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A characterisation of the ellipsoid in terms of concurrent sections

G. R. BURTON AND P. MANI

Dedicated to Hugo Hadwiger on his seventieth birthday

1. Introduction

The ellipsoid has the property that parallel pairs of its sections are directly homothetic. It has been known for some time that this property characterises the ellipsoid among finite-dimensional convex bodies; some early proofs of this are referred to in Bonneson and Fenchel [4], page 142. Recently, Aitchison, [1] and [2], has proved some stronger converse results involving only sections close to the boundary. Our main result characterises the ellipsoid in terms of the property that its parallel sections through a pair of fixed points are directly homothetic; this answers affirmatively a conjecture proposed by P. Gruber at Oberwolfach in 1974.

THEOREM 1. *Let $2 \le k < d$, let K be a convex body in E^d, and let a_1 and a_2 be distinct points of E^d. Suppose that for every k-flat Λ through the origin in E^d, $(a_1 + \Lambda) \cap K$ is directly homothetic to $(a_2 + \Lambda) \cap K$. Then K is an ellipsoid.*

We must, of course, regard the empty set as being directly homothetic to itself. Rogers [8] and Burton [5] have shown that a convex body is determined up to direct homothety when its sections through a fixed point p are known up to direct homothety. However, the body may not be determined up to a homothety which preserves p; Burton conjectured that this indeterminacy could only occur for the ellipsoid. Our second result proves this conjecture, and is deduced from Theorem 1.

THEOREM 2. *Let $2 \le k < d$, let K and K' be convex bodies in E^d, and let p and p' be points of E^d. Suppose that for every k-flat Λ through the origin in E^d, $(p + \Lambda) \cap K$ is directly homothetic to $(p' + \Lambda) \cap K'$. Then there is a directly homothetic map Γ of E^d such that $\Gamma(K) = K'$. If $\Gamma(p) \ne p'$, then K and K' are ellipsoids.*

A special case of Theorem 2, which assumed K was centrally symmetric and that $p \notin K$, was given by Burton [5]. Using Theorem 1, we are also able to re-prove the False Centre Theorem of Aitchison, Petty, Rogers [3] and Larman [7]:

FALSE CENTRE THEOREM. *Let $2 \leq k < d$, let K be a convex body in E^d and let p be a point of E^d. Suppose that $\Lambda \cap K$ is centrally symmetric whenever Λ is a k-flat of E^d containing p. Then K is centrally symmetric. If p is not the centre of K, then K is an ellipsoid.*

2. Proof of Theorem 2 and the False Centre Theorem

In this section, we show how Theorem 2 and the False Centre Theorem follow from Theorem 1.

LEMMA 2.1. *Let $2 \leq k < d$ and let K and K' be convex bodies in E^d. Suppose that $\pi(K)$ is directly homothetic to $\pi(K')$ whenever π is an orthogonal projection on a k-flat. Then K is directly homothetic to K'.*

Proof. If π is an orthogonal projection on a linear 2-flat, then there is an orthogonal projection ϕ on a linear k-flat such that $\pi = \pi \circ \phi$. Thus $\pi(K)$ is directly homothetic to $\pi(K')$. It therefore suffices to consider the case $k = 2$, which Rogers [8] has done.

LEMMA 2.2. *Let $2 \leq k < d$, let K and K' be convex bodies in E^d and let p and p' be points of E^d. Suppose that $(p + \Lambda) \cap K$ is directly homothetic to $(p' + \Lambda) \cap K'$ whenever Λ is a k-flat through the origin in E^d. Then K is directly homothetic to K'.*

Proof. The case $k = 2$ has been considered by Rogers [8] and Burton [5]. Suppose $k > 2$, and let π be an orthogonal projection on a linear $(d - k + 2)$-flat Φ. If λ is a linear 2-flat in Φ, then $\Lambda = \pi(p) + \lambda + \Phi^\perp$ and $\Lambda' = \pi(p') + \lambda + \Phi^\perp$ are parallel k-flats which contain p and p' respectively. So $\Lambda \cap K$ is directly homothetic to $\Lambda' \cap K'$, and $(\pi(p) + \lambda) \cap \pi(K) = \pi(\Lambda \cap K)$ is directly homothetic to $(\pi(p') + \lambda) \cap \pi(K')$. Thus $\pi(K)$ is directly homothetic to $\pi(K')$. It follows from Lemma 2.1 that K is directly homothetic to K'.

Proof of Theorem 2. By Lemma 2.2 there is a direct homothety Γ such that $\Gamma(K) = K'$. Suppose that $\Gamma(p) \neq p'$. Let Λ be any linear k-flat in E^d. Then $(p + \Lambda) \cap K$ is directly homothetic to $(p' + \Lambda) \cap K'$, so $(\Gamma(p) + \Lambda) \cap K'$ is directly homothetic to $(p' + \Lambda) \cap K'$. It now follows from Theorem 1 that K' is an ellipsoid.

