha \notin \overline{\operatorname{Im} d_{\mathscr{F}} \oplus \operatorname{Ker} \delta_{\mathscr{F}}}$, by the Hahn–Banach theorem there exists a closed hyperplane L that contains $\overline{\operatorname{Im} d_{\mathscr{F}} \oplus \operatorname{Ker} \delta_{\mathscr{F}}}$ and such that $\alpha \notin L$. If β is orthogonal to L, $\beta \in (\overline{\operatorname{Im} d_{\mathscr{F}}})^{\perp}$ and $\beta \notin \operatorname{Ker} \delta_{\mathscr{F}}$, contradition. Now, an analogous argument finishes the proof of (2).

We are assuming, first, that $\mathscr{F}$ is transversally parallelizable, that is, there exist foliated vector fields $Z_1, \ldots, Z_n$ such that their images generate $T_x M / T_x \mathscr{F}$ for all $x \in M$. The closures of the leaves of the foliation are the fibres of the *basic fibration* associated to $\mathscr{F}$ [12],

$$\pi : M \to W,$$

and the foliation defines by restriction to each of the fibres a Lie $\mathbf{g}$-foliation, where $\mathbf{g}$ is the *structural Lie algebra*.

Let v be an invariant transverse volume for $\mathscr{F}$, $v \in \Omega^n_B(M)$, defining the nonzero class in $E_2^{n,0}(\mathscr{F})$. We can choose the form v to be orthogonal to $d(\Omega^{n-1}_B(M))$. This is trivial if the leaves of $\mathscr{F}$ are dense, as then $d(\Omega^{n-1}_B(M)) = 0$. The general case requires the use of the structure of the fibration $\pi : M \to W$ defined by the closures of the leaves of $\mathscr{F}$. Consider the filtration defined in Section 1, but now associated to the fibration π, and so we can speak about the forms of type $(p, q)_\pi$, and differentials d_π, ${}_\pi d_{1,0}$, and so on.

Let ω be the image by π of the volume form on W. The condition $E_2^{n,0}(\mathscr{F}) \neq 0$ is equivalent to the following [4]: $\mathbf{g}$ is unimodular and there exists a form λ such that $v = \lambda \wedge \omega$ is an invariant transverse volume for $\mathscr{F}$ and satisfying

(1) λ is of type $(0, s)_\pi$, $s = \dim \mathbf{g}$, and
(2) $d_\pi \lambda = 0$ and ${}_\pi d_{1,0} \lambda = 0$.

Let $f = \int_\pi *(\lambda \wedge \omega) \wedge \lambda$, where $\int_\pi$ is the integral along the fibres. Then $f \in C^\infty(W)$ and $f(x) \neq 0$ for all x. Now, the volume $v_0 = (\lambda \wedge \omega)/f$ is orthogonal to $d(\Omega^{n-1}_B(M))$. In fact, a form $\gamma \in \Omega^{n-1}_B(M)$ can be written as

$$\gamma = \sum_{k=1}^{n} \{(i_{Z_k}\lambda) \wedge f_k \omega + \lambda \wedge g_k i_{Z_k} \omega\}$$

where $f_k, g_k \in C^\infty(W)$. Then,

$$d\gamma = \pm \lambda \wedge d\left(\sum_{k=1}^{n} g_k i_{Z_k} \omega\right) = \lambda \wedge d(\pi^* \eta)$$

with $\eta \in \Omega^*(W)$, and

$$\left\langle \frac{1}{f}(\lambda \wedge \omega), d\gamma \right\rangle = \left\langle \frac{1}{f}(\lambda \wedge \omega), \lambda \wedge d(\pi^*\eta) \right\rangle$$
$$= \pm \int_M \left\{ * \frac{1}{f}(\lambda \wedge \omega) \right\} \wedge \lambda \wedge d\pi^*\eta$$
$$= \int_W \frac{1}{f} d\eta \int_\pi *(\lambda \wedge \omega) \wedge \lambda = \int_W d\eta = 0.$$

With this choice of v_0, the form $\chi = *v_0$ is positive along the leaves and defines a nonzero class in $\mathbb{E}_2^{0,m}(\mathscr{F})$, i.e., $d_1(\chi) \in \bar{O}_{\mathscr{F}}^{1,m}$. In fact, let γ be a $(n-1)$-basic form. We have

$$\langle d_{1,0}(\chi), *\gamma \rangle = \langle d(\chi), *\gamma \rangle = \pm \langle v_0, d\gamma \rangle = 0,$$

and, by (2),

$$d_{1,0}(\chi) \in \overline{d_{\mathscr{F}} \Omega^{1,m-1}(M)}.$$

Let $[\Gamma] \in E_2^{0,m}(\mathscr{F})$ be the class corresponding to χ by the isomorphism $E_2^{0,m}(\mathscr{F}) \cong \mathbb{E}_2^{0,m}(\mathscr{F})$. We have

$$\Gamma = \chi + \eta, \qquad \text{with } \eta \in \bar{O}_{\mathscr{F}}^{0,m}.$$

Since χ is positive along the leaves, we can take some form $\alpha \in d_0(E_0^{0,m-1}(\mathscr{F}))$ such that $\Gamma + \alpha$ is close enough to χ so that $\Gamma + \alpha$ is also positive along the leaves. But $\Gamma + \alpha$ also defines $[\Gamma]$ and then $\mathscr{F}$ is minimal by the criterion of Rummler–Sullivan.

