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Remarks on the closest packing of convex discs

L. Fejes Tóth

To Professor H. Hadwiger on his seventieth birthday

In the Euclidean plane let P be a packing of congruent replicas of a convex disc c. Let $r = r(P)$ be the supremum of the radii of those circles which have no common point with any disc of P. The smaller r is the "closer" is the packing. Thus $1/r$ can be considered as a measure of the closeness. If r for a certain packing $\bar{P} = \bar{P}(c)$ attains its infimum $\bar{r}$, then we speak of a closest packing or, in short, a close packing. A simple example for a close packing is given by a packing of unit circles in which each circle is touched by six others. Here we have

$$\bar{r} = \frac{2}{\sqrt{3}} - 1.$$

The above definitions can be extended to more general spaces. In Euclidean 3-space the closest packing of equal balls was determined by Böröczky [1]: the centres of the balls form a body-centred cubic lattice. The paper [2] deals with the same problem in spherical 2-space.

For the density of a packing of convex discs various results are known. In this paper we want to discuss similar problems for the closeness. Let us recall some theorems concerning the density [3, 4]. We shall denote a domain and its area by the same symbol.

THEOREM 1. *If d is the density of a packing of congruent replicas of a convex disc c and H is the hexagon of least area circumscribed about c then $d \le c/H$.*

Theorem 1 implies

THEOREM 2. *The density of an arbitrary packing of congruent centro-symmetric convex discs cannot exceed the density of the densest lattice-packing of the discs.*

Theorem 2 implies

THEOREM 3. *The density of an arbitrary packing of translates of a convex disc cannot exceed the density of the densest lattice-packing of the discs.*

We start with the following

Remark 1. In a packing of congruent convex discs let r be the supremum of the radii of those circles which have no points in common with any of the discs. Let H be the hexagon of least area circumscribed about a disc. Let $h(x)$ be a hexagon of greatest area inscribed in the parallel domain of distance x of a disc. Then $h(r) \geq H$.

Since $h(x)$ is a strictly increasing function, the above inequality gives a lower bound for r, i.e. an upper bound for the closeness $1/r$.

The proof rests on Theorem 1 and an analogous theorem for the covering [3, 5]: If D is the density of a covering of the plane with non-crossing congruent replicas of a convex disc c and h is a hexagon of maximal area inscribed in c then $D \geq c/h$.

The term that two discs cross means that removing their intersection causes both discs to fall into disjoint pieces.

If d is the density of the packing considered in Remark 1 then we have $d \leq c/H$. On the other hand, let us observe that the parallel-domains of the discs at distance r cover the plane. The density of the parallel-domains is equal to $(c_r/c)d$, where c_r is the area of a parallel-domain. Since the parallel-domains of the same distance of two arbitrary non-overlapping convex domains do not cross, we can apply the above inequality for the covering density:

$$\frac{c_r}{c}\, d \geq c_r/h(r).$$

Thus we have $c/h(r) \leq d \leq c/H$ which implies the inequality to be proved.

If for a certain disc c H is a plane-filler and for a certain value r_0 the hexagon $h(r_0)$ is identical with H then $\bar{r} = r_0$. There is a great variety of discs with this property. The simplest example is the circle. The closest packing of such discs arises by tiling the plane with congruent replicas of H and inscribing in each hexagon a disc.

Remark 2. The statement arising from Theorem 2 by replacing the words "density" and "densest" by "closeness" and "closest" is false.

We shall show this by a special packing of directly and oppositely congruent discs. The question whether the statement under consideration becomes true by replacing the word "congruent" by "directly congruent" is still open.

Let $u = ABCDEF$ be a centro-symmetric hexagon such that $AB > BC = CD$

and $\sphericalangle ABC = \sphericalangle BCD = 135°$. Let $v = A'A''BC'C''D'D''EF'F''$ be a centrosymmetric decagon arising from u by cutting off at the corners A, C, D and F small triangles such that $A'A = AA'' = CC''$ and

$$C'C = \left(1 - \frac{\sqrt{2}}{2}\right) AA'', = \rho,$$

where ρ is the radius of the incircle of the triangle $A'AA''$. We claim that in any lattice-packing of translates of v there is a gap into which a circle of radius greater than ρ can be inserted.

Obviously, we can restrict ourselves to gaps bounded by three mutually touching translates of v. Again, we can restrict ourselves to such positions of the decagons in which the whole side $A'A''$ (or, which is the same, the whole side $D'D''$) belongs to the boundary of the gap, because otherwise the gap is "bigger" than the triangle $A'AA''$ (Fig. 1). Now we have only to check that in such a position the whole triangle $A'AA''$ belongs to the gap and that from among the two points at which the incircle of $A'AA''$ touches the sides $A'A$ and AA'' one is always in the interior of the gap (Fig. 2).

We continue to construct a packing of congruent replicas of v with a closeness equal to $1/\rho$.