LEMMA 2.3. *Let $2 \leq k < d$, let K be a convex body in E^d and let $p \in E^d$. If $\Lambda \cap K$ is centrally symmetric for every k-flat Λ which contains p, then K is centrally symmetric.*

Proof. If Λ is a k-flat which contains p, then $\Lambda \cap K$ is centrally symmetric, so $(-\Lambda) \cap (-K)$ is a translate of $\Lambda \cap K$, and $-p \in -\Lambda$. By Lemma 2.2, $-K$ is directly homothetic to K. Comparing diameters, $-K$ is a translate of K, so K is centrally symmetric.

Proof of the False Centre Theorem. By Lemma 2.3, K has a centre of symmetry a, say. Suppose $a \neq p$. Consider a linear k-flat Λ. Then $(2a - p + \Lambda) \cap K$ is a central reflection of $(p + \Lambda) \cap K$ which is centrally symmetric, so $(2a - p + \Lambda) \cap K$ is a translate of $(p + \Lambda) \cap K$. It now follows from Theorem 1 that K is an ellipsoid.

3. Reduction of Theorem 1 to 3 dimensions

In this section we shall suppose that Theorem 1 holds for $k = 2$, $d = 3$, and we shall deduce the result for general k and d.

First assume that K, a_1 and a_2 satisfy the hypothesis of Theorem 1 with $k = 2$, $d \geq 3$. Let φ be any 2-flat which contains a_1 and intersects $\operatorname{int} K$. Then φ is contained in a 3-flat Φ which contains a_2. Let Λ_1 and Λ_2 be parallel 2-flats in Φ which contain a_1 and a_2 respectively. Then $\Lambda_1 \cap K$ is directly homothetic to $\Lambda_2 \cap K$; since $\Lambda_1 \cap (\Phi \cap K) = \Lambda_1 \cap K$ and $\Lambda_2 \cap (\Phi \cap K) = \Lambda_2 \cap K$, we can apply the 3-dimensional case of Theorem 1 to show that $\Phi \cap K$ is an ellipsoid. Thus $\varphi \cap K$ is an ellipse, for every 2-flat φ which contains a_1 and intersects the interior of K. It now follows that K is an ellipsoid; an elementary proof of this is given by Burton [5], generalising a result in Busemann [6], page 91, which referred only to sections through an interior point.

Now consider the case $2 < k < d$. Let π be the orthogonal projection on a linear $(d - k + 2)$-flat Φ of E^d, and suppose initially that $\pi(a_1) \neq \pi(a_2)$. Consider a linear 2-flat Λ in Φ. By considering $(a_1 + \Lambda + \Phi^{\perp}) \cap K$ and $(a_2 + \Lambda + \Phi^{\perp}) \cap K$ we find that $(\pi(a_1) + \Lambda) \cap \pi(K)$ is directly homothetic to $(\pi(a_2) + \Lambda) \cap \pi(K)$. It now follows from the cases already considered that $\pi(K)$ is an ellipsoid. By continuity, this holds for all $(d - k + 2)$-dimensional orthogonal projections π. Hence K is an ellipsoid; this may be deduced by dualizing the above-mentioned result about sections in Busemann's book.

4. Theorem 1 in 3 dimensions

Throughout the rest of the paper, K will be a fixed convex body in E^3, and a_1 and a_2 will be distinct points of E^3 such that for every plane Λ containing 0, $(a_1+\Lambda)\cap K$ is directly homothetic to $(a_2+\Lambda)\cap K$.

The purpose of Lemmas 4.1 to 4.8 will be to show that aff$\{a_1, a_2\}$ intersects the boundary of K in two smooth exposed points, and that when K has been projectively transformed so that its support planes at these points are parallel, its sections parallel to these planes are directly homothetic and have collinear centres of symmetry. The approach during some of these Lemmas resembles that of Aitchison, Petty, Rogers [3] and Larman [7].

LEMMA 4.1. *The line-segment* $[a_1, a_2]$ *contains inner points of* K.

Proof. First consider the possibility that $[a_1, a_2]\cap K=\phi$. We could then choose a support plane Λ of K which contained a_1 say, but which separated a_2 from K. Thus $a_1\in\Lambda\cap K$ while $(a_2-a_1+\Lambda)\cap K=\phi$ which is impossible. So $[a_1, a_2]\cap K\neq\phi$. If $[a_1, a_2]\cap K=\{a_1\}$, then a_2 would lie in a plane $a_2+\Lambda$ which was disjoint from K, and yet $a_1\in(a_1+\Lambda)\cap K$, which is impossible. So K contains relatively interior points of $[a_1, a_2]$.

Let us suppose that $[a_1, a_2]\cap \operatorname{int} K=\phi$, so that a_1 and a_2 lie in a support plane H of K. If $a_1\notin K$, then there would be a plane Λ containing a_1, and having direction close to that of H, such that $\Lambda\cap K=\phi$ but $(a_2-a_1+\Lambda)\cap K\neq\phi$. Thus $[a_1, a_2]\subset H\cap K$.

