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A toroidal compactification of the Fermi surface for the discrete Schrödinger operator

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1. Introduction

Let$^{\circ\circ}\psi\!\neq a_1\mathbb{Z} \oplus a_2\mathbb{Z} \oplus a_3\mathbb{Z}$ be a lattice in $\mathbb{R}^3$ and q a real valued square-integrable function on the torus $\mathbb{R}^3\backslash\Gamma$. For each $\xi = (\xi_1, \xi_2, \xi_3) \in S^1 \times S^1 \times S^1$ the self-adjoint boundary value problem, called the independent electron approximation of solid state physics (see [1]),

$$(-\Delta + q)\psi = \lambda\psi,$$
$$\psi(x+\gamma) = \xi_1^{\gamma_1}\xi_2^{\gamma_2}\xi_3^{\gamma_3}\psi(x) \quad \forall\gamma \in \Gamma,$$

has discrete spectrum, denoted by

$$E_1(\xi) \le E_2(\xi) \le E_3(\xi) \le \cdots$$

The (physical) Fermi surface for energy λ is the set

$$F_{\mathrm{phys},\lambda}(q) := \{\xi \in S^1 \times S^1 \times S^1 \mid E_n(\xi) = \lambda \text{ for some } n \ge 1\}.$$

In [3] one defines the complex Fermi surface by

$$F_\lambda(q) := \{(\xi_1, \xi_2, \xi_3) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \mid \text{there exists a non-trivial function } \psi \text{ in } H^2_{\mathrm{loc}}(\mathbb{R}^3) \text{ solving the above boundary value problem}\}.$$

Clearly $F_\lambda(q)$ contains all points that can be reached by analytic continuation of $F_{\mathrm{phys},\lambda}(q)$. Using regularized determinants (see [7]) it can be shown, that $F_\lambda(q)$ is a complex analytic hypersurface in $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$. In [3] it was shown that for potentials $q(x)$ of the form $p_1(x_1) + p_2(x_2) + p_3(x_3)$ or $p_1(x_1) + p_2(x_2, x_3)$ the surface $F_\lambda(q)$ is irreducible, i.e. in this case $F_{\mathrm{phys},\lambda}(q)$, if it is a nonempty set of dimension two, determines $F_\lambda(q)$ uniquely.

In this paper we consider a discrete version and show that for each (complex) potential q, the Fermi surface is always irreducible.

So let $V : \mathbb{Z}^3 \to \mathbb{C}$ an arbitrary function periodic with respect to the lattice Γ. Furthermore let Δ be the discrete Laplace operator defined by

$$(\Delta\psi)(m, n, p) = \psi(m-1, n, p) + \psi(m+1, n, p) + \psi(m, n-1, p) + \psi(m, n+1, p) + \psi(m, n, p-1) + \psi(m, n, p+1)$$

for functions $\psi : \mathbb{Z}^3 \to \mathbb{C}$.

We are interested in the spectral problem

$$(-\Delta + V)\psi = \lambda\psi$$

with boundary conditions

$$\psi(m + a_1, n, p) = \xi_1\psi(m, n, p), \qquad \psi(m, n + a_2, p) = \xi_2\psi(m, n, p),$$
$$\psi(m, n, p + a_3) = \xi_3\psi(m, n, p)$$

for functions $\psi : \mathbb{Z}^3 \to \mathbb{C}$ and $(\lambda, \xi_1, \xi_2, \xi_3) \in \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$, and define as above the complex Fermi surface $F_\lambda(V)$ for this discrete problem (see [4]). Furthermore we assume that a_1, a_2 and a_3 are relatively prime, positive natural numbers greater to two.

Due to the boundary conditions the spectral problem can be written in terms of the values $\psi(m, n, p)$ with $1 \le m \le a_1$, $1 \le n \le a_2$, $1 \le p \le a_3$. The Fermi surface $F_\lambda(V)$ is then given by the vanishing of the determinant of a certain $a_1a_2a_3 \times a_1a_2a_3$-matrix, or concrete, it is the zero-set of a polynomial P in the variables ξ_1, ξ_1^{-1}, ξ_2, ξ_2^{-1} and ξ_3, ξ_3^{-1}, where the coefficients depend on λ:

$$\begin{aligned} P = &(-1)^{a_2a_3(a_1-1)}\xi_1^{a_2a_3} + (-1)^{a_2a_3(a_1-1)}\xi_1^{-a_2a_3} \\ &+ (-1)^{a_1a_3(a_2-1)}\xi_2^{a_1a_3} + (-1)^{a_1a_3(a_2-1)}\xi_2^{-a_1a_3} \\ &+ (-1)^{a_1a_2(a_3-1)}\xi_3^{a_1a_2} + (-1)^{a_1a_2(a_3-1)}\xi_3^{-a_1a_2} + \cdots \end{aligned}$$

lower order terms, i.e. an algebraic surface in $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$.

For potentials $V = C = \text{constant}$, $F_\lambda(V)$ can be calculated explicitly, using discrete Fourier analysis. Namely let μ_{a_i} be the multiplicative group of a_i-th root of unity. Then for $\rho = (\rho_1, \rho_2, \rho_3) \in \mu_{a_1} \times \mu_{a_2} \times \mu_{a_3}$ consider the set

$$\tilde{F}_\lambda(V = C) = \bigcup_\rho \left\{(z_1, z_2, z_3) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \,\Big|\, \sum_{i=1}^{3} ((\rho_i z_i)^{-1} + (\rho_i z_i)) = \lambda - C\right\}.$$

Now $\mu_{a_1} \times \mu_{a_2} \times \mu_{a_3}$ acts on $\tilde{F}_\lambda(V = C)$ by $\rho \cdot z = (\rho_1 z_1, \rho_2 z_2, \rho_3 z_3)$. Then one has $F_\lambda(V) = \tilde{F}_\lambda(V = C)/\mu_{a_1} \times \mu_{a_2} \times \mu_{a_3}$, and so $F_\lambda(V)$ is irreducible.

It is well known, that the one-dimensional Bloch variety $B(W)$, defined by (see [4], [8])

$$B(W) = \{(\xi, \lambda) \in \mathbb{C}^* \times \mathbb{C} \mid \text{there exists a non-trivial function } \psi : \mathbb{Z} \to \mathbb{C} \text{ solving}$$
$$-[\psi(m-1) + \psi(m+1)] + W(m)\psi(m) = \lambda\psi(m), \psi(m+a) = \psi(m)\},$$

where $W : \mathbb{Z} \to \mathbb{C}$ has period a, is a hyperelliptic curve of arithmetic genus $a - 1$, which can be compactified by adding two smooth points.

