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Intersection homology operations

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§1. Introduction

In this paper we construct Steenrod squares in intersection homology,

$$Sq^i : IH^m_{\bar{a}}(X; \mathbb{Z}/(2)) \to IH^{m+1}_{\bar{b}}(X; \mathbb{Z}/(2))$$

for any topological pseudomanifold X. Here, $\bar{a}$ and $\bar{b}$ are perversities ([GM1], [GM2]) with

$$b(c) \geq 2a(c) \text{ for each } c \geq 2.$$

These homomorphisms are natural with respect to normally nonsingular maps, and they agree with the usual Steenrod squares on the normalization of X when $\bar{a} = \bar{b} = \bar{0}$. They also satisfy a Cartan formula.

If X is an n-dimensional $\mathbb{Z}/(2)$-Witt space ([S], [GM2]) then the "middle" intersection homology group $IH^*_{\bar{m}}(X; \mathbb{Z}/(2))$ satisfies Poincaré duality. Thus the Steenrod square

$$Sq^i : IH^{n-i}_{\bar{m}}(X; \mathbb{Z}/(2)) \to H_0(X; \mathbb{Z}/(2)) \to \mathbb{Z}/(2)$$

may be used to define (in the usual way) a Wu class $Iv \in IH^*_{\bar{m}}(X; \mathbb{Z}/(2))$ and an intersection homology Whitney class $Iw = Sq(Iv)$.

For piecewise linear pseudomanifolds X, we give a combinatorial formula for this intersection homology Whitney class, and compare it with Sullivan's Whitney class for Euler spaces.

The intersection homology Whitney class Iw does not normally lift to intersection homology (even if X is a complex algebraic variety.) However the single characteristic number

$$I\chi(X; \mathbb{Z}/(2)) = Iw_n \cdot Iw_0 = \sum_i \text{rank } IH^i_{\bar{m}}(X; \mathbb{Z}/)2))$$

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determines the cobordism class of X in the Witt-space cobordism groups of P. Siegel ([S]).

The results in this paper on Steenrod operations and Wu classes may be considered as part of a program to describe ways in which the intersection homology groups of certain singular spaces behave like the ordinary homology groups of a nonsingular space ([CGM] §1). It remains as open question whether there is an intersection homology - analogue to the rational homotopy theory of Sullivan. For example, one would like to know when Massey triple products are defined in intersection homology and whether they always vanish on a (singular) projective algebraic variety (see [DGMS]).

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§2. Intersection homology sheaves

In this chapter we summarize basic material from [GM1], [GM2] and fix notation which will be used throughout this paper.

2.1. Let X denote an n-dimensional topological pseudomanifold, with singular set $\Sigma \subset X$. By *sheaf* we shall mean a sheaf of $\mathbb{Z}/2\mathbb{Z}$ modules on X.

Choose a topological stratification

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_{n-2} = \Sigma \subset X_n = X$$

by closed subsets X_i of dimension $\leq i$. ([GM1]), [GM]). Thus, each $x \in X_i - X_{i-1}$ has a fundamental neighborhood U_x which is homeomorphic (by a stratum preserving homeomorphism) to $\mathbb{R}^i \times \text{cone}(L)$ where L is the (topologically stratified) link of the stratum $X_i - X_{i-1}$.

For any perversity $\bar{a} = (a(2), a(3), a(4), \ldots)$ there is a bounded complex of injective sheaves $\mathbf{IC}^\cdot_{\bar{a}}$ which is constructible with respect to this stratification and is uniquely determined up to chain homotopy by the following conditions:

(a) $\mathbf{IC}^i_{\bar{a}} = 0$ for all $i < 0$

(b) $\mathbf{IC}^\cdot_{\bar{a}} \mid (X - \Sigma) \cong \mathbb{Z}/(2)_{X-\Sigma}$

(c) For all $c \geq 2$ and for any $x \in X - X_{n-c-1}$,
$\mathcal{H}^i(U_x; \mathbf{IC}^\cdot_{\bar{a}}) = 0$ whenever $i \geq a(c) + 1$.

(d) For all $c \geq 2$ and for any $x \in X - X_{n-c-1}$,
$\mathcal{H}_c^i(U_x; \mathbf{IC}_{\bar{a}}^{\cdot}) = 0$ whenever $i \leq n - c + a(c) + 1$.

(Here U_x denotes a fundamental neighborhood of x, of the type considered above. $\mathcal{H}^i$ denotes hypercohomology and $\mathcal{H}_c^i$ denotes hypercohomology with compact support.)

The cohomology groups of the complex of global sections,

$$\cdots \to \Gamma(X; \mathbf{IC}_{\bar{a}}^{i-1}) \to \Gamma(X; \mathbf{IC}_{\bar{a}}^{i}) \to \Gamma(X; \mathbf{IC}_{\bar{a}}^{i+1}) \to \cdots$$

are the intersection homology groups of X.

2.2 In this section we give an explicit construction of the sheaves $\mathbf{IC}_{\bar{a}}^{\cdot}$.

If $\mathbf{A}^{\cdot}$ is a complex of sheaves and $p \in \mathbb{Z}$, Deligne defines ([GM2]) the complexes $\tau_{\leq p}\mathbf{A}^{\cdot}$ and $\tau^{\leq p}\mathbf{A}^{\cdot}$ as follows:

$$(\tau_{\leq p}\mathbf{A})^j = \begin{cases} 0 & \text{for } j > p \\ \mathbf{ker}\, d & \text{for } j = p \\ \mathbf{A}^j & \text{for } j < p \end{cases}$$

$$(\tau^{\leq p}\mathbf{A})^j = \begin{cases} 0 & \text{for } j > p+1 \\ \mathbf{Im}\, d & \text{for } j = p+1 \\ \mathbf{A}^j & \text{for } j \leq p \end{cases}$$

Clearly, $\tau_{\leq p}\mathbf{A} \subset \tau^{\leq p}\mathbf{A}$ and this inclusion induces isomorphisms on cohomology.

Now let $\mathbf{I}^{\cdot}$ denote a fixed injective resolution of the constant sheaf $\mathbb{Z}/(2)$ over X. Let $\mathbf{I}_k^{\cdot}$ denote its restriction to the open set $U_k = X - X_{n-k}$. Define $\mathbf{A}_k^{\cdot}$ inductively by the rules

(a) $\mathbf{A}_2^{\cdot} = \mathbf{I}_2^{\cdot}$
(b) $\mathbf{A}_{k+1}^{\cdot} = (\tau_{\leq a(k)} i_{k*} \mathbf{A}_k^{\cdot}) \otimes \mathbf{I}_{k+1}^{\cdot}$

where $i_k : U_k \to U_{k-1}$ is the inclusion. Then $\mathbf{I}C_{\bar{a}}^{\cdot} = \mathbf{A}_{n+1}^{\cdot}$ is the intersection homology complex.