Consider the possibility that $H\cap K$ is a facet of K. Choose a line l through 0 which is parallel to H, and so that $(a_1+l)\cap K$ and $(a_2+l)\cap K$ are disjoint, the former being a line-segment. We can suppose that $\infty\geqslant\sigma\geqslant 1$, where σ is the ratio of the length of $(a_1+l)\cap K$ to that of $(a_2+l)\cap K$. Let c_1 and c_2 be corresponding end-points of $(a_1+l)\cap K$ and $(a_2+l)\cap K$ respectively. For each plane Λ which contains l but is not parallel to H, we have

$$\Lambda\cap(-c_1+K)=\sigma(\Lambda\cap(-c_2+K)).$$

In particular this shows that $\sigma\neq\infty$. Let b be a point of $H\cap K$ for which $b\cdot(c_1-c_2)$ is maximal, and let (b_n) be a sequence in $K\backslash H$ which converges to b. Let Λ_n be a plane which contains l and satisfies $b_n\in a_2+\Lambda_n$. Then

$$\sigma(b_n-c_2)+c_1\in(a_1+\Lambda_n)\cap K,$$

so taking the limit

$$\sigma(b-c_2)+c_1 \in H \cap K.$$

This is impossible since

$$[\sigma(b-c_2)+c_1]\cdot[c_1-c_2]=b\cdot(c_1-c_2)+\|c_1-c_2\|^2 \\ +(\sigma-1)(b-c_2)\cdot(c_1-c_2)>b\cdot(c_1-c_2).$$

Hence $H \cap K$ is a line-segment. Let l be a line through 0 such that a_1+l contains inner points of K. Consideration of parallel sections of K which contain $(a_1+l)\cap K$ and $(a_2+l)\cap K$ respectively shows that $(a_2+l)\cap K$ is a proper line-segment. We shall suppose $\sigma \geqslant 1$, where σ is the ratio of the length of $(a_1+l)\cap K$ to that of $(a_2+l)\cap K$. Then for every plane Λ containing l but not parallel to $H \cap K$, we have

$$\Lambda \cap(-a_1+K)=\sigma[\Lambda \cap(-a_2+K)].$$

Let b be the point of $H \cap K$ for which $b\cdot(a_1-a_2)$ is maximal and let (b_n) be a sequence in $K\backslash[l+\operatorname{aff}(H \cap K)]$ which converges to b. Let Λ_n be the plane which contains l and satisfies $b_n \in a_2+\Lambda_n$. Arguing as for the case above, we find

$$\sigma(b-a_2)+a_1 \in H \cap K$$

and

$$[\sigma(b-a_2)+a_1]\cdot(a_1-a_2)>b\cdot(a_1-a_2).$$

We conclude that $[a_1, a_2]$ contains inner points of K, completing the proof.

We shall work with Cartesian coordinates, and write $e_1=(1,0,0)$, $e_2=(0,1,0)$ and $e_3=(0,0,1)$. Whenever $S \subset\{1,2,3\}$, we write $L_S=\operatorname{lin}\{e_i: i \in S\}$ and denote by π_S the orthogonal projection on L_S. In view of Lemma 4.1, we may assume after an affine transformation that $K \cap \operatorname{aff}\{a_1, a_2\}=[0, e_1]$ and that L_{23} supports K at 0. Let Z be a support plane of K at e_1. We can also assume that $a_1\cdot e_1< a_2\cdot e_1$.

From Lemma 4.1 and the observation that is impossible for exactly one of a_1 and a_2 to lie in K, we have:

Remark. Either $a_1\cdot e_1<0<1<a_2\cdot e_1$ or $0 \leq a_1\cdot e_1<a_2\cdot e_1 \leq 1$.

LEMMA 4.2. *The point* $\frac{1}{2}(a_1+a_2)$ *is interior to* K.

Proof. Suppose this is false, so a_1 and a_2 are not in K. We may suppose $\frac{1}{2}(a_1+a_2)\in[0, a_1]$. Since $(a_1+L_{23})\cap K=\phi$, we have $(a_2+L_{23})\cap K=\phi$. Let l be a line through 0 in L_{23}, and let g be a non-zero vector in L_{23} whose direction is perpendicular to l. Consider a plane $\Lambda\neq L_{23}$ which contains l, and points $x_i\in a_i+\Lambda$ with $0\leq x_i\cdot e_i\leq a_2\cdot e_1$ for $i=1,2$. Then $|x_2\cdot g|\leq|x_1\cdot g|$, and equality can only occur if x_1 and x_2 lie in $\frac{1}{2}(a_1+a_2)+L_{23}$ in which case $\frac{1}{2}(a_1+a_2)=0$. Consider now the case when Λ and x_2 are chosen so that x_2 is a point of K for which $x_2\cdot g$ is maximal. Let x_1 be any point of $(a_1+\Lambda)\cap K$, which must be non-empty, so that $|x_1\cdot g|\leq|x_2\cdot g|$. The above argument shows that $|x_1\cdot g|=|x_2\cdot g|$, $\frac{1}{2}(a_1+a_2)=0$ and x_1 and x_2 are both in L_{23}. Then $(a_1+\Lambda)\cap K\subset L_{23}$, so $(a_2+\Lambda)<K\subset L_{23}$ also. This shows that the two support lines of $F=L_{23}\cap K$ parallel to l are distinct and at equal distances from 0. Varying l, we find that F is a facet of K and $F=-F$. Notice that every support plane of K through a_2 intersects K in a subset of L_{23}. Return to a fixed l and g. Let $0<\alpha<1$, and let x_2^* be a point of $(\alpha e_1+L_{23})\cap K$ for which $|x_2^*\cdot g|$ is maximal. Then the plane H_2 which contains a_2 and x_2^*+l intersects the relative interior of F. Comparing intersections with F, the section $G=(a_1-a_2+H_2)\cap K$ is a translate of $H_2\cap K$ by a vector in L_{23}, so G contains a point x_1^* of $(\alpha e_1+L_{23})\cap K$. The considerations of the first paragraph show that $|x_1^*\cdot g|>|x_2^*\cdot g|$ which is a contradiction.