In this paper we construct a compactification $F_\lambda(V)_{\text{comp}}$ of $F_\lambda(V)$, such that the generic points of $F_\lambda(V)_{\text{comp}}$ are smooth points of $F_\lambda(V)_{\text{comp}}$.

THEOREM 1. *$F_\lambda(V)_{\text{comp}} - F_\lambda(V)$ is the union of twenty algebraic curves due to twenty one-dimensional spectral problems*:

(i) *eight rational curves $Q_1, \dots, Q_8$ with $(a_1 - 1)(a_2 - 1)(a_3 - 1) + \Sigma_{i \neq j}(a_i - 1)(a_j - 1)$ ordinary double points. These curves do not depend on the potential V.*

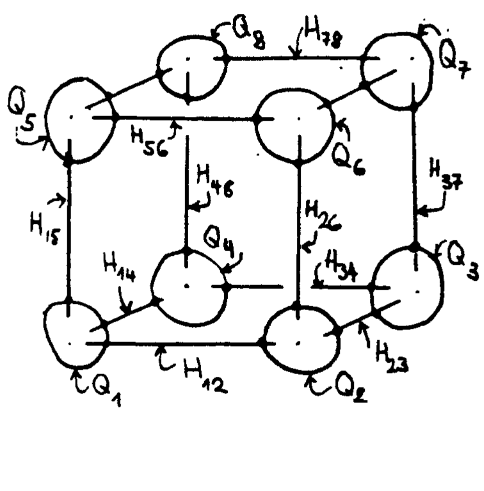
(ii) *Twelve hyperelliptic curves $H_{ij} : H_{12}, H_{34}, H_{56}, H_{78}$ (resp. $H_{14}, H_{58}, H_{23}, H_{67}$; resp. $H_{15}, H_{26}, H_{37}, H_{48}$) of arithmetic genus $a_1 - 1$ (resp. $a_2 - 1$, resp. $a_3 - 1$), each isomorphic to the one-dimensional Bloch variety $B(W)$, where W is the averaged potential*

$$W(\cdot) = \frac{1}{a_2 a_3} \sum_{i=1}^{a_2} \sum_{k=1}^{a_3} V(\cdot, i, k)$$

$$\left(\textit{resp.}\ \frac{1}{a_1 a_3} \sum_{i=1}^{a_1} \sum_{k=1}^{a_3} V(i, \cdot, k);\ \textit{resp.}\ \frac{1}{a_1 a_2} \sum_{i=1}^{a_1} \sum_{k=1}^{a_2} V(i, k, \cdot)\right).$$

(iii) *$F_\lambda(V)_{\text{comp}}$ is smooth on all smooth points of $F_\lambda(V)_{\text{comp}} - F_\lambda(V)$.*

(iv) *All the above curves intersect transversally, only on smooth points of $F_\lambda(V)_{\text{comp}}$ and the intersection pattern is given by the following picture*:



As an immediate consequence we get

THEOREM 2. *The Fermi surface $F_\lambda(V)$ is irreducible.*

Naively one could try to compactify $F_\lambda(V)$ by embedding $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and closing the Fermi surface in there. This doesn't work, since the new points added to $F_\lambda(V)$ are highly singular. Instead we construct (as in [2] and [8]) a compact three-dimensional torus embedding X_Σ, such that

$$F_\lambda(V) \subset \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \subset X_\Sigma.$$

A torus embedding X_Σ is a scheme such that algebraic torus $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ can be embedded in X_Σ in a way, that the action of the algebraic torus can be extended to the whole torus embedding X_Σ. Information and facts about torus embeddings can be found in [5], [6] and [9]. The closure of $F_\lambda(V)$ in this space X_Σ (after resolution of certain singular points of X_Σ) is the compactified Fermi surface $F_\lambda(V)_{\mathrm{comp}}$.

Furthermore we not only construct the torus embedding X_Σ, but also an infinite-dimensional vector bundle Y, the vectorspace F of all functions $\psi : \mathbb{Z} \to \mathbb{C}$ as fiber, on X_Σ. On $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times F \subset Y$ we have four commutating operators:

$$-\Delta + V - \lambda 1, \qquad S^{(a_1,0,0)} - \xi_1 1, \qquad S^{(0,a_2,0)} - \xi_2 1, \qquad S^{(0,0,a_3)} - \xi_3 1,$$

for $(\xi_1, \xi_2, \xi_3) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ (here $S^{(\alpha,\beta,\gamma)}$ denotes the shift operator in direction (α, β, γ)). The Fermi surface is then the support of the bundle

$$\{(\xi_1, \xi_2, \xi_3, \psi) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times F \mid \text{the above four operators have a common kernel } \psi\}.$$

By extending this bundle to the whole X_Σ the rational and hyperelliptic curves mentioned in Theorem 1 will appear in a straightforward way.

Let us close by mentioning, that a similar construction was worked out in detail for the two-dimensional Bloch variety in [2].

2. The construction of the toroidal octahedron

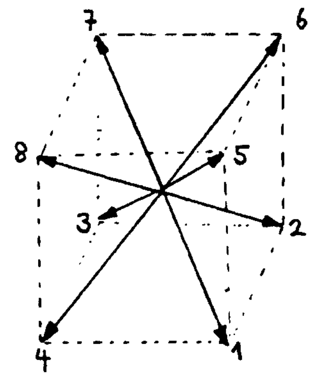
In this chapter we construct the three-dimensional torus embedding, in which $F_\lambda(V)$ will lie. Consider the eight vectors $1, 2, \ldots, 8$ in $\mathbb{R}^3$ given by

$1 := (a_1, a_2, a_3), \quad 2 := (-a_1, a_2, a_3),$

$3 := (-a_1, -a_2, a_3), \quad 4 := (a_1, -a_2, -a_3),$

$5 := (a_1, a_2, -a_3), \quad 6 := (-a_1, a_2, -a_3),$

$7 := (-a_1, -a_2, -a_3), \quad 8 := (a_1, -a_2, -a_3).$



We introduce the following notation:

σ^i means the strongly convex polyhedral cone generated by the vector i.

So for example $\sigma^{12} = \{t(a_1, a_2, a_3) + s(-a_1, a_2, a_3) \mid s, t \in \mathbb{R}_{\geq 0}\}$, where $\mathbb{R}_{\geq 0}$ denotes the non-negative real numbers.