Remarks: 1. The tensor product with $\mathbf{I}_{k+1}^{\cdot}$ is formed in step (b) because it injectively resolves the sheaf $\tau_{\leq a(k)} i_* \mathbf{I}_k^{\cdot}$ in a canonical way.

2. The truncation functor $\tau^{\leq a(k)}$ could be used instead of $\tau_{\leq a(k)}$.

3. Indexing schemes: In this paper we will use "cohomology" notation for the intersection homology groups and sheaves. This means that $\mathbf{IC}_{\bar{a}}^{\cdot} \mid (X - \Sigma) \cong \mathbb{Z}/(2)$

in degree 0. The hypercohomology of the complex $IC^{\cdot}_{\bar{a}}$ is denoted

$$\mathscr{H}^i(X;\mathbf{IC}^{\cdot}_{\bar{a}}) = IH^i_{\bar{a}}(X)$$

If X is an n dimensional piecewise linear pseudomanifold then the intersection homology groups $IH^{\bar{a}}_i(X)$ defined geometrically in [GM1] may be identified with the hypercohomology with compact support

$$\mathscr{H}^{n-i}_c(X;\mathbf{IC}^{\cdot}_{\bar{a}})$$

as in [GM2]. For compact X we shall use both notations $IH^{\bar{a}}_i(X)$ and $IH^{n-i}_{\bar{a}}(X)$.

2.3. Multiplication on the nonsingular part,

$$\mathbb{Z}/(\mathbf{2})_{X-\Sigma}\otimes\mathbb{Z}/(\mathbf{2})_{X-\Sigma}\to\mathbb{Z}/(\mathbf{2})_{X-\Sigma}$$

extends in a unique way to a product structure

$$\mathbf{IC}^{\cdot}_{\bar{a}}\otimes\mathbf{IC}^{\cdot}_{\bar{b}}\to\mathbf{IC}^{\cdot}_{\bar{a}+\bar{b}}$$

whenever $\bar{a}+\bar{b}$ is a perversity. If $\bar{a}+\bar{b}=\bar{t}=(0,1,2,3\ldots)$ then this product is a Verdier dual pairing, i.e., the associated map

$$\mathbf{IC}^{\cdot}_{\bar{a}}\to R\,\mathbf{Hom}\,(\mathbf{IC}^{\cdot}_{\bar{b}},\mathbf{D}^{\cdot}_X)$$

is a quasi-isomorphism. (Here $\mathbf{D}^{\cdot}_X$ is the dualizing complex in the derived category of constructible sheaves of $\mathbb{Z}/(2)$-modules on X). In particular, for compact X,

$$IH^{\bar{a}}_i(X;\mathbb{Z}/(2))\cong\mathrm{Hom}\,(IH^{\bar{b}}_{n-i}(X;\mathbb{Z}/(2)),\mathbb{Z}/(2)).$$

§3. Steenrod squares

In this chapter we show how to define, for any perversity $\bar{a}$, mod 2 Steenrod operations

$$Sq^i: IH^i_{\bar{a}}(X;\mathbb{Z}/(2))\to IH^{i+j}_{\bar{b}}(X;\mathbb{Z}/(2))$$

where $b(c)\geq 2a(c)$ for each c. These operations are compatible with the usual Steenrod operations in cohomology.

The Steenrod squares do not usually define "operations" on intersection homology. This can be seen from a simple example: suppose X is a 6 dimensional piecewise linear pseudomanifold with an isolated singularity x_0 and suppose $v \in IH_4^{\bar{m}}(X)$ is a homology class which is represented by a P.L. cycle Z which contains x_0. (Here $\bar{m}$ is the "middle" perversity of [GM1].) Then $Sq^2(v) = v \cdot v$ is represented by $Z \cap Z'$ where Z' is a cycle transverse but homologous to Z ([MC1]). This means Z' may also contain the singular point $\{x_0\}$, so the intersection $Z \cap Z'$ does also. However $Z \cap Z'$ is a 2-dimensional cycle and in order that a 2 dimensional cycle represent an element of $IH_2^{\bar{m}}(X)$ it must not contain the stratum $\{x_0\}$. Thus, Sq^2 does not lift to an operation on $IH^{\bar{m}}(X)$ unless all the intersection homology classes of dimension 4 can be "moved away" from the singular point $\{x_0\}$, i.e., unless $IH_4(X, x - x_0) = 0$.

3.1. In this section we review the construction of Steenrod squares as found in Bredon [B] §20. Fix a topological pseudomanifold X, and let $\mathbf{I}^\cdot$ be an injective resolution of the constant sheaf $\mathbb{Z}/(2)$ on X. Bredon defines a sequence of sheaf morphisms

$$h_m : \bigoplus_{p+q=n} \mathbf{I}^p \otimes \mathbf{I}^q \to \mathbf{I}^{n-m}$$

which (do not commute with the differentials but) are determined "up to homotopy" (see §3.6) by the conditions

(a) h_0 is induced from multiplication

$$\mathbb{Z}/(2) \otimes \mathbb{Z}/(2) \to \mathbb{Z}/(2)$$

(b) $h_m + h_m\tau = dh_{m-1} + h_{m+1}d$

where $\tau : I^p \otimes I^q \to I^q \otimes I^p$ switches the factors.

The Steenrod squares are defined as follows: If U is any open subset of X, and $a \in \Gamma(U, \mathbf{I}^p)$ is a section such that $da = 0$ then

$$St^i(a) = h_{p-i}(a \otimes a) \in \Gamma(U, \mathbf{I}^{p+i})$$

is also a cycle. Furthermore, if $a = db$ then

$$St^i(a) = dh_{p-i}(b \otimes db) + dh_{p-i-1}(b \otimes b) + 2dh_{p-i-2}(b \otimes b).$$

Using relation (b) above, it follows easily that St^i induces a homomorphism,

$$Sq^i : H^p(U) \to H^{p+i}(U)$$

which is the Steenrod squaring operation.