Consider a unit vector $u\in L_{23}$, write $P(u)=\text{lin}\{u, e_1\}$ and write $v(\varphi, u)=\cos\varphi e_1+\sin\varphi u$ for real φ. The section $P(u)\cap K$ has two one-sided tangent rays at e_1; let the one which lies in the half-plane $\{x\in P(u): x\cdot u\geq 0\}$ be parallel to the vector $w_1(u)$, having $w_1(u)\cdot u=1$. The other ray will then be parallel to the vector $w_1(-u)$. In the same way define the vector $w_0(u)$ corresponding to a tangent ray at 0.

For small positive φ let $a_i+\text{lin}\{v(\varphi, u)\}$ intersect K in the line-segment $[b_i(\varphi, u), c_i(\varphi, u)]$ where $(b_i(\varphi, u)-c_i(\varphi, u))\cdot e_1>0$, for $i=1,2$.

We find that

$$b_i(\varphi, u)=\begin{cases} e_1+(a_i\cdot e_1-1)\varphi w_1(-u)+0(\varphi) & \text{if} \quad a_i\cdot e_1>1 \\ e_1+(1-a_i\cdot e_1)\varphi w_1(u)+0(\varphi) & \text{if} \quad a_i\cdot e_1\leq 1 \end{cases}$$

$$c_i(\varphi, u)=\begin{cases} (a_i\cdot e_1)\varphi w_0(-u)+0(\varphi) & \text{if} \quad a_i\cdot e_1\geqslant 0 \\ (-a_i\cdot e_1)\varphi w_0(u)+0(\varphi) & \text{if} \quad a_i\cdot e_1<0. \end{cases}$$

As $\varphi\to 0^+$, $\varphi^{-1}(b_2(\varphi, u)-b_1(\varphi, u))$ approaches a limit

$$z_1(u)=\begin{cases} (a_2\cdot e_1-1)w_1(-u)-(1-a_1\cdot e_1)w_1(u) & \text{if} \quad a_1\cdot e_1<0<1<a_2\cdot e_1 \\ ((a_1-a_2)\cdot e_1)w_1(u) & \text{if} \quad 0\leq a_1\cdot e_1<a_2\cdot e_1\leq 1, \end{cases}$$

and $\varphi^{-1}(c_2(\varphi, u)-c_1(\varphi, u))$ approaches a limit

$$z_0(u)=\begin{cases}(a_2\cdot e_1)w_0(-u)+(a_1\cdot e_1)w_0(u) & \text{if} \quad a_1\cdot e_1<0<1<a_2\cdot e_1\\ ((a_2-a_1)\cdot e_1)w_0(-u) & \text{if} \quad 0\le a_1\cdot e_1<a_2\cdot e_1\le 1.\end{cases}$$

LEMMA 4.3. *The vectors* $z_1(u)$ *and* $z_0(u)$ *are nowhere zero continuous functions of* u. *For* $i=0, 1$, *if* $z_i(u)$ *is a multiple of* $z_i(-u)$ *then* $w_i(u)=-w_i(-u)$.

Proof. Continuity follows from the continuity of w_1 and w_0. Since $a_2\cdot e_1>a_1\cdot e_1$, z_1 and z_0 are non-vanishing. Suppose that $z_1(u)=\lambda z_1(-u)$. In the case $0\le a_1\cdot e_1<a_2\cdot e_1\le 1$ it is immediate that $w_1(u)$ is a multiple of $w_1(-u)$ and comparing the scalar products with u we obtain $w_1(u)=-w_1(-u)$. If $a_1\cdot e_1<0<1<a_2\cdot e_1$ and $w_1(u)$ is not a multiple of $w_1(-u)$, we find

$$\lambda=\frac{a_2\cdot e_1-1}{a_1\cdot e_1-1}=\frac{a_1\cdot e_1-1}{a_2\cdot e_1-1}$$

Then $\lambda=-1$ and $a_2\cdot e_1-1=1-a_1\cdot e_1$. This contradicts Lemma 4.2, so $w_1(u)$ is a multiple of $w_1(-u)$, and it follows that $w_1(u)=-w_1(-u)$. The case $i=0$ is similar.