We define the fan Σ to be the collection of the six three-dimensional cones σ^{1256}, σ^{2367}, σ^{3478}, σ^{1458}, σ^{5678} and σ^{1234} and all it's faces. There are two-dimensional faces as σ^{12} or σ^{15}, one-dimensional faces as σ^1, σ^2 and one zero-dimensional face $\sigma^0 = \{0\}$. We call the torus embedding X_Σ associated to this fan toroidal octahedron. It is compact (see [5]). Explicitly X_Σ is given by a coordinate covering $(X_\sigma)_{\sigma \in \Sigma}$. The (X_σ)'s are (quasi)-affine varieties defined by

$$X_\sigma = \operatorname{Spec} \mathbb{C}[\ldots, \xi_1^{r_1} \xi_2^{r_2} \xi_3^{r_3}, \ldots],$$

where $\mathbb{C}[\ldots, \xi_1^{r_1} \xi_2^{r_2} \xi_3^{r_3}, \ldots]$ is the algebra generated by the polynomials $\xi_1^{r_1} \xi_2^{r_2} \xi_3^{r_3}$ with (r_1, r_2, r_3) in $\mathbb{Z}^3$ such that $\langle (r_1, r_2, r_3), \sigma \rangle \geq 0$.

If σ^α and σ^β are two cones in Σ, then the charts X_{σ^α} and X_{σ^β} are patched together along $X_{\sigma^\alpha} \cap X_{\sigma^\beta}$. So, for example $X_{\sigma^{12}}$ and $X_{\sigma^{13}}$ are glued together on $X_{\sigma^{12}} \cap X_{\sigma^{13}} = X_{\sigma^1}$.

Clearly we have $X_{\sigma^0} = \operatorname{Spec} \mathbb{C}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}, \xi_3, \xi_3^{-1}] = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$. So we embed the Fermi surface $F_\lambda(V)$ by the inclusions

$$F_\lambda(V) \subset \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* = X_{\sigma^0} \subset X_\Sigma$$

in the toroidal octahedron.

In the following we analyze X_Σ.

Since the action of the algebraic torus $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ on itself can be extended to X_Σ, the toroidal octahedron is a disjoint union of $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ orbits. There is now a one-to-one correspondence between these orbits in X_Σ and the cones $\sigma \in \Sigma$; so we can label an orbit by a cone $\sigma : \mathbb{O}_\sigma$. Furthermore we can organize this labeling such that (see [5]):

$$\mathbb{O}_\sigma \subset X_\Sigma, \qquad \dim_{\mathbb{C}} \mathbb{O}_\sigma = 3 - \dim_{\mathbb{R}} \sigma,$$

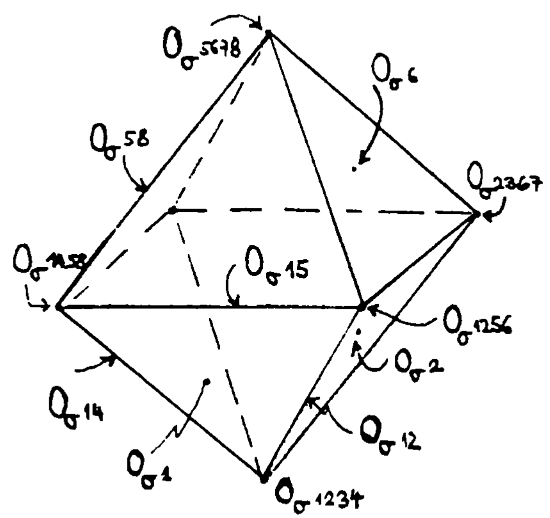
and $\mathbb{O}_\sigma = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \cdot \lambda_b(0)$, where $\lambda_b(0)$ is the point

$$\lim_{t \to 0} X_\sigma(t) = \lim_{t \to 0} \operatorname{Spec} \mathbb{C}[\ \dots, \xi_1^{r_1} \xi_2^{r_2} \xi_3^{r_3}, \dots\ ]|_{\text{setting } \xi_i = t^{b_i}},$$

where $b = (b_1, b_2, b_3) \in \mathbb{Z}^3$ is a point in the interior of σ.

It is easy to draw a schematic picture of the toroidal octahedron X_Σ (compare with [2] and [7]):

The "interior" of this octahedron represents $\mathbb{O}_{\sigma^0} = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$.



Using the symmetry of the fan Σ we can restrict to the orbits $\mathbb{O}_{\sigma^0}$, $\mathbb{O}_{\sigma^1}$, $\mathbb{O}_{\sigma^{12}}$ and $\mathbb{O}_{\sigma^{1234}}$. Let y_0, z_0 be integers with $a_2 y_0 + a_3 z_0 = 1$.

LEMMA 1. (i) $X_{\sigma^0} = \operatorname{Spec} \mathbb{C}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}, \xi_3, \xi_3^{-1}] = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$.

(ii) $X_{\sigma^1} = \operatorname{Spec} \mathbb{C}[\xi_1^{-1}(\xi_2^{y_0}\xi_3^{z_0})^{a_1}, \xi_1(\xi_2^{y_0}\xi_3^{z_0})^{-a_1}, \xi_2^{y_0}\xi_3^{z_0}, \xi_2^{-a_3}\xi_3^{a_2}, \xi_2^{a_3}\xi_3^{-a_2}]$, *i.e.* X_{σ^1} *is isomorphic to* $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ *with local coordinates*

$$u_1 := \xi_1^{-1}(\xi_2^{y_0}\xi_3^{z_0})^{a_1} \in \mathbb{C}^*, \qquad v_1 := \xi_2^{y_0}\xi_3^{z_0} \in \mathbb{C} \quad \text{and} \quad w_1 := \xi_2^{-a_3}\xi_3^{a_2} \in \mathbb{C}^*.$$

Furthermore $\mathbb{O}_{\sigma^1} = \{(u_1, w_1) \in \mathbb{C}^* \times \mathbb{C}^*, v_1 = 0\}$.

(iii) $X_{\sigma^{12}} = \operatorname{Spec} \mathbb{C}[\xi_1^{-1}(\xi_2^{y_0}\xi_3^{z_0})^{a_1}, \xi_1(\xi_2^{y_0}\xi_3^{z_0})^{a_1}, \xi_2^{y_0}\xi_3^{z_0}, \xi_2^{-a_3}\xi_3^{a_2}, \xi_2^{a_3}\xi_3^{-a_2}]$ *i.e.* $X_{\sigma^{12}}$ *isomorphic to* $\operatorname{Spec} \mathbb{C}[x, y, w, z, z^{-1}]/\langle xy = w^{2a_1}\rangle$. $\mathbb{O}_{\sigma^{12}} = \{z \in \mathbb{C}^*, x = y = w = 0\}$, *i.e. each point of* $\mathbb{O}^{\sigma^{12}}$ *is singular of type* A_{2a_1-1}.

Proof

(i) Trivial.