3.2. The following construction is an important step in extending the Steenrod operations to the intersection homology sheaves.

Suppose $\tilde{\mathbf{A}}^{\cdot}$ and $\tilde{\mathbf{B}}^{\cdot}$ are complexes of sheaves on a pseudomanifold X, and suppose a sequence of sheaf morphisms

$$\tilde{J}_m : \bigoplus_{p+q=n} \tilde{\mathbf{A}}^p \otimes \tilde{\mathbf{A}}^q \to \tilde{\mathbf{B}}^{n-m}$$

have been defined for all integers m, such that

(a) $\tilde{J}_m = 0$ for all $m < 0$

(b) $d\tilde{J}_{m+1} + \tilde{J}_{m+1}d = \tilde{J}_m + \tilde{J}_m\tau$

where τ switches the factors. Let $\mathbf{I}^{\cdot}$ denote an injective resolution of the constant sheaf $\mathbb{Z}/(2)$. Let $\mathbf{A}^{\cdot} = \tilde{\mathbf{A}}^{\cdot} \otimes \mathbf{I}^{\cdot}$ and $\mathbf{B}^{\cdot} = \tilde{\mathbf{B}}^{\cdot} \otimes \mathbf{I}^{\cdot}$ denote the corresponding injective resolutions of $A^{\cdot}$ and $B^{\cdot}$.

DEFINITION. The sheaf morphism

$$J_m : \bigoplus_{p+q=n} \mathbf{A}^p \otimes \mathbf{A}^q \to \mathbf{B}^{n-m}$$

induced from $\{\tilde{J}_m\}$ is given by the following formula: For any open set $U \subset X$,

$$J_m((a \otimes u) \otimes (b \otimes v)) = \sum_{i=0}^{m} \tilde{J}_i \tau^{m-i}(a \otimes b) \otimes h_{m-i}\tau^{m-i}(u \otimes v)$$

whenever $a, b \in \Gamma(U, \tilde{\mathbf{A}}^{\cdot})$; $u, v \in \Gamma(U, \mathbf{I}^{\cdot})$ are homogeneous elements such that

$$\deg(a) + \deg(u) = p, \qquad \deg(b) + \deg(v) = q.$$

(Here, τ switches factors, and h_m are the sheaf morphisms of Bredon, see §3.1.)

PROPOSITION. *The maps J_m also satisfy the relations*

(a) $J_m = 0$ *for all* $m < 0$

(b) $dJ_{m+1} + J_{m+1}d = J_m + J_m\tau$

Proof. Direct calculation.

3.3. In this section we will restrict the maps h_m of §3.1 to the nonsingular part $X - \Sigma$ of X, and show that they naturally induce maps on the intersection homology sheaves.

Suppose $\bar{a}$ and $\bar{b}$ are perversities with $2a(c) \le b(c)$ for each c. Let $\mathbf{A}_k^{\cdot}$ and $\mathbf{B}_k^{\cdot}$ be the corresponding intersection homology complexes over the open sets $U_k = X - X_{n-k}$, as in §2.

PROPOSITION. *Suppose sheaf maps*

$$J_{m,k} : \mathbf{A}_k^{\cdot} \otimes \mathbf{A}_k^{\cdot} \to \mathbf{B}_k^{\cdot}[-m]$$

have been defined for each m such that

(a) $J_{m,k} = 0$ *whenever* $m < 0$

(b) $dJ_{m+1,k} + J_{m+1,k}d = J_{m,k} + J_{m,k}\tau$ (*where τ switches factors*).

Then each $J_{m,k}$ extends in a natural way to a sheaf map

$$J_{m,k+1} : \mathbf{A}_{k+1}^{\cdot} \otimes \mathbf{A}_{k+1}^{\cdot} \to \mathbf{B}_{k+1}^{\cdot}[-m]$$

which is defined over U_{k+1}, and these maps also satisfy the equations (a) *and* (b) *above* (*but with k replaced by $k+1$*).

Proof. Apply i_{k*} to each of the sheaves. We obtain a diagram

$$\begin{array}{ccc}
(\tau_{\le a(k)} i_{k*}\mathbf{A}_k^{\cdot}) \otimes (\tau_{\le a(k)} i_{k*}\mathbf{A}_k^{\cdot}) & \overset{\tilde{J}_{m,k+1}}{\dashrightarrow} & (\tau_{\le b(k)} i_{k*}\mathbf{B}_k^{\cdot})[-m] \\
\downarrow & \searrow^{\phi} & \downarrow \\
i_{k*}(\mathbf{A}_k^{\cdot}) \otimes i_{k*}(\mathbf{A}_k^{\cdot}) = i_{k*}(\mathbf{A}_k^{\cdot} \otimes \mathbf{A}_k^{\cdot}) & \underset{J_{m,k}}{\longrightarrow} & i_{k*}\mathbf{B}_k^{\cdot}[-m]
\end{array}$$

But $(\tau_{\le b(k)} i_{k*}\mathbf{B}_k^{\cdot})[-m]$ is a subcomplex of $i_{k*}\mathbf{B}_k^{\cdot}[-m]$, and the image of ϕ lies in this subcomplex. (This is obvious except when $m = 0$. But h_0 is a chain map so it takes $\ker(d) \otimes \ker(d)$ to $\ker(d)$.) Thus we have found sheaf morphisms

$$\tilde{J}_{m,k+1} : \tilde{\mathbf{A}}_{k+1}^{\cdot} \otimes \tilde{\mathbf{A}}_{k+1}^{\cdot} \to \tilde{\mathbf{B}}_{k+1}^{\cdot}[-m]$$

satisfying (a) and (b) above, where

$$\tilde{A}^{\cdot}_{k+1} = \tau_{\leq a(k)} i_{k*} \mathbf{A}^{\cdot}_k \quad \text{and} \quad \tilde{\mathbf{B}}^{\cdot}_{k+1} = \tau_{\leq b(k)} i_{k*} \mathbf{B}^{\cdot}_k.$$

The construction of §3.2 now gives canonical extensions of the $\tilde{J}_{m,k+1}$ to the injective resolutions,

$$J_{m,k+1} : \mathbf{A}^{\cdot}_{k+1} \otimes \mathbf{A}^{\cdot}_{k+1} \to \mathbf{B}^{\cdot}_{k+1}[-m]$$

as desired.