When l and m are distinct coplanar lines, let $\mathfrak{P}[l, m]$ be the pencil of lines determined by l and m; that is, if $l\cap m\neq\phi$, $\mathfrak{P}[l, m]$ is the family of all lines which contain $l\cap m$, while if l is parallel to m, then $\mathfrak{P}[l, m]$ is the family of all lines parallel to l and m. Write

$$m_0(u)=\text{lin}\,\{z_0(u)\}, \quad \text{and} \quad m_1(u)=e_1+\text{lin}\,\{z_1(u)\}.$$

LEMMA 4.4. *For each unit vector* $u\in L_{23}$ *there is a plane* $\Pi(u)$ *which contains* L_1, *and such that every point of* $\Pi(u)\cap\text{bd}\,K$ *belongs to a line of* $\mathfrak{P}[m_0(u), m_1(u)]$ *which supports* K.

Proof. Fix u and define

$$l_0(\varphi)=\text{aff}\,\{c_1(\varphi, u), c_2(\varphi, u)\}$$
$$l_1(\varphi)=\text{aff}\,\{b_1(\varphi, u), b_2(\varphi, u)\}$$

for small positive φ. As $\varphi\to 0^+$, the lines $l_0(\varphi)$ and $l_1(\varphi)$ tend to $m_0(u)$ and $m_1(u)$ respectively. Let Θ_φ be the orthogonal projection on $\text{lin}\,\{v(\varphi, u)\}^\perp$.

For small positive φ, $\Theta_\varphi(a_1)$ and $\Theta_\varphi(a_2)$ are distinct relatively interior points of $\Theta_\varphi(K)$. We can therefore choose distinct parallel chords $I_1(\varphi)$ and $I_2(\varphi)$ of $\Theta_\varphi(K)$ which contain $\Theta_\varphi(a_1)$ and $\Theta_\varphi(a_2)$ respectively, and which are divided in the same ratio by these points. Write

$$H_i(\varphi)=\Theta_\varphi^{-1}(\operatorname{aff} I_i(\varphi)),$$

which contains a_i, and let Δ_φ be the direct homothety such that

$$\Delta_\varphi[H_2(\varphi)\cap K]=H_1(\varphi)\cap K.$$

Then $\Theta_\varphi\Delta_\varphi(a_2)$ must divide $I_1(\varphi)$ in the same ratio in which $\Theta_\varphi(a_2)$ divides $I_2(\varphi)$, so $\Theta_\varphi\Delta_\varphi(a_2)=\Theta_\varphi(a_1)$. Thus Δ_φ preserves $P(u)$. In particular,

$$\Delta_\varphi(b_2(\varphi, u))=b_1(\varphi, u) \tag{1}$$

$$\Delta_\varphi(c_2(\varphi, u))=c_1(\varphi, u). \tag{2}$$

Choose a sequence $(\varphi(n))$ of positive numbers tending to zero so that $H_2(\varphi(n))$ converges to a plane $\Pi(u)$ which contains L_1.

Consider $x\in\Pi(u)\cap\operatorname{bd} K$, and choose $x(n)\in H_2(\varphi(n))\cap\operatorname{bd} K$ so that $x(n)\to x$ as $n\to\infty$. Let

$$y(n)=\Delta_{\varphi(n)}(x(n))\in H_1(\varphi(n))\cap\operatorname{bd} K$$

and write $k(n)=\operatorname{aff}\{x(n), y(n)\}$. Then $\Delta_{\varphi(n)}$ preserves $k(n)$, and in view of (1) and (2), $k(n)\in\mathfrak{P}[l_0(\varphi(n)), l_1(\varphi(n))]$. As $n\to\infty$, $x(n)$ and $y(n)$ tend to x, and since $k(n)\cap\operatorname{int} K$ lies between $H_1(\varphi(n))$ and $H_2(\varphi(n))$, $k(n)$ tends to a support line k of K at x, with $k\in\mathfrak{P}[m_0(u), m_1(u)]$.

Let Γ be the set of unit vectors u in L_{23} for which $P(u)$ is parallel to two edges of $\pi_{23}(K)$, or $P(u)$ contains a point collinear with each of two edges of $\pi_{23}(K)$. Clearly Γ is countable and $-\Gamma=\Gamma$. When $u\notin\Gamma$, there is exactly one plane $\Pi(u)$ as described in Lemma 4.4.

LEMMA 4.5. *If u is a unit vector in $L_{23}\backslash\Gamma$, then $\Pi(u)=\Pi(-u)$.*

Proof. Let h and k be support lines of $\pi_{23}(K)$ at points p and q respectively in $\Pi(u)$, such that h and k are images under π_{23} of lines in $\mathfrak{P}[m_0(u), m_1(u)]$ which support K. Suppose $\Pi(-u)\neq\Pi(u)$, and that $\Pi(-u)$ intersects relbd $\pi_{23}(K)$ at points p' and q' which lie on the same sides of $P(u)$ as p and q respectively. Define support lines h' and k' of $\pi_{23}(K)$ at p' and q' in the same manner as above, with u replaced by $-u$.