Finally, if $\mathscr{F}$ is an arbitrary Riemannian foliation, we consider the principal bundle (P, π, M) and the transversally parallelizable foliation $\tilde{\mathscr{F}}$, as in Section 2. Integration along the fibres of $\pi : P \to M$, after exterior multiplication with the invariant volume form along the fibres, assigns m-forms on M positive along the fibres of $\mathscr{F}$ to m-forms on P positive along the leaves of $\tilde{\mathscr{F}}$. The computations in Section 2 permits us to conclude the

MINIMALITY THEOREM. *Let M be a smooth closed orientable manifold and $\mathscr{F}$ an oriented Riemannian foliation. There exists a Riemannian metric on M for which the leaves are minimal submanifolds iff the basic cohomology of maximal dimension is nonzero.*

4. Duality theorem

DUALITY THEOREM. *If M is a smooth closed orientable manifold and $\mathcal{F}$ is a Riemannian foliation, then*

$$E_2^{p,q}(\mathcal{F}) \cong E_2^{n-p,m-q}(\mathcal{F}).$$

This Theorem reduces now to the Duality Theorem proved by J. A. Álvarez López [1], [3]. He defines a filtration in the complex of currents (Ω', d') in M, obtaining a spectral sequence (E^r, d^r) which converges to $H(\Omega', d')$, and he proves that there exist regularization operators which are adjoint of continuous filtration-preserving operators in $\Omega(M)$, resulting in an isomorphism between E_2 and E^2. Finally, he has duality isomorphisms $\mathbb{E}_2^{p,q}(\mathcal{F}) \cong \mathbb{E}_2^{n-p,m-q}(\mathcal{F})$.

REFERENCES

[1] ÁLVAREZ-LÓPEZ, J. A., "Sucesión espectral asociada a foliaciones Riemannianas". Publ. del Dpto. de Geometría y Topología de Santiago de Compostela *72* (1987).

[2] ÁLVAREZ-LÓPEZ, J. A., *A finiteness theorem for the spectral sequence of a Riemannian foliation.* Illinois J. of Math. *33* (1989), 79–92.

[3] ÁLVAREZ-LÓPEZ, J. A., *Duality in the spectral sequence of Riemannian foliations.* Amer. J. of Math. *111* (1989) 905–926.

[4] EL KACIMI-ALAOUI, A. and HECTOR, G., *Decomposition de Hodge basique pour un feuilletage Riemannien.* Ann. Inst. Fourier de Grenoble *36* (1986), 207–227.

[5] GREUB, W., HALPERIN, S. and VANSTONE, R., "Connections, Curvature and Cohomology". Academic Press, New York, 1976.

[6] GROTHENDIECK, A., "Topological Vector Spaces". Gordon and Breach, London, 1973.

[7] HAEFLIGER, A., *Some remarks on foliations with minimal leaves.* J. Diff. Geom. *15* (1986), 269–284.

[8] KAMBER, F. and TONDEUR, Ph., *Foliations and metrics.* Proc. of the 1981–1982 Year in Differential Geometry, Univ. Maryland, Birkhäuser. Progress in Math. *32* (1983), 103–152.

[9] MACIAS, E., "Las cohomologias diferenciable, continua y discreta de una variedad foliada". Publ. del Dpto. de Geometria y Topologia de Santiago de Compostela *60* (1983).

[10] MACIAS, E. and MASA, X., *Cohomologia diferenciable en variedades foliadas.* Actas de las IX Jornadas Matemáticas Hispano-Lusas. Universidad de Salamanca, 1982, vol. II, pp. 533–536.

[11] MASA, X., *Cohomology of Lie foliations.* Research Notes in Math. vol. 32. Differential Geometry. Pitman Advanced Publishing Program (1985), pp. 211–214.

[12] MOLINO, P., *Riemannian foliations.* Progress in Mathematics, Birkhäuser, 1988.

[13] ROBERTSON, A. P. and ROBERTSON, W. J., "Topological Vector Spaces". Cambridge Univ. Press, 1973.

[14] RUMMLER, H., *Quelques notions simples en géométrie Riemannienne et leurs applications aux feuilletages compacts.* Comment. Math. Helvetici *54* (1979), 224–239.

[15] SARKARIA, K. S., *A finiteness theorem for foliated manifolds.* J. Math. Soc. Japan *30* (1978), 687–696.

[16] SULLIVAN, D., *A homological characterization of foliations consisting of minimal surfaces*. Comment. Math. Helv. *54* (1979), 218–223.

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