COROLLARY. *If* $2\bar{a}(c) \leq \bar{b}(c)$ *for all c, then the maps* h_m *defined by Bredon have canonical extensions*

$$J_m : \mathbf{IC}^{\cdot}_{\bar{a}} \otimes \mathbf{IC}_{\bar{a}} \to \mathbf{IC}^{\cdot}_{\bar{b}}[-m]$$

such that

(a) $J_m = 0$ *for all* $m < 0$

(b) $J_{m+1} d + dJ_{m+1} = J_m + J_m \tau$

(c) $J_m \mid (X - \Sigma) = h_m \mid (X - \Sigma)$.

3.4. Suppose $\bar{a}$ and $\bar{b}$ are perversities such that

$$2a(c) \leq b(c) \text{ for each } c.$$

We define Steenrod operations for any open set U X,

$$Sq^r : IH^s_{\bar{a}}(U) \to IH^{s+r}_{\bar{b}}(U)$$

as follows: if $a \in \Gamma(U, \mathbf{IC}^{\cdot}_{\bar{a}})$ let

$$St^r(a) = h_{s-r}(a \otimes a).$$

The same calculation as §3.1 shows that St^r induces a homomorphism Sq^r on cohomology.

Remarks. 1. Suppose $z \in IH^s_{\bar{a}}(X; \mathbb{Z}/(2))$. If $r > s$ then $Sq^r(z) = 0$. If $r = s$ then $Sq^r(z) = z \cdot z \in IH^{2s}_{2\bar{a}}(X; \mathbb{Z}/(2))$.

2. The method of [GM2] §4 can be used to show the homomorphism $Sq^r : IH^s_{\bar{a}}(X) \to IH^{r+s}_{\bar{b}}(X)$ is topologically invariant and does not depend on the choice of stratification of X.

3.5. It is easy to see from the method of §3.3 that J_m is defined naturally as a morphism

$$\mathbf{IC}^{\cdot}_{\bar{a}} \otimes \mathbf{IC}^{\cdot}_{\bar{a}} \to \mathbf{IC}^{\cdot}_{\bar{b}}$$

where $b(k) = 2a(k) - m$ for each k. (One must replace the complex $\mathbf{B}^{\cdot}_{k+1}$ by the quasi-isomorphic complex $\tau^{\geq b(k)} i_{k*}\mathbf{B}^{\cdot}_k$ in the proof of Prop. 3.3.)

Problem. Can one use this fact to (a) lift the Steenrod squares

$$Sq^r : IH^s_{\bar{a}} \to IH^{s+r}_{\bar{b}}$$

to a perversity $\bar{b} < 2\bar{a}$ and to (b) lift the corresponding Whitney classes of §5.2 to intersection homology?

Now suppose $\bar{a} \leq \bar{b}$ are perversities, and X is locally $(\bar{a}, \bar{b})$-acyclic, i.e.,

$$IH^{a(k+1)}_{\bar{a}}(L) = IH^{a(k+2)}_{\bar{a}}(L) = \cdots = IH^{b(k)}_{\bar{a}}(L) = 0,$$

whenever L is the link of a codimension k stratum. This implies that the natural homomorphism

$$IH^*_{\bar{a}}(X) \to IH^*_{\bar{b}}(X)$$

is an isomorphism ([GM2] §5.5). For which perversities $\bar{a} \leq \bar{b}$ is it possible to multiply the Whitney classes of a locally $(\bar{a}, \bar{b})$-acyclic space X, and obtain cobordism invariant characteristic numbers?

3.6. In this section we show that the maps J_m of §3.3 are essentially unique.

PROPOSITION. *Let $\bar{a}$ and $\bar{b}$ be perversities such that $2a(k) \leq b(k)$ for all k. Suppose $\mathbf{A}^{\cdot}$ and $\mathbf{B}^{\cdot}$ are complexes of injective sheaves which are quasi-isomorphic to $\mathbf{IC}^{\cdot}_{\bar{a}}$ and $\mathbf{IC}^{\cdot}_{\bar{b}}$ respectively. Suppose $K_m : \mathbf{A}^{\cdot} \otimes \mathbf{A}^{\cdot} \to \mathbf{B}^{\cdot}[-m]$ is a system of morphisms such that*

(a) $K_m = 0$ *for all* $m < 0$

(b) $dK_{m+1} + K_{m+1}d = K_m + K_m\tau$

(c) $K_0 \mid (X-\Sigma)$ *induces the multiplication map on the cohomology sheaves over the nonsingular part* $X-\Sigma$ *of* X

$$K_0 \mid (X-\Sigma): \mathbb{Z}_2 \otimes \mathbb{Z}_2 \to \mathbb{Z}_2.$$

Suppose $J_m : \mathbf{A}^\cdot \otimes \mathbf{A}^\cdot \to \mathbf{B}^\cdot[-m]$ *is another system of morphisms which also satisfy* (a), (b), *and* (c). *Then there exists a system of morphisms*

$$D_m : \mathbf{A}^\cdot \otimes \mathbf{A}^\cdot \to \mathbf{B}^\cdot[-m]$$

such that

$$J_m - H_m = D_{m+1}d + dD_{m+1} + D_m + D_m\tau$$

(*Consequently, if* ξ *is a section of* $\mathbf{A}^\cdot$ *such that* $d\xi = 0$ *then* $J_m(\xi \otimes \xi) - H_m(\xi \otimes \xi) = dD_{m+1}(\xi \otimes \xi)$ *so* $Sq^*(\xi)$ *is independent of choices.*)

Proof. First we show that J_0 and K_0 are chain homotopic. The multiplication on the nonsingular part $X-\Sigma$ has a unique lift in $D^b(X)$ to a morphism

$$\phi : \mathbf{IC}^\cdot_{\bar{a}} \otimes \mathbf{IC}^\cdot_{\bar{a}} \to \mathbf{IC}^\cdot_{\bar{b}}$$

by [GM2] §5.1 and §1.15. Since $\mathbf{A}^\cdot$ and $\mathbf{B}^\cdot$ are injective, they are homotopy equivalent to $\mathbf{IC}_{\bar{a}}$ and $\mathbf{IC}_{\bar{b}}$ respectively. The morphism ϕ then corresponds to a unique homotopy class of maps from $\mathbf{A}^\cdot \otimes A^\cdot \to \mathbf{B}^\cdot$. But J_0 and K_0 are both in this homotopy class.