Since $u\in\Gamma$, we can suppose that $h\cap\pi_{23}(K)=\{p\}$. Choose a projective transformation T of L_{23}, which preserves all lines through the origin, such that $T(h)$

and $T(k)$ are parallel to $\text{lin}\{u\}$. Then $T(h')$ is not parallel to $T(h)$, so $T(h')$ intersects $\text{lin}\{u\}$. But $T(k')$ is either equal to $T(k)$ or intersects $\text{lin}\{u\}$ on the opposite side of 0 from $T(h')$, since $T(p')$ and $T(q')$ are on opposite sides of $\Pi(u)$. This shows that h' and k' are neither both parallel to $\text{lin}\{u\}$ nor concurrent at a point of $\text{lin}\{u\}$, which is inconsistent with Lemma 4.4. We conclude that $\Pi(u) = \Pi(-u)$.

LEMMA 4.6. *The points* 0 *and* e_1 *are smooth on* K.

Proof. Suppose this fails, and let $b \in \{0, e_1\}$ be non-smooth. Then for all unit vectors u in L_{23}, apart possibly from those in a certain two element set Δ, b, is a non-smooth point of $P(u) \cap K$. For such u, $w_1(u) \neq -w_1(-u)$ if $b = e_1$ or $w_0(u) \neq -w_0(-u)$ if $b = 0$, so that by Lemma 4.3 $z_1(u)$ is not a multiple of $z_1(-u)$ or $z_0(u)$ is not a multiple of $z_0(-u)$; in either case, $\mathfrak{P}[m_0(u), m_1(u)] \neq \mathfrak{P}[m_0(-u), m_1(-u)]$. Write $\mathfrak{T}(u)$ for the family of lines in $\mathfrak{P}[m_0(u), m_1(u)]$ which support K. We show that it is possible to define a continuously varying plane $\Phi(u)$ for unit vectors $u \in L_{23} \backslash \Delta$, such that $\Phi(u) = \Pi(u)$ when $u \notin \Gamma$. Suppose this is impossible, so there are sequences $(u_n), (u_n^*)$ of unit vectors in $L_{23} \backslash \Gamma$ which converge to a vector $u \notin \Delta$, and so that $\Pi(u_n)$ and $\Pi(u_n^*)$ converge to distinct planes Π and Π^* respectively. By continuity, and since $\Pi(u_n) = \Pi(-u_n)$, we find that each relative boundary point of $\Pi \cap K$ belongs to a line in $\mathfrak{T}(u)$ and to a line in $\mathfrak{T}(-u)$. Similarly each relative boundary point of $\Pi^* \cap K$ belongs to a line in $\mathfrak{T}(u)$ and to a line in $\mathfrak{T}(-u)$. Since $\mathfrak{T}(u) \neq \mathfrak{T}(-u)$ this is impossible, for the conical or cylindrical surfaces whose families of edges are $\mathfrak{T}(u)$ and $\mathfrak{T}(-u)$ are completely determined by their intersections with the planes Π and Π^*. We deduce the existence of $\Phi(u)$ as claimed; note that each relative boundary point of $\Phi(u) \cap K$ belongs to a line in $\mathfrak{T}(u)$ and to a line in $\mathfrak{T}(-u)$. It is clear that if u^* is a unit vector in $L_{23} \backslash \Delta$ and u is sufficiently close to $\Phi(u^*)$, then $\Phi(u) \neq \Phi(u^*)$. Hence we can choose an arc Σ of unit vectors in $L_{23} \backslash \Delta$ so that $\Phi(u)$ attains more than one value for $u \in \Sigma$. Choose by continuity an interior point u' of Σ such that $\Phi(u)$ is non-constant on every neighbourhood of u' in Σ.

By continuity we can choose a neighbourhood U of b in $\text{bd}\, K$ and a neighbourhood S of u' in Σ such that for every $x \in U$ and $u \in S$, x lies on distinct lines from $\mathfrak{P}[m_0(u), m_1(u)]$ and from $\mathfrak{P}[m_0(-u), m_1(-u)]$ that define a plane which intersects the interior of K. If $u \in S$ and $x \in U \cap \Phi(u)$ then x lies on distinct lines from $\mathfrak{T}(u)$ and from $\mathfrak{T}(-u)$ that define a plane which intersects the interior of K, so x is non-smooth. By choice of u', it follows that the non-smooth points of K contain a non-empty open subset of the boundary of K. This is impossible since almost all boundary points of K are smooth. We conclude that 0 and e_1 are smooth points of K.

Recall the support plane Z which was defined before Lemma 4.2. Observe now that $m_1(u) = Z \cap P(u)$ and $m_0(u) = L_{23} \cap P(u)$ for all unit vectors $u \in L_{23}$. We may assume that $Z \cap L_{23}$ is either empty or is parallel to L_2. Then there is a projective transformation T having the form

$$T(x) = (1 + \delta(x \cdot e_3))^{-1} x$$

such that $T(Z)$ is parallel to L_{23}.