(ii) Clearly the three vectors $(1, -a_1 y_0, -a_1 z_0)$, $(0, y_0, z_0)$, $(0, -a_3, a_2)$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^3$. So each $(r_1, r_2, r_3) \in \mathbb{Z}^3$ can be written as

$$(r_1, r_2, r_3) = r_1(1, -a_1 y_0, -a_1 z_0) + s(0, y_0, z_0) + t(0, -a_3, a_2) \tag{1}$$

with $s, t \in \mathbb{Z}$. Now $\langle (r_1, r_2, r_3), \sigma^1 \rangle \geq 0$ exactly if $a_1 r_1 + a_2 r_2 + a_3 r_3 \geq 0$. But s is equal to $a_1 r_1 + a_2 r_2 + a_3 r_3$ by (1). Therefore X_{σ^1} is given as stated in the lemma. Computing $\mathbb{O}_{\sigma^1}$ is straightforward.

(iii) We have two $\mathbb{Z}$-bases of $\mathbb{Z}^3$; first the three vectors $(1, -a_1 y_0, -a_1 z_0)$, $(0, y_0, z_0)$, $(0, -a_3, a_2)$ and second the vectors $(1, a_1 y_0, a_1 z_0)$, $(0, y_0, z_0)$, $(0, -a_3, a_2)$. So for each $(r_1, r_2, r_3) \in \mathbb{Z}^3$ we can write

$$(r_1, r_2, r_3) = r_1(1, -a_1 y_0, -a_1 z_0) + s(0, y_0, z_0) + t(0, -a_3, a_2), \tag{2}$$

$$(r_1, r_2, r_3) = r_1(1, a_1 y_0, a_1 z_0) + \tilde{s}(0, y_0, z_0) + \tilde{t}(0, -a_3, a_2), \tag{3}$$

with $s, t, \tilde{s}, \tilde{t} \in \mathbb{Z}$. Since $\langle (r_1, r_2, r_3), \sigma^{12} \rangle \geq 0$ if and only if $a_1 r_1 + a_2 r_2 + a_3 r_3 \geq 0$ and $-a_1 r_1 + a_2 r_2 + a_3 r_3 \geq 0$ it follows with (2) and (3) that

$$a_1 r_1 + a_2 r_2 + a_3 r_3 = s = 2a_1 r_1 + \tilde{s} \geq 0,$$

$$a_1 r_1 + a_2 r_2 + a_3 r_3 = \tilde{s} = -2a_1 r_1 + s \geq 0.$$

Let first be $r_1 \geq 0$, then both inequalities are fulfilled exactly if $\tilde{s} \geq 0$. Secondly let $r_1 \leq 0$, then the necessary and sufficient condition is $s \geq 0$. This proves (iii) (again $\mathbb{O}_{\sigma^{12}}$ is easy to calculate).

We do not need the chart $X_{\sigma^{1234}}$ since we have:

LEMMA 2. *The closure* $\overline{F_\lambda(V)}$ *of the Fermi surface* $F_\lambda(V)$ *in* X_Σ *doesn't intersect the zero-dimensional orbits.*

Proof. It is enough to show that $\mathbb{O}_{\sigma^{1234}} \cap \overline{F_\lambda(V)} = \emptyset$. Since $\dim_{\mathbb{C}} \mathbb{O}_{\sigma^{1234}} = 0$ the (singular) point $\mathbb{O}_{\sigma^{1234}}$ has coordinates (in $X_{\sigma^{1234}}$) $\xi_1^{r_1}\xi_2^{r_2}\xi_3^{r_3} = 0$ for all $(r_1, r_2, r_3) \in \mathbb{Z}^3$ with $\langle (r_1, r_2, r_3), \sigma^{1234} \rangle \geq 0$.

Clearly $\xi_3 \in \mathbb{C}$ is a coordinate of $X_{\sigma^{1234}}$, so the polynomial P, defining $F_\lambda(V)$, has a pol in ξ_3 of order $a_1 a_2$. On the other hand since $P = \Sigma_{ijk}\, a_{ijk}(\lambda)\xi_1^i \xi_2^j \xi_3^k$ with (due to the boundary conditions defining the Fermi surface) $a_1|i| + a_2|j| + a_3|k| \le a_1 a_2 a_3$ it follows that each summand $\xi_1^i \xi_2^j \xi_3^{k+a_1a_2} \neq 1$ of the polynomial $\xi_3^{a_1a_2}P$ is a coordinate of $X_{\sigma^{1234}}$.

Therefore the closure of $F_\lambda(V)$ in X_Σ lying in the chart $X_{\sigma^{1234}}$ is given by the equation

$$\xi_3^{a_1a_2}P = 0.$$

But $\xi_3^{a_1a_2}P|_{O_{\sigma^{1234}}} = (-1)^{a_1a_2(a_3-1)} \neq 0$.

Motivated from this lemma we are only are interested in the closure $\overline{F_\lambda(V)}$ of $F_\lambda(V)$ in

$$X_\Sigma^* = X_\Sigma - \{\text{union of the zero-dimensional orbits}\}.$$

3. The compactification

We consider the compactified Fermi surface as the solution of a spectral problem on a vectorbundle Y of infinite rank on X_Σ^*.

We define by F the infinite-dimensional vector space of all functions $\psi : \mathbb{Z}^3 \to \mathbb{C}$. The vectorbundle $\pi : Y \to X_\Sigma^*$ will be trivial over each affine part X_σ of X_Σ^* $(\sigma \in \Sigma)$.

On $Y|_{X_{\sigma^0}} = X_{\sigma^0} \times F = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times F$ we have four commutating operators

$$T - \lambda 1 := -\Delta + V - \lambda 1, \qquad S^{(a_1,0,0)} - \xi_1 1, \qquad S^{(0,a_2,0)} - \xi_2 1, \qquad S^{(0,0,a_3)} - \xi_3 1,$$

for $(\xi_1, \xi_2, \xi_3) \in X_{\sigma^0}$.