We now follow Bredon [B] §20.7. Let D_1 be a homotopy between J_0 and K_0. Thus

$$(J_0 - K_0)(1+\tau) = D_1 d(1+\tau) + dD_1(1+\tau)$$

or

$$(J_1 - K_1 - D_1(1+\tau))d + d(J_1 - K_1 - D_1(1+\tau)) = 0.$$

Thus, $J_1 - K_1 - D_1(1+\tau)$ is a chain map and gives an element of $\mathrm{Hom}_{D^b(X)}(\mathbf{A}^\cdot \otimes \mathbf{A}^\cdot, B^\cdot[-1])$. The same argument as [GM2] §1.15, §5.1 shows that this element is determined by its action on the cohomology sheaves over the nonsingular part of X. But this action is 0. So $H_1 - K_1 - D_1(1+\tau)$ is homotopic to 0 by some homotopy D_2. Continuing in this way the maps D_m can be defined inductively.

3.7. In this section we show the Steenrod squares are compatible with the canonical maps between intersection homology groups with different perversities.

PROPOSITION. *Suppose $\bar{a} \leq \bar{c}$ and $\bar{b} \leq \bar{d}$ are perversities such that $2\bar{a}(k) \leq b(k)$ and $2c(k) \leq d(k)$ for each k. Then the following diagram commutes:*

$$\begin{array}{ccc} IH^s_{\bar{a}}(X) & \xrightarrow{\beta} & IH^s_{\bar{c}}(X) \\ \downarrow{\scriptstyle Sq^r} & & \downarrow{\scriptstyle Sq^r} \\ IH^{s+r}_{\bar{b}}(X) & \xrightarrow{\beta} & IH^{s+r}_{\bar{d}}(X) \end{array}$$

Furthermore, if $\bar{a} = \bar{b} = \bar{0}$ then $Sq^r : IH^s_{\bar{0}}(X) \to IH^{s+r}_{\bar{0}}(X)$ coincides with the usual Steenrod square on the (ordinary) cohomology of the normalization of X.

Proof. Let $\mathbf{A}^\cdot_k$, $\mathbf{B}^\cdot_k$, $\mathbf{C}^\cdot_k$ and $\mathbf{D}^\cdot_k$ denote the corresponding complexes of sheaves on the open set U_k (see §2). One checks by induction that the following diagram of sheaf maps commutes:

$$\begin{array}{ccc} \mathbf{A}^\cdot_k \otimes \mathbf{A}^\cdot_k & \xrightarrow{\beta\otimes\beta} & \mathbf{C}^\cdot_k \otimes \mathbf{C}^\cdot_k \\ {\scriptstyle J_{m,k}}\downarrow & & {\scriptstyle J_{m,k}}\downarrow \\ \mathbf{B}^\cdot_k[-m] & \xrightarrow{\beta} & \mathbf{D}^\cdot_k[-n] \end{array}$$

The case $k = 2$ is trivial. The maps β are inclusions of complexes, so the inductive hypothesis is easily verified.

Now suppose that X is normal and $\bar{a} = \bar{b} = \bar{0}$. The injective complexes $\mathbf{I}^\cdot$ and $\mathbf{IC}^\cdot_{\bar{a}}$ are quasi isomorphic. Thus there is a homotopy equivalence $\phi : \mathbf{I}^\cdot \to \mathbf{IC}^\cdot_{\bar{a}}$ and a homotopy inverse $\psi : \mathbf{IC}_{\bar{a}} \to \mathbf{I}^\cdot$. Apply the uniqueness result (§3.6) to the systems of morphisms $\{J_m\}$ (from §3.3) and $\{\phi h_m \psi\}$. We conclude that they determine the same Steenrod squares.

3.8. In this paragraph we show that the Steenrod squares satisfy a Cartan formula.

PROPOSITION. *Suppose $\bar{a}$ and $\bar{b}$ are perversities such that $b(k) \geq 2a(k)$ for each k. Suppose $\xi \in H^t(X)$ and $\eta \in IH^s_{\bar{a}}(X)$. Then the following equality holds in $IH^{r+s+t}_{\bar{b}}(X)$.*

$$Sq^r(\xi \cdot \eta) = \sum_j Sq^j(\xi) \cdot Sq^{r-j}(\eta).$$

Proof. The proof is similar to [B] §20.11.

Consider the family of morphisms of sheaves

$$K_m : (\mathbf{I}^\cdot \otimes \mathbf{IC}^\cdot_{\bar{a}}) \otimes (\mathbf{I}^\cdot \otimes \mathbf{IC}^\cdot_{\bar{a}}) \to \mathbf{I}^\cdot \otimes \mathbf{IC}^\cdot_{\bar{b}}$$

which assigns to a homogeneous section $u \otimes a \otimes v \otimes b$ the section

$$K_m(u \otimes a \otimes v \otimes b) = \sum_{i=0}^{m} h_i \tau^i (u \otimes v) \otimes J_{m-i} \tau^{m-i}(a \otimes b).$$

A direct calculation shows that

$$dK_{m+1} + K_{m+1} d = K_m + K_m \tau$$

and that K_0 induces the multiplication map on the cohomology sheaves over the nonsingular part $X - \Sigma$ of X.

Let $\phi : \mathbf{I}^\cdot \otimes \mathbf{IC}^\cdot_{\bar{a}} \to \mathbf{IC}^\cdot_{\bar{a}}$ be the quasi-isomorphism which is induced from multiplication on the nonsingular part of X (and which induces the product $H^* \otimes IH^*_{\bar{a}} \to IH^*_{\bar{a}}$). If we apply the uniqueness result (§3.6) to the systems of morphisms, $J_m \circ (\phi \otimes \phi)$ and $\phi \circ K_m$, we obtain morphisms

$$D_m : (\mathbf{I}^\cdot \otimes \mathbf{IC}^\cdot_{\bar{a}}) \otimes (\mathbf{I}^\cdot \otimes \mathbf{IC}^\cdot_{\bar{a}}) \to \mathbf{IC}^\cdot_{\bar{b}}$$

such that

$$J_m \circ \phi \otimes \phi - \phi \circ K_m = D_{m+1} d + d D_{m+1} + D_m + D_m \tau$$

Now suppose u and a are sections of $\mathbf{I}^t$ and $\mathbf{IC}^s_{\bar{a}}$ respectively, and that $du = 0$ and $da = 0$. Then

$$\begin{aligned} Sq^r([u] \cdot [a]) &= [J_{s+t-r}(\phi(u \otimes a) \otimes \phi(u \otimes a))] \\ &= \left[\phi \sum_{i=0}^{s+t-r} h_i (u \otimes u) \otimes J_{s+t-r-i}(a \otimes a)\right] \\ &\quad + [dD_{s+t-r+1}(u \otimes a \otimes u \otimes a)] \\ &= \sum_{j=0}^{r} [Sq^j(u)] \cdot [Sq^{r-j}(a)] \end{aligned}$$

where $[a]$ denotes the homology class represented by the section a.