LEMMA 4.7. *$T(K)$ is bounded, and the sections of $T(K)$ parallel to L_{23} are directly homothetic and have centres of symmetry of L_1.*

Proof. To prove that $T(K)$ is bounded, it will be sufficient to suppose that $Z \cap L_{23} \neq \phi$ and to prove that $Z \cap L_{23} \cap K = \phi$. Let us assume this is false. First consider the possibility that $Z \cap L_{23} \cap K$ is a line-segment I, and choose a relatively interior point x of I. By Lemma 4.4 there is a plane Λ which contains L_1, such that every point of $\Lambda \cap \operatorname{bd} K$ lies on a support line of K containing x. If Φ is a plane containing I which also contains an inner point of K, then at most one end point of $\Phi \cap \Lambda \cap K$ lies on a support line of $\Phi \cap K$ through x, which is a contradiction. We may therefore assume that $Z \cap L_{23} \cap K$ is a single point y. Let Φ be the plane $\operatorname{lin}\{e_1, y\}$. Then by Lemma 4.4 every point of $\Phi \cap \operatorname{bd} K$ lies on support lines of K through each point x of $Z \cap L_{23} \setminus \{y\}$; if we let x approach y, we find that Φ is a support plane of K, contradicting the fact that L_1 contains inner points of K. Hence $T(K)$ is bounded.

Consider any unit vector $u \in L_{23}$, and let $\mathfrak{T}(u)$ be the family of all support lines of $T(K)$ which are parallel to $\operatorname{lin}\{u\}$. The family

$$\mathfrak{T}_0(u) = \{T^{-1}(k) : k \in \mathfrak{T}(u)\}$$

consists of those support lines of K which belong to $\mathfrak{P}[m_0(u), m_1(u)]$, and by Lemma 4.4 there is a plane $\Pi(u)$ which contains L_1, such that every point of $\Pi(u) \cap \operatorname{bd} K$ belongs to a member of $\mathfrak{T}_0(u)$. Then every point of $\Pi(u) \cap \operatorname{bd} T(K)$ belongs to a line in $\mathfrak{T}(u)$, since $T\Pi(u) = \Pi(u)$, modulo missing points at infinity.

Choose $0 < \xi < \xi' < 1$ and let

$$P = \operatorname{relbd}(-\xi e_1 + T(K)) \cap L_{23}$$

$$P' = \operatorname{relbd}(-\xi' e_1 + T(K)) \cap L_{23}$$

$$t(\theta) = \cos\theta e_2 + \sin\theta e_3$$

and suppose the curves P and P' are described by the points $\rho(\theta)t(\theta)$ and $\rho'(\theta)t(\theta)$ respectively, where ρ and ρ' are positive, for real θ.

If u is a unit vector in L_{23} and $\rho(\theta)t(\theta)$ is the unique point of contact of a support line of P parallel to $\text{lin}\{u\}$, then $\rho(\theta)t(\theta)$, $\rho(\theta+\pi)t(\theta+\pi)$ and $\rho'(\theta)t(\theta)$ all belong to $\Pi(u)$. So $\rho(\theta+\pi)t(\theta+\pi)$ and $\rho'(\theta)t(\theta)$ lie in support lines of P and P' respectively parallel to lin $\{u\}$. So if $\rho(\theta)t(\theta)$ is an exposed point of P, then the set of tangent lines to P at $\rho(\theta)t(\theta)$, the set of tangent lines to P at $\rho(\theta+\pi)t(\theta+\pi)$ and the set of tangent lines to P' at $\rho'(\theta)t(\theta)$ are just translates of one another. By approximation, it follows also that if $\rho(\theta)t(\theta)$ and $\rho(\varphi)t(\varphi)$ are the end points of an edge I of P, then $\rho(\theta+\pi)t(\theta+\pi)$ and $\rho(\varphi+\pi)t(\varphi+\pi)$ lie in a support line of P parallel to I, and that $\rho'(\theta)t(\theta)$ and $\rho'(\varphi)t(\varphi)$ lie in a support line of P' parallel to I. Hence for every θ, the sets of tangent lines to P at $\rho(\theta)t(\theta)$, to P at $\rho(\theta+\pi)t(\theta+\pi)$ and to P' at $\rho'(\theta)t(\theta)$ are just translates of one another. We deduce that

$$\frac{1}{\rho(\theta)}D_+\rho(\theta)=\frac{1}{\rho(\theta+\pi)}D_+\rho(\theta+\pi)=\frac{1}{\rho'(\theta)}D_+\rho'(\theta)$$

$$\frac{1}{\rho(\theta)}D_-\rho(\theta)=\frac{1}{\rho(\theta+\pi)}D_-\rho(\theta+\pi)=\frac{1}{\rho'(\theta)}D_-\rho'(\theta)$$

where D_+ and D_- denote differentiation on the right and left respectively with respect to θ. Hence

$$\frac{d}{d\theta}(\rho(\theta)/\rho(\theta+\pi))=0=\frac{d}{d\theta}(\rho(\theta)/\rho'(\theta)),$$

whence $\rho(\theta)/\rho'(\theta)$ and $\rho(\theta)/\rho(\theta+\pi)$ are constants. So P is directly homothetic to P' and $-P=cP$ for some positive c; comparing diameters we find $c=1$. This proves the Lemma.