DEFINITION. The (uncompactified) Fermi surface $F_\lambda(V)$ is the support of the bundle

$$\{(\xi_1, \xi_2, \xi_3, y_0) \in X_{\sigma^0} \times F \mid \text{the above four operators have a common kernel } \psi_0\}.$$

By symmetry and lemma 2 (since we want the closure $\overline{F_\lambda(V)}$ of $F_\lambda(V)$ in X_Σ^* to coincide with the support of bundle on X_Σ^*) it is enough to extend the vectorbundle $Y|_{X_{\sigma^0}}$ on X_{σ^1} and $X_{\sigma^{12}}$. We give the transition functions, using lemma 1:

(i) $X_{\sigma^1} = \operatorname{Spec} \mathbb{C}[u_1, u_1^{-1}, v_1, w_1, w_1^{-1}]$, i.e. the coordinates are $(u_1, v_1, w_1) \in$

$\mathbb{C}^* \times \mathbb{C} \times \mathbb{C}^*$. We now identify $(\xi_1, \xi_2, \xi_3, \psi_0) \in X_{\sigma^0} \times F$ with $(u_1, v_1, w_1, \psi_1) \in X_{\sigma^1} \times F$ on $X_{\sigma^1} \cap X_{\sigma^0} = X_{\sigma^0}$ (or equivalently on $v_1 \neq 0$) by

$$u_1 = \xi_1^{-1}(\xi_2^{y_0}\xi_3^{z_0})^{a_1}, \qquad v_1 = \xi_2^{y_0}\xi_3^{z_0}, \qquad w_1 = \xi_2^{-a_3}\xi_3^{a_2}$$

(this is the coordinate change on X_Σ from X_{σ^o} to X_{σ^1}) and

$$\psi_0(m, n, p) = v_1^{m+n+p}\psi_1(m, n, p).$$

(ii) On $X_{\sigma^{12}}$ we have coordinates $(x, y, w, z) \in \mathbb{C}^3 \times \mathbb{C}^*$ with $xy = w^{2a_1}$. Identify $(\xi_1, \xi_2, \xi_3, \psi_0) \in X_{\sigma^0} \times F$ with $(x, y, w, z) \in X_{\sigma^{12}} \times F$ on $X_{\sigma^{12}} \cap X_{\sigma^0}$ (i.e. on $w \neq 0$) by

$$x = \xi_1^{-1}(\xi_2^{y_0}\xi_3^{z_0})^{a_1}, \qquad y = \xi_1(\xi_2^{y_0}\xi_3^{z_0})^{a_1}, \qquad w = \xi_2^{y_0}\xi_3^{z_0}, \qquad z = \xi_2^{-a_3}\xi_3^{a_2},$$

and

$$\psi_0(m, n, p) = w^{m+n+p}x^{-m/a_1}\psi_{12}(m, n, p).$$

Denote by $\overline{F_\lambda(V)}$ the closure of $F_\lambda(V)$ in X^*_Σ.

PROPOSITION 1. *$(\overline{F_\lambda(V)} - F_\lambda(V)) \cap X_{\sigma^{12}}$ is the union of two rational curves Q_1 and Q_2 with the following properties;*

(i) *Q_i $(i = 1, 2)$ has $(a_1 - 1)(a_2 - 1)(a_3 - 1) + \Sigma_{i \neq j}(a_i - 1)(a_j - 1)$ ordinary double points. Q_i does not depend on the potential V.*

(ii) *$Q_1 \cap Q_2$ is a point $(=: P_{12})$. P_{12} lies on the singular orbit $\mathbb{O}_{\sigma^{12}}$.*

(iii) *$\overline{F_\lambda(V)}$ is smooth on all smooth points of $Q_1 \cup Q_2 - \{P_{12}\}$.*

Remark. Since P_{12} is singular, we will resolve this point. The strict transformation of $\overline{F_\lambda(V)}$ on the exceptional divisor is then one of the hyperelliptic curves mentioned in theorem 1 of the introduction.

PROPOSITION 2. *Q_1 is given on the chart X_{σ^1} as the support of the bundle $(u_1, v_1, w_1, \psi_1) \in Y$ with*

$$(S^{(-1,0,0)} + S^{(0,-1,0)} + S^{(0,0,-1)})\psi_1 = 0,$$

$$S^{(-a_1,0,0)}\psi_1 = u_1\psi_1, \qquad S^{(0,-a_2a_3,a_2a_3)}\psi_1 = w_1\psi_1, \qquad S^{(0,a_2y_0,a_3z_0)}\psi_1 = \psi_1$$

with $v_1 = 0$.

We first prove Proposition 2, then Proposition 1 will follow easily.

Proof of proposition 2. The spectral problem on $X_{\sigma^0} \times F$

$$T\psi_0 = \lambda\psi_0, \qquad S^{(a_1,0,0)}\psi_0 = \xi_1\psi_0, \qquad S^{(0,a_2,0)}\psi_0 = \xi_2\psi_0, \qquad S^{(0,0,a_3)}\psi_0 = \xi_3\psi_0$$

can be written alternatively as

$$TS^{(0,a_2y_0,a_3z_0)}\psi_0 = \lambda\xi_2^{y_0}\xi_3^{z_0}\psi_0, \qquad S^{(-a_1,0,0)}\psi_0 = \xi_1^{-1}\psi_0,$$
$$S^{(0,a_2y_0,a_3z_0)}\psi_0 = \xi_2^{y_0}\xi_3^{z_0}\psi_0, \qquad S^{(0,-a_2a_3,a_2a_3)}\psi_0 = \xi_2^{-a_3}\xi_3^{a_2}\psi_0,$$

since the shift operators are invertible and the vectors $(-a_1, 0, 0), (0, a_2y_0, a_3z_0), (0, -a_3a_2, a_3a_2)$ are also a basis for the lattice Γ.

By the construction of the vectorbundle Y these four equations transform to

$$-(S^{(-1,a_2y_0,a_3z_0)}\psi_1 + v_1^2S^{(1,a_2y_0,a_3z_0)}\psi_1 + S^{(0,a_2y_0-1,a_3z_0)}\psi_1$$
$$+ v_1^2S^{(0,a_2y_0+1,a_3z_0)}\psi_1 + S^{(0,a_2y_0,a_3z_0-1)}\psi_1 + v_1^2S^{(0,a_2y_0,a_3z_0+1)}\psi_1)$$
$$+ v_1VS^{(0,a_2y_0,a_3z_0)}\psi_1 = v_1\lambda S^{(0,a_2y_0,a_3z_0)}\psi_1,$$
$$S^{(-a_1,0,0)}\psi_1 = u_1\psi_1, \qquad S^{(0,a_2y_0,a_3z_0)}\psi_1 = \psi_1, \qquad S^{(0,-a_2a_3,a_2a_3)}\psi_1 = w_1\psi_1,$$

on $X_{\sigma^1} \times F$.

But $X_{\sigma^1} - X_{\sigma^0} = \{v_1 = 0\}$ and on the open set $X_{\sigma^1} \cup X_{\sigma^0} = X_{\sigma^0}$ by the continuity of the transition function the spectral problems on $Y|_{X_{\sigma^0}}$ and $Y|_{X_{\sigma^1}}$ coincide.