Besides the tiling with translates of the hexagon u there is another regular tiling consisting of alternate rows of translates of u and of translates of oppositely congruent replicas of u.This tiling generates a packing of congruent replicas of v in which there are equal gaps consisting of two triangles congruent with $A'AA''$

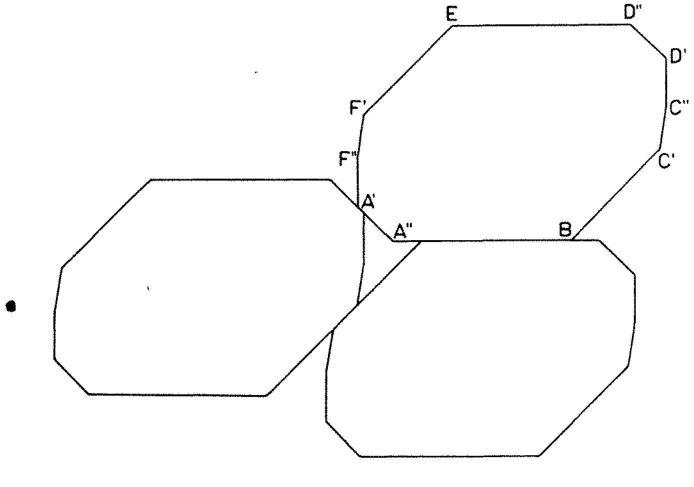

Figure 1

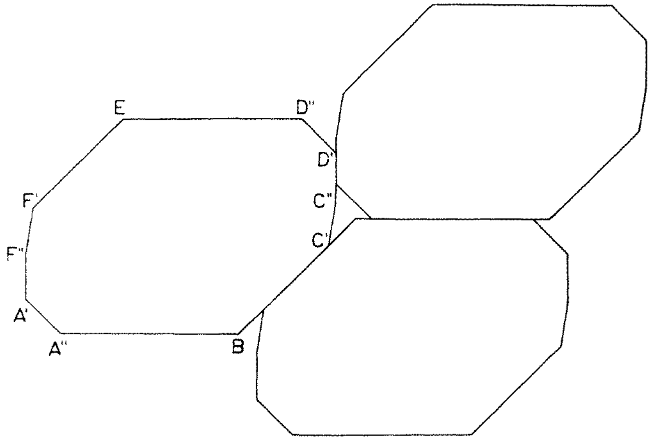

Figure 2

and $C'CC''$ put side by side so that A and C coincide and C' lies on a side of $A'AA''$ (Fig. 3). Consequently C' lies on the incircle of $A'AA''$. Thus the biggest circle contained in a gap is identical with the incircle of $A'AA''$.

This completes the proof of Remark 2.

In constrast with Theorem 2, there is an analogue of Theorem 3 for the closeness which we phrase as

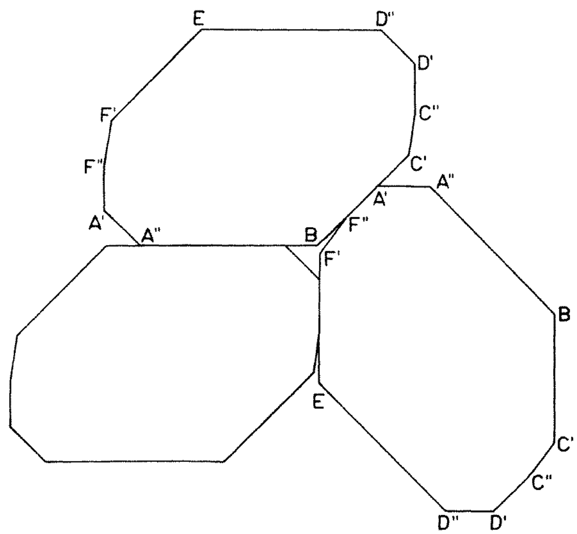

Figure 3

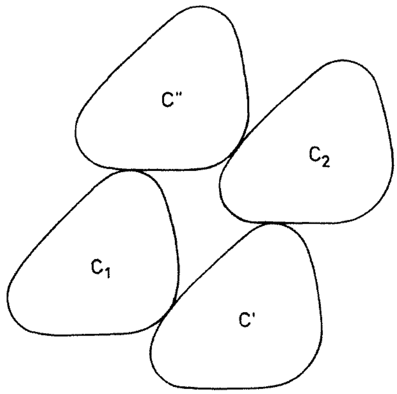

Figure 4

Remark 3. The closeness of an arbitrary packing of translates of a convex disc cannot exceed the closeness of the closest lattice-packing of the discs.

Let $c_1, c_2, \ldots$ be translates of a convex disc c forming a packing P. We may assume that in P there are two discs, say, c_1 and c_2 sufficiently near to one another in the following sense. There are two non-overlapping translates c' and c'' of c both touching simultaneously c_1 and c_2 (Fig. 4). Otherwise we could dilate the discs in the same ratio until the desired situation ensues. By a subsequent contraction we obtain a closer packing of translates of c than the original one.

If c is not strictly convex it may occur that the positions of c' and c'' are not uniquely determined. In this case let c'' be the image of c_1 under the trnslation $c' \rightarrow c_2$.

Obviously, none of the discs $c_3, c_4, \ldots$ can reach into the domain q enclosed by c_1, c', c_2 and c''. (In general, q is a curvilinear quadrangle which can degenerate into two curvilinear triangles.) Thus $r = r(P)$ is at least as great as the radius r_0 of the biggest circle contained in q. On the other hand, we have for the lattice-packing L generated by any three of c_1, c', c_2 and c'' $r(L) = r_0$. Thus we have, in accordance with Remark 3, $r(P) \geq r(L)$.

The above considerations show that Remark 3 remains valid if we measure the closeness of a packing instead of circles by means of an arbitrary figure, say by the supremum of the area of the ellipses contained in the gaps of the packing.

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