§4. Open questions on the geometry of Steenrod operations

4.1. A homology operation which doubles perversity can be constructed using the geometric technique outlined by McCrory [MC2] §6, i.e., by dualizing the construction in [SE]VII.1. Does this agree with the operations Sq^i defined in §3? An investigation of this question might lead one to study a Smith theory of involutions for the intersection homology groups.

4.2. It would be interesting to study the relationship between the operations Sq^r and the "branch point" operations of [MC2] and [HMC]. Intuitively, $Sq^*(\xi)$ represents the Whitney class of the "normal bundle" of a cycle ξ in a space X. (It is precisely this when ξ and X are manifolds.) The "branch point operation" $\bar{S}^*(\xi)$ represents the Whitney class of the "inverse tangent bundle" of ξ. One might hope for a Whitney duality formula relating these operations.

4.3. The following question is due to R. MacPherson:

Steenrod operations (in ordinary cohomology) arise as an obstruction to finding a cochain-level representation of the cup product which is both commutative and everywhere defined. If we take an everywhere defined product (as in sheaf theory, or by using front and back faces of simplices in the singular theory) then it fails to be commutative, and the amount by which it fails is precisely the Steenrod square. If instead we take a commutative product on the cochain level (as in the geometric intersection of *transverse* cochains [G], [GM1]) then it fails to be everywhere defined. Is it possible to use this second choice of product to give a geometric construction of the Steenrod operations in intersection homology, as the amount by which the product fails to be globally defined?

§5. Witt spaces and Wu classes

5.1. Throughout this chapter we shall assume X is a locally compact n-dimensional piecewise linear pseudomanifold.

DEFINITION. [S], [GM2] X is a $\mathbb{Z}/(2)$-Witt space if, for some (and hence for every) stratification of X, and for every stratum of *odd* codimension c in that stratification,

$$IH^l_{\bar{m}}(L;\mathbb{Z}/(2))=0$$

where L is the link of that stratum and $c=2l+1$.

Remark. It follows ([S]) that the natural map

$$IH^*_{\bar{m}}(X;\mathbb{Z}/(2)) \to IH^*_{\bar{n}}(X;\mathbb{Z}/(2))$$

is an isomorphism, so $IH^{\bar{m}}_*(X;\mathbb{Z}/(2))$ is self-dual.

For the rest of this chapter IH^* will be used to denote the intersection homology with middle perversity, $IH^*_{\bar{m}}$.

DEFINITION. A Witt space with boundary $(X, \partial X)$ is a compact pseudomanifold X with collared boundary ∂X such that both $X-\partial X$ and ∂X are $\mathbb{Z}/(2)$-Witt spaces. We shall say two compact $\mathbb{Z}/(2)$-Witt spaces X_1 and X_2 are cobordant if there is a $\mathbb{Z}/(2)$ Witt space with boundary $(X, \partial X)$ such that $\partial X = X_1 \cup X_2$. The technique of [S] gives:

PROPOSITION. *The cobordism group of* n*-dimensional* $\mathbb{Z}/(2)$*-Witt spaces is*

$$\Omega^n_{\text{Witt}} = \begin{cases} 0 \text{ for n odd} \\ \mathbb{Z}/(2) \text{ for n even.} \end{cases}$$

The cobordism class of a compact n*-dimensional Witt space* X *is determined by the single characteristic number*

$$I\chi(X;\mathbb{Z}/(2)) \equiv \sum_{i=0}^{n} \text{rank } IH^i(X;\mathbb{Z}/(2)) \qquad (\text{mod } 2)$$

Remark. The cobordism groups of rational-Witt spaces were calculated [S] to coincide with the higher Mischenko–Witt groups of $\mathbb{Q}$, [R] [Mis].

Remark. It is interesting to compare the $\mathbb{Z}/(2)$ – Witt space cobordism groups to the $\mathbb{Z}/(2)$ – Euler space cobordism groups of Akin and Sullivan [A]. The $\mathbb{Z}/(2)$-Euler space cobordism class of an Euler space X is completely determined by the (usual) mod 2 Euler characteristic of X. McCrory showed [MC3] that each Whitney class defines a homology operation in Euler space bordism theory. We do not know whether there is an analogous operation in Witt-space bordism theory.

5.2. In this section we define Wu classes in intersection homology and Whitney classes in ordinary homology for $\mathbb{Z}/(2)$-Witt spaces, using the original method of Wu. We will allow the n-dimensional Witt space X to be noncompact in this section, and use $IH^*_c(X)$ to denote the intersection homology with compact supports.

Let $\alpha : IH_c^*(X) \to \mathbb{Z}/(2)$ denote the augmentation, i.e., $\alpha(\xi)=0$ unless $\xi \in IH_c^n(X)$ and in that case $\alpha(\xi)$ is the number of points in any cycle representation of ξ. This augmentation is defined for any perversity.

DEFINITION. The intersection homology Wu class, $Iv^* \in IH^*(X)$ is the unique class such that, for all $\xi \in IH_c^*(X)$ the following formula holds:

$$\alpha(Sq(\xi)) = \alpha(Iv^* \cdot \xi)$$

where

$$Sq = 1 + Sq^1 + Sq^2 + \cdots$$

Following Wu we define the intersection homology Whitney class to be

$$IW(X) = Sq(Iv^*) \in H_{\bar{t}}^*(X) = H_{n-*}^{BM}(X)$$

The Whitney class is an element of the (Borel–Moore) homology of X with closed supports. If X is compact we shall write $IW_i(X)$ for the component of $IW(X)$ in $H_i(X)$.

Remarks. 1. $Iv^*(X)$ and $Iw(X)$ are topological invariant of X since the squaring operations on the intersection homology sheaves are topologically invariant.

2. $Iv^j(X)=0$ for all $j > n/2$.