Therefore $(\overline{F_\lambda(V)} - F_\lambda(V)) \cap X_{\sigma^1}$ is the support of the following spectral problem

$$(S^{(-1,a_2y_0,a_3z_0)} + S^{(0,a_2y_0-1,a_3z_0)} + S^{(0,a_2y_0,a_3z_0-1)})\psi_1 = 0,$$
$$S^{(-a_1,0,0)}\psi_1 = u_1\psi_1, \qquad S^{(0,a_2y_0,a_3z_0)}\psi_1 = \psi_1, \qquad S^{(0,-a_2a_3,a_2a_3)}\psi_1 = w_1\psi_1,$$

which leads immediately to proposition 2.

Proof of proposition 1. (i) Clearly Q_1 does not depend on V. To calculate the genus of this curve, we consider a covering of Q_1 with $a_1a_2a_3$ sheets, as in [4].

Let μ_{a_1} (resp. $\mu_{a_2a_3}$) be the multiplicative group of a_1th (resp. a_2a_3th) root of unity and $(z_1^{-a_1}, z_2^{-a_2a_3}) := (u_1, w_1)$. Then the functions $e_\rho(z)(m, -n, n) = (\rho_1z_1)^m(\rho_2z_2)^n$ with $\rho = (\rho_1, \rho_2) \in \mu_{a_1} \times \mu_{a_2a_3}$ form a basis of the vectorspace of functions

$$\psi : \mathbb{Z}^2 \to \mathbb{C}, (m, -n, n) \to \psi(m, -n, n) \quad \text{with } S^{(-a_1,0,0)}\psi_1 = u_1\psi_1,$$

$$S^{(0,-a_2a_3,a_2a_3)}\psi_1 = w_1\psi_1.$$

The operator $S^{(-1,a_2y_0,a_3z_0)} + S^{(0,a_2y_0-1,a_3z_0)} + S^{(0,a_2y_0,a_3z_0-1)}$ diagonalizes in this basis and considering the covering

$$c : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^*, (z_1, z_2) \to (z_1^{-a_1}, z_2^{-a_2a_3}) = (u_1, w_1)$$

$\tilde{Q}_1 = c^{-1}(Q_1)$ is given by

$$\bigcup_\rho \{(z_1, z_2) \in \mathbb{C}^* \times \mathbb{C}^* \mid (\rho_1 z_1)^{-1} + (\rho_2 z_2)^{a_2y_0-1} + (\rho_2 z_2)^{a_2y_0} = 0\}.$$

This means Q_1 is rational. Let $C_\rho = \{(\rho_1 z_1)^{-1} + (\rho_2 z_2)^{a_2y_0-1} + (\rho_2 z_2)^{a_2y_0} = 0\}$. Perform the changes of coordinates

$$z_1 = y^{-a_2y_0}x^{-a_3z_0}, \qquad z_2 = yx^{-1}$$

C_ρ transforms to

$$\rho_1^{-1}xy + \rho_2^{-a_3z_0}x + \rho_2^{a_2y_0}y = 0.$$

Since a_2 and a_3 are relatively prime we have $\mu_{a_2a_3} = \mu_{a_2} \otimes \mu_{a_3}$ i.e. each $\rho_2 \in \mu_{a_2a_3}$ can be written as $\tilde{\rho}_2\tilde{\rho}_3^{-1}$ with $\tilde{\rho}_2 \in \mu_{a_2}, \tilde{\rho}_3 \in \mu_{a_3}$. Therefore $C_\rho = C_{(\rho_1,\tilde{\rho}_2,\tilde{\rho}_3)}$ is given by

$$\rho_1^{-1}xy + \tilde{\rho}_2^{-1}x + \tilde{\rho}_3^{-1}y = 0.$$

For $\rho \in \mu_{a_1} \times \mu_{a_2} \times \mu_{a_3}$ the action on $\tilde{Q}_1$ is given by $\rho \cdot (x, y) = (\rho_1^{-1}\tilde{\rho}_2 x, \rho_1^{-1}\tilde{\rho}_3 y)$, and we have $\rho \cdot C_{\rho'} = C_{\rho\rho'}$. Now by Bézout's Theorem C_1 and C_ρ intersect transverse in $\mathbb{C}^* \times \mathbb{C}^*$ in exactly one point, given by

$$x(\rho) = -\frac{\tilde{\rho}_3^{-1} - \tilde{\rho}_2^{-1}}{\rho_1^{-1} - \tilde{\rho}_2^{-1}}, \qquad y(\rho) = -\frac{\tilde{\rho}_2^{-1} - \tilde{\rho}_3^{-1}}{\rho_1^{-1} - \tilde{\rho}_3^{-1}}$$

if ρ is not of the form $(1, 1, \tilde{\rho}_3), (1, \tilde{\rho}_2, 1), (\rho_1, 1, 1)$. In this second case we have $C_1 \cap C_\rho \cap (\mathbb{C}^* \times \mathbb{C}^*) = \varnothing$. To prove (i) it remains to show that

$$(x(\rho), y(\rho)) \neq (x(\rho'), y(\rho')) \qquad \text{for } \rho \neq \rho',$$

because then Q_1 has exactly

$$a_1a_2a_3 - 1 - (a_1 - 1) - (a_2 - 1) - (a_3 - 1)$$

ordinary double points.

Observe that $arg\, x(\rho) = \alpha =$ angle in $\tilde{\rho}_2^{-1}$ of the triangle $\rho_1^{-1}, \tilde{\rho}_2^{-1}, \tilde{\rho}_3^{-1}$. Put $\rho_1^{-1} = e^{2\pi i m_1/a_1}$, $\rho_1'^{-1} = e^{2\pi i m_1'/a_1}$, $\tilde{\rho}_3^{-1} = e^{2\pi i m_3/a_3}$ and $\tilde{\rho}_3'^{-1} = e^{2\pi i m_3'/a_3}$ with $m_i, m_i' \in \{0, \ldots, a_i - 1\}$. One shows

$$\alpha = \pi \left| \frac{m_3}{a_3} - \frac{m_1}{a_1} \right|,$$

so if $x(\rho) = x(\rho')$ then $a_3(m_1 \pm m_1') = a_1(m_3 \pm m_3')$. Since a_1 and a_3 are relatively prime it follows that

$$(m_1 \pm m_1', m_3 \pm m_3') \in \{(0, 0), \pm(a_1, a_3)\}.$$

In the first case we get either $\rho_1 = \rho_1'$, $\rho_3 = \tilde{\rho}_3'$ or $\rho_1 = \rho_1'^{-1}, \tilde{\rho}_3 = \tilde{\rho}_3'^{-1}$. If $\rho_1 = \rho_1', \tilde{\rho}_3 = \tilde{\rho}_3'$ using $x(\rho) = x(\rho')$ we have $\tilde{\rho}_2 = \tilde{\rho}_2'$. If on the other hand $\rho_1 = \rho_1'^{-1}, \tilde{\rho}_3 = \tilde{\rho}_3'^{-1}$ by assuming $arg\, y(\rho) = arg\, y(\rho')$ it follows $\tilde{\rho}_2 = \tilde{\rho}_2'^{-1}$, i.e. $x(\rho)$ is real, so $\alpha \in \{0, \pi)$ and therefore $\rho_1 = \tilde{\rho}_3 = 1$ which contradicts $y(\rho) \in \mathbb{C}^*$. The cases $m_i \pm m_i' = \pm a_i$ are treaten similarly.

(ii) The spectral problem on $X_{\sigma 12} \times F$ is given in the coordinates (x, y, w, z) of $X_{\sigma 12}$ by

$$TS^{(0, a_2 y_0, a_3 z_0)}\psi_0 = \lambda w \psi_0, \qquad S^{(-a_1, 0, 0)}\psi_0 = x w^{-a_1}\psi_0,$$

$$S^{(0, a_2 y_0, a_3 z_0)}\psi_0 = w\psi_0, \qquad S^{(0, -a_2 a_3, a_2 a_3)}\psi_0 = z\psi_0.$$

By the construction of the vectorbundle the last three equations transform to

$$S^{(-a_1, 0, 0)}\psi_{12} = x w^{-a_1}\psi_{12}, \qquad S^{(0, a_2 y_0, a_3 z_0)}\psi_{12}, \qquad S^{(0, -a_2 a_3, a_2 a_3)}\psi_{12} = z\psi_{12}$$

The first equation gives (on $x = y = w = 0$), using $S^{(0, a_2 y_0, a_3 z_0)}\psi_{12} = \psi_{12}$,

$$(S^{(0, -1, 0)} + S^{(0, 0, -1)})\psi_{12} = 0,$$

i.e. $S^{(0, -1, 1)}\psi_{12} = -\psi_{12}$.

It follows $S^{(0, -a_2 a_3, a_2 a_3)}\psi_{12} = (-1)^{a_2 a_3}\psi_{12}$, which leads to $z = (-1)^{a_2 a_3}$. This means $\overline{F_\lambda(V)} \cap \mathbb{O}_{\sigma 12} =$ one point (P_{12}) with coordinates $x = y = w = 0$, $z = (-1)^{a_2 a_3}$.

(iii) On $X_{\sigma 1}, \overline{F_\lambda(V)}$ is given as the zero set of polynomial $P(u_1, u_1^{-1}, w_1, w_1^{-1}, v_1) = Q(u_1, u_1^{-1}, w_1, w_1^{-1}) + v_1 R(u_1, u_1^{-1}, w_1, w_1^{-1}, v_1)$, where the zero set of Q describes Q_1. So $\overline{F_\lambda(V)}$ is smooth on the smooth points of $Q_1 \subset (X_{\sigma 1} \cap \{v_1 = 0\})$.

Now we resolve the singular point P_{12} of type A_{2a_1-1}. Its coordinates are

$$x = y = w = 0, \qquad z = (-1)^{a_2 a_3}.$$

Blowing up this point in $\mathbb{C}^4$ a_1-times, the exceptional divisor is the transverse union of a_1 hyperplanes E_i $(i = 1, \ldots, a_1)$, where the E_i is the exceptional divisor of the ith blowing up.

PROPOSITION 3. *The blowing up of $\overline{F_\lambda(V)}$ at the point P_{12} intersects only the exceptional divisor E_{a_1}. The strict transform of $\overline{F_\lambda(V)}$ (on E_{a_1}) is a hyperelliptic curve of arithmetic genus $a_1 - 1$. The blowing up of $\overline{F_\lambda(V)}$ is smooth on all smooth points of this curve. Furthermore the curve is determined by the following one-dimensional spectral problem*

$$S^{(-a_1,0,0)}\psi = x_{a_1}\psi, \qquad S^{(0,a_2y_0,a_3z_0)}\psi = \psi, \qquad S^{(0,-1,1)}\psi = -\psi,$$

$$\begin{aligned} &-\psi(m-1,n,p) - \psi(m+1,n,p) \\ &+\frac{1}{a_2a_3}\left(\sum_{i=1}^{a_2}\sum_{j=1}^{a_3} V(m,i,j)\right)\psi(m,n,p) \\ &= \tilde{z}\psi(m,n,p) \end{aligned}$$

where the coordinates $\tilde{z}, x_{a_1}$ are defined by resolving the point P_{12}:

$$\begin{aligned} &w = \mu, \qquad x = \mu^{a_1}x_{a_1}, \qquad y = \mu^{a_1}y_{a_1}, \\ &(1 + (-1)^{a_2a_3-1}z) = a_2a_3(-1)^{a_2y_0}\mu(\tilde{z} - \lambda) \end{aligned}$$

(here $\mu = 0$ is the exceptional divisor E_{a_1}).

Due to the shift operators $S^{(0,a_2y_0,a_3z_0)}$ and $S^{(0,-1,1)}$ the curve on E_{a_1} is already determined by the values of $\psi(m,n,p)$ on the line spanned by the vector $(a_1, 0, 0)$. Therefore Proposition 3 and Proposition 2 prove the theorems in the introduction.

Proof. We first calculate the strict transform of $\overline{F_\lambda(V)}$ on E_{a_1}. Blowing up P_{12} a_1-times, we get the coordinates

$$\begin{aligned} &w = \mu, \qquad x = \mu^{a_1}x_{a_1}, \qquad y = \mu^{a_1}y_{a_1}, \\ &(1 + (-1)^{a_2a_3-1}z) = a_2a_3(-1)^{a_2y_0}\mu(\tilde{z} - \lambda) \end{aligned}$$

Denote by U the chart generated by the coordinates $(\mu, x_{a_1}, y_{a_1}, \tilde{z})$, i.e.

$$U = \{(\mu, x_{a_1}, y_{a_1}, \tilde{z}) \in \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C} \mid x_{a_1} y_{a_1} = 1\}.$$

Now $X_{\sigma 0} \cap U = X_{\sigma 0}$ and define the transition function for the vectorbundle Y by

$$\psi_0(m, n, p) = \mu^{n+p} \psi(m, n, p).$$

The spectral problem on $X_{\sigma 0} \times F$ is given in the coordinates of U by

$$TS^{(0, a_2 y_0, a_3 z_0)} \psi_0 = \mu \lambda \psi_0 \tag{1}$$

$$S^{(-a_1, 0, 0)} \psi_0 = x_{a_1} \psi_0, \qquad S^{(0, a_2 y_0, a_3 z_0)} \psi_0 = \mu \psi_0, \tag{2), (3}$$

$$\mu^{-1}\{1 + (-1)^{a_2 a_3 - 1} S^{(0, -a_2 a_3, a_2 a_3)}\} \psi_0 = a_2 a_3 (-1)^{a_2 y_0} (\tilde{z} - \lambda) \psi_0 \tag{4}$$