3. If X is a $\mathbb{Z}/(2)$-homology manifold then $Iv^*(X)$ and $IW(X)$ agree with the usual Wu and Whitney classes.

4. $Iw(X)$ does not necessarily lift to $IH^*(X)$, even if X is a complex algebraic variety. For example take X to be the Thom space of the negative line bundle $E \to \mathbb{CP}^4$ whose first chern class is -2. Then $IW_2(X)$ is nonzero in $H_2(X)$. However, the map $IH^8(X) \to H_2(X)$ is zero. (see also §5.5)

5.3. In this section we calculate the pullback of the intersection homology Whitney class under a normally nonsingular map.

THEOREM. *Suppose X and Y are $\mathbb{Z}/(2)$-Witt spaces, and $f: X \to Y$ is a normally nonsingular map* ([FM], [G], [GM2]) *with normal bundle ν. Then the following equation holds in $IH^*(X)$:*

$$f^*(IW(Y)) = W(\nu) \cdot IW(X)$$

where $W(\nu)$ is the Whitney class (in $H^*(X)$) of the normal bundle ν.

Proof. We will prove this formula for compact X in two special cases,

Case 1. F is a normally nonsingular inclusion

Case 2. f is a projection $M \times Y \to Y$ where M is a smooth manifold.

The general case follows from these because any normally nonsingular map can be factored into a composition of these two types.

Case 1. By restricting to a tubular neighborhood of X in Y, we may suppose that f is the inclusion of the zero section X into a vector bundle $\pi : Y \to X$ (which, therefore, coincides with ν). Then it suffices to show that

$$IW(Y) = \pi^*(W(\nu)) \cdot \pi^*(IW(X))$$

(Here $IW(Y)$ is an element of the closed support homology of Y or, equivalently, of the relative homology $H_*(Y, Y-X)$.

Let $\alpha : IH^*(X) \to \mathbb{Z}/(2)$ be the augmentation.

LEMMA 1. *Define $R \in IH^*(X)$ to be the unique class which satisfies the following equation for all $\beta \in IH^*(X)$,*

$$\alpha(W(\nu) \cdot Sq(\beta)) = \alpha(\beta \cdot R)$$

Then $\pi^*(R)$ is the Wu class of Y.

Proof. Let α' denote the augmentation on $IH_c^*(Y)$. Let $\phi : IH^*(X) \to IH_c^*(Y)$ be the Thom isomorphism, with Thom class $U = \phi(1)$. For any $\beta' \in IH_c^*(Y)$ we can write $\beta' = \phi(\beta) = \pi^*(\beta) \cdot U$ for some $\beta \in IH^*(X)$. Therefore,

$$\begin{aligned} \alpha'(Sq(\beta')) &= \alpha'(\pi^* Sq(\beta) \cdot Sq(U)) \\ &= \alpha'(\pi^* Sq(\beta) \cdot \phi(W(\nu)) \\ &= \alpha'(Sq(\beta) \cdot W(\nu)) \\ &= \alpha'(\beta \cdot R) \\ &= \alpha'(\beta' \cdot \pi^*(R)). \qquad \text{Q.E.D.} \end{aligned}$$

It follows that $IW(Y) = \pi^* Sq(R)$, so we must show that the following equation holds on $IH^*(X)$:

$$Sq(R) = W(\nu) \cdot Sq(Iv(X)).$$

LEMMA 2. *The nondegenerate bilinear pairing*

$$H^*(X)\times H_*(X)\to \mathbb{Z}/(2)$$

(which is given by $\langle a, b\rangle = \alpha(a\cdot b)$) is compatible with the nondegenerate bilinear pairing

$$IH^*(X)\times IH^*(X)\to \mathbb{Z}/(2)$$

(which is given by $\langle a, b\rangle = \alpha(a\cdot b)$) with respect to the canonical maps $H^*(X)\xrightarrow{A} IH^*(X)\xrightarrow{B} H_*(X)$.

Proof. Obvious.

Remark. It follows that $\langle A(a), b\rangle = \langle a, B(b)\rangle$ for any $a\in H^*(X)$ and $b\in IH^*(X)$. Thus, A and B are adjoints with respect to these inner products.

We may unambiguously define the adjoint

$$Sq^*: H^*(X)\to IH^*(X)$$

by the formula

$$\langle b, Sq(a)\rangle = \langle Sq^*(b), a\rangle$$

for any $b\in H^*(X)$ and $a\in IH^*(X)$.

LEMMA 3. $Sq(R) = W(\nu)\cdot Sq(Ic(X))$

Proof. We shall show that for any $\beta\in H^*(X)$, the following formula holds:

$$\langle \beta, Sq(R)\rangle = \langle \beta, W(\nu)\cdot Sq(Iv(X))\rangle.$$

We shall use $\bar{W}$ to denote the cohomology class $Sq^{-1}(W(\nu))$. This is well defined because Sq is invertible when considered as an operation on ordinary cohomology. Now calculate

$$\begin{aligned}\langle \beta, Sq(R)\rangle &= \langle \beta, Sq\,Sq^*W(\nu)\rangle \text{ since } R = Sq^*W(\nu)\\ &= \langle Sq\,Sq^*\beta, Sq\bar{W}\rangle \text{ since } W = Sq(\bar{W})\\ &= \alpha(Sq(Sq^*(\beta)\cdot \bar{W})) \text{ by Cartan formula}\\ &= \alpha(Sq^*(\beta)\cdot \bar{W}\cdot Iv(X))\\ &= \langle Sq^*(\beta), \bar{W}\cdot Iv(X)\rangle\\ &= \langle \beta, w(\nu)\cdot Sq(Iv(X))\rangle \text{ as desired.}\end{aligned}$$

This concludes the proof of Case 1.

Case 2. Suppose $f: M\times Y \to Y$ is the projection to the second factor, where M is a smooth manifold. Then $\nu^{-1} = \pi^*(TM)$ so we must show $IW(M\times Y) = f^*(IW(Y))\cdot \pi^*(W(M))$ where $\pi: M\times Y\to M$ is the projection to the first factor. From the Kunneth formula for middle intersection homology ([GM2]) and the Cartan formula for Sq, it follows that the intersection homology Wu class of $M\times Y$ is the product of the Wu classes of M and Y and, therefore, (by the Cartan formula again) the intersection homology Whitney class is the product of the Whitney classes of M and of Y. This completes the proof in Case 2.

5.4. In this section we give a combinatorial formula for the intersection homology Whitney class of a compact piecewise linear pseudomanifold.