Using the transition function the equations (1), (2) and (3) transform to

$$\begin{aligned}
&- S^{(0, a_2 y_0 - 1, a_3 z_0)} \psi - S^{(0, a_2 y_0, a_3 z_0 - 1)} \psi \\
&+ \mu\{-S^{(-1, a_2 y_0, a_3 z_0)} \psi - S^{(1, a_2 y_0, a_3 z_0)} \psi + V S^{(0, a_2 y_0, a_3 z_0)} \psi\} \\
&- \mu^2\{S^{(0, a_2 y_0 + 1, a_3 z_0)} \psi + S^{(0, a_2 y_0, a_3 z_0 + 1)} \psi\} = \mu \lambda \psi
\end{aligned}$$

and

$$S^{(-a_1, 0, 0)} \psi = x_{a_1} \psi, \qquad S^{(0, a_2 y_0, a_3 z_0)} \psi = \psi.$$

Therefore on $E_{a_1} = \{\mu = 0\}$ we have

$$S^{(0, -1, 1)} \psi = -\psi, \qquad S^{(-a_1, 0, 0)} \psi = x_{a_1} \psi, \qquad S^{(0, a_2 y_0, a_3 z_0)} \psi = \psi.$$

To explore (4) observe that

$$1 + (-1)^{a_2 a_3 - 1} S^{(0, -a_2 a_3, a_2 a_3)} = \sum_{i=0}^{a_2 a_3 - 1} (-1)^i (S^{i(0, -1, 1)} + S^{(i+1)(0, -1, 1)})$$

On the other hand we have form (1)

$$\begin{aligned}
(S^{i(0, -1, 1)} + S^{(i+1)(0, -1, 1)}) \psi_0 &= -S^{(-1, -i, i+1)} \psi_0 - S^{(1, -i, i+1)} \psi_0 \\
&- S^{(0, -i, i+2)} \psi_0 - S^{(0, -i+1, i+1)} \psi_0 + (V(m, n-i, p+i+1) - \lambda) S^{(0, -i, i+1)} \psi_0
\end{aligned}$$

Thus

$$\mu^{-1}\{1+(-1)^{a_2a_3-1}S^{(0,-a_2a_3,a_2a_3)}\}\psi_0=\mu^{-1}\sum_{i=0}^{a_2a_3-1}(-1)^i\{-\mu^{n+p+1}(S^{(-1,-i,i+1)}\psi$$
$$+S^{(1,-i,i+1)}\psi)-\mu^{n+p+2}(S^{(0,-i,i+2)}\psi+S^{(0,-i+1,i+1)}\psi)$$
$$+\mu^{n+p+1}(V(m,n-i,p+1+i)-\lambda)S^{(0,-i,i+1)}\psi\}$$

So on $\mu=0$ (4) transforms to

$$\sum_{i=0}^{a_2a_3-1}(-1)^i\{-(S^{(-1,-i,i+1)}-S^{(1,-i,i+1)}+(V(m,n-i,p+1+i)$$
$$-\lambda)S^{(0,-i,i+1)}\psi\}=a_2a_3(-1)^{a_2y_0}(\tilde{z}-\lambda)\psi.$$

Since $S^{(0,-1,1)}\psi=-\psi$ we get

$$-a_2a_3S^{(-1,0,0)}\psi-a_2a_3S^{(1,0,0)}\psi-\lambda a_2a_3S^{(0,0,0)}\psi+\sum_{i=0}^{a_2a_3-1}V(m,n-i,p+i)\psi$$
$$=a_2a_3(-1)^{a_2y_0}(\tilde{z}-\lambda)S^{(0,0,-1)}\psi.$$

But $S^{(0,0,-1)}=S^{(0,-a_2y_0,-a_3z_0)}S^{-(0,-a_2y_0,a_2y_0)}$ and we have

$$-S^{(-1,0,0)}\psi-S^{(1,0,0)}\psi+\frac{1}{a_2a_3}\sum_{i=0}^{a_2a_3-1}V(m,n-i,p+i)\psi=\tilde{z}\psi.$$

Now a_2 and a_3 are relatively prime, therefore we get the desired spectral problem as in proposition 3.

Let now π_i be the ith blowing up of the point P_{12} and E_i the exceptional divisor. So we have

$$w=\mu,\qquad x=\mu^ix_i,\qquad y=\mu^iy_i,\qquad (1+(-1)^{a_2a_3-1}z)=a_2a_3(-1)^{a_2y_0}\mu(\tilde{z}-\lambda).$$

Let $U_i=\{(\mu,x_i,y_i,\tilde{z})\in\mathbb{C}^4\mid x_iy_i=\mu^{2a_1-2i}\}$ be the new chart. On $U_i\cap X_{\sigma 0}=X_{\sigma 0}$ define the transition function $\psi_0(m,n,p)=\mu^{n+p}\psi_i(m,n,p)$. The spectral problem on $X_{\sigma 0}\times F$ is given by the equations (1), (3), (4) and

$$S^{(-a_1,0,0)}S^{(a_1-i)(0,a_2y_0,a_3z_0)}\psi_0=x_i\psi_0$$
$$S^{(a_1,0,0)}S^{(a_1-i)(0,a_2y_0,a_3z_0)}\psi_0=y_i\psi_0$$

The last two equations give on the exceptional divisor $E_i = \{\mu = 0\}$
$x_i = y_i = 0 \quad \text{for } i \neq a_1$, i.e.

$$\overline{\pi^{-1}(\overline{F_\lambda(V)} - P_{12})} \cap E_i = (E_i)_{\text{singular}}$$

Denote by H_{12} the above hyperelliptic curve. Now $F_\lambda(V)_{\text{comp}}$ is smooth on the smooth points of $H_{12} - (H_{12} \cap Q_1 \cap Q_2)$ as in proposition 2. Observe that $Q_1 \subset X_{\sigma 12}$ lies in the plane $x = 0$, so by the blowing-ups Q_1 intersects H_{12} transversally at $x_{a_1} = 0$ (and similarly Q_2 intersects H_{12} at $x_{a_1} = y_{a_1}^{-1} = \infty$), i.e. on (see the introduction) a smooth point of H_{12}. This proves proposition 3.

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