LEMMA. *Suppose X is a compact $\mathbb{Z}/(2)$-Witt space. Then*

$$IW_0(X) \equiv I\chi(X;\mathbb{Z}/(2)) = \sum_i \operatorname{rank} IH^i(X;\mathbb{Z}/(2) \qquad (\operatorname{mod} 2).$$

Proof. If $n = \dim(X)$ is odd then $I\chi(X) = 0$ by Poincaré duality, while $IW_0(X) = 0$ by remark (2) above. If $\dim(X)$ is even (say $n = 2l$) then $IW_0(X) = Iv^l\cdot Iv^l$ and $I\chi(X) \equiv \operatorname{rank} IH^l(X;\mathbb{Z}/(2))$ (mod 2). By Milnor [Mil], $IH^l(X;\mathbb{Z}/(2))$ breaks into an orthogonal direct sum

$$\langle e_i\rangle \oplus \langle e_2\rangle \oplus \cdots \oplus \langle e_r\rangle \oplus H$$

where $\langle e_i\rangle$ is a one dimensional subspace generated by a vector e_i such that $e_i^2 = 1$, and where H is hyperbolic. (i.e., $h\cdot h = 0$ for all $h\in H$.) This means that H is even dimensional, and $Iv^l = e_1 + e_2 + \cdots + e_r$. Therefore, $IW_0 = e_1^2 + e_2^2 + \cdots + e_r^2 \equiv r \equiv \operatorname{rank}(IH^l(X))$ (mod 2) as desired.

THEOREM. *Suppose X is a compact n-dimensional $\mathbb{Z}/(2)$-Witt space. Then $IW(X)$ equals the Whitney class $W_*(f)$ which corresponds to the constructible function $f(x) = I\chi(X, X-x) = \sum_{i=0}^n \operatorname{rank} IH_i^{\bar{m}}(X, X-x;\mathbb{Z}/(2))$ (as defined by Fulton and MacPherson* [FM]).

Proof. The proof is almost the same as [FM] §6.3.2 which was due originally to R. Thom [T].

First we check that $IW_0(X) = W_0(f)$, i.e., that both Whitney classes have the same Euler characteristic. Consider the spectral sequence for $IH^*(X)$ which is associated to the complex of sheaves $\mathbf{IC}^\cdot$ ([GM2]). We have $E_1^{p,q} = C^p(X;\mathbf{IH}^q)$

where $\mathbf{IH}^q$ represents the local intersection homology sheaf. By the preceding lemma, $IW_0(X) = I\chi(X) = \chi(E_1^{p,q}) = \sum_{p,q} \text{rank } C^p(X; \mathbf{IH}^q)$ (if these are all finite dimensional). Choose any triangulation of X to compute these cochain groups. Each simplex σ will contribute a tern

$$\sum_q \text{rank } IH^q(X, X - \hat{\sigma}) = f(\hat{\sigma})$$

where $\hat{\sigma}$ is the barycentre of t. Therefore,

$$IW_0(X) = \sum_\sigma f(\hat{\sigma})$$

which is the formula for $W_0(f)$ in [FM] §6.1.1.

Now we shall show, for each cohomology class $\xi \in H^*(X; \mathbb{Z}/(2))$ that $\langle \xi, IW(X)\rangle = \langle \xi, W_*(f)\rangle$. By cobordism theory, ξ is the Thom class of some normally nonsingular map $g: Y \to X$ with some virtual normal bundle ν. Therefore,

$$\begin{aligned} \langle \xi, IW(X)\rangle &= \langle g^*(IW(X)), [Y]\rangle \\ &= \langle w(\nu) \cdot IW(Y), [Y]\rangle && \text{by §5.3} \\ &= \langle w(\nu) \cdot W(g^*(f)), [Y]\rangle && \text{by induction} \\ &= \langle g^*(W(f)), [Y]\rangle && \text{by [FM]} \\ &= \langle \xi, W(f)\rangle && \text{Q.E.D.} \end{aligned}$$

COROLLARY 1. *If X is a complex algebraic variety then $IW_j(X) = 0$ whenever j is odd.*

Proof. Let f be the constructible function

$$f(x) = I\chi(X, X - x).$$

Then

$$\begin{aligned} IW(X) &= W_*(f) \\ &= C_*(f) \pmod 2 \end{aligned}$$

where C_* is the homology chern class of MacPherson [M].

COROLLARY 2. *Let K' be the first barycentric subdivision of any triangulation of a compact Witt space X. Then $IW_j(X)$ is represented by the chain which is* *the sum of all the j-simplices $\sigma \in K'$ such that $I\chi(X, X-x)=1$ for any point x in the interior of σ.*

COROLLARY 3. *Suppose a compact Witt space X can be stratified with even dimensional strata $\{S_\alpha\}$. Then there exist numbers $\{F_\alpha\}$ and $\{G_\alpha\}$ (in $\mathbb{Z}/(2)$) such that*

$$W_*(X)=\sum_\alpha F_\alpha IW(\bar{S}_\alpha)$$

and

$$IW(X)=\sum_\alpha G_\alpha W_*(\bar{S}_\alpha).$$

(*Here W_* denotes the Sullivan Whitney class* [Su] *of a* mod 2 *Euler space.*)

Proof. For each stratum S_α consider the $\mathbb{Z}/(2)$-valued constructible functions f_α and g_α which are supported on the closure $\bar{S}_\alpha$ and are defined by

$$f_\alpha(x)=I\chi(\bar{S}_\alpha, \bar{S}_\alpha - x) \qquad (\text{mod } 2)$$
$$g_\alpha(x)=\chi(\bar{S}_\alpha, \bar{S}_\alpha - x) \qquad (\text{mod } 2)$$

for any $x \in \bar{S}_\alpha$. If $x \in S_\alpha$ then $f_\alpha(x)=g_\alpha(x)=1$. Therefore, $\{f_\alpha\}$ and $\{g_\alpha\}$ are both bases for the space of $\mathbb{Z}/(2)$-valued functions on X which are constructible with respect to the stratification $\{S_\alpha\}$. Therefore, we can find numbers F_α and G_α so that

$$1=\sum_\alpha F_\alpha f_\alpha$$

and

$$I\chi(X, X-x)=\sum_\alpha G_\alpha g_\alpha.$$

However, each $\bar{S}_\alpha$ is simultaneously a $\mathbb{Z}/(2)$-Witt space and a $\mathbb{Z}/(2)$-Euler space so each of the functions f_α and g_α satisfy the local Euler condition of [FM]. Therefore, we can apply W_* to each of these equations, which gives the desired formula.

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