LEMMA 4.8. 0 *and* e_1 *are exposed points of* K.

Proof. We suppose the Lemma is false, and assume without loss of generality that 0 is not an exposed point of K. In view of Lemma 4.7, 0 must be a relatively interior point of a facet F of K, with $F\subset L_{23}$, and 0 is the centre of symmetry of $T(F)$. Let $\{b,c\}=\{a_1,a_2\}$ rearranged so that $\|b\|\geq\|c\|$. Consider a line $l\subset L_{23}$ which intersects F in a single point. Let H be aff $(\{b\}\cup l)$, $H'=c-b+H$ and let $l'=H'\cap L_{23}$. Then $H'\cap K$ is directly homothetic to $H\cap K$, and l' is parallel to l, so if l' intersects K, $l'\cap K$ must be a single point; in any case, it follows that l' does not intersect the relative interior of F. Since l' is distinct from l, and has no greater distance from 0 than l has, it follows that l' is on the opposite side of 0 from l. This shows that $a_2 \cdot e_1 > 1 > 0 > a_1 \cdot e_1$, and that the other support line of F parallel to l has no greater distance from 0 than l has. Varying l and taking limits, we find that the support function h of F satisfies $h(u) \geq h(-u)$ for all $u \in L_{23}$, so that $h(u) = h(-u)$. Hence F has 0 as centre of symmetry. Returning to the consideration of the line l, we now find that l' supports F, so $l = -l'$ and therefore $a_2 = -a_1$. This is impossible by Lemma 4.2.

LEMMA 4.9. *If* $a_1 \cdot e_1 < 0 < 1 < a_2 \cdot e_1$ *then* Z *is parallel to* L_{23}.

Proof. Let M_1 and N_1 be the two support planes of K which contain $a_1 + L_2$, and write

$$M_2 = a_2 - a_1 + M_1, N_2 = a_2 - a_1 + N_1$$

so that M_2 and N_2 are also support planes of K; for if say M_2 did not support K, a suitable slight alteration in the directions of M_2 and M_1 would yield parallel planes containing a_2 and a_1 respectively, with exactly one of these planes intersecting K, which is impossible. Suppose that Z is not parallel to L_{23}, so that T maps the plane at infinity onto a translate Λ of L_{12}. Then

$$\overline{(T(M_1))} \cap \overline{(T(M_2))} = f + L_2, \overline{(T(N_1))} \cap \overline{(T(N_2))} = g + L_2$$

where f and g are points of $\Lambda \cap L_{13}$, and the bars indicate closure. The planes $T(M_1)$ and $T(N_1)$ support $T(K)$ and are symmetrically placed about L_{12} by Lemma 4.7.Hence the triangle conv $\{f, g, a_1\}$ is isosceles with base $[f, g]$. Similarly conv $\{f, g, a_2\}$ is isosceles with base $[f, g]$. This is impossible since $[a_1, a_2]$ is parallel to $[f, g]$. Thus Z is parallel to L_{23}.

We now abandon all the notation which has accumulated so far, with the exception of a_i, K introduced at the beginning of section 4, L_S, π_S, e_i, Z introduced after Lemma 4.1 and T introduced before Lemma 4.6. Write $P = \pi_{23} T(K)$, so that

$$(\xi e_1 + L_{23}) \cap T(K) = \xi e_1 + k(\xi) P$$

for $0 \leq \xi \leq 1$, where k is a continuous concave non-negative function, $k(0) = k(1) = 0$, $\xi^{-1} k(\xi) \to \infty$ as $\xi \to 0^+$ and $(1-\xi)^{-1} k(\xi) \to \infty$ as $\xi \to 1^-$. Whenever x is a compact convex set, let $h[X, .]$ denote the support function of X. Our aim in Lemmas 4.10 to 4.14 will be to show that P is an ellipse.

Choose a non-zero vector $y \in L_{23}$ such that lin $\{y\}$ intersects the relative boundary of P at smooth points. Let v be a vector in L_{23} such that $v \cdot y = 0$ and $h[P, v] = \|v\|^2$. Choose β with $0 < |\beta| < 1$ which will be fixed for some time. Let $R(\xi)$, for $0 < \xi < 1$, be the line such that

$$T(R(\xi)) = \xi e_1 + \beta k(\xi) v + \text{lin}\,\{y\},$$

equality being modulo missing points at infinity. Write

$$H_2(\xi) = \text{aff}\,(\{a_2\} \cup R(\xi))$$
$$H_1(\xi) = a_1 - a_2 + H_2(\xi).$$

Let Φ_ξ be the unique direct homothety of E^3 which satisfies

$$\Phi_\xi[H_2(\xi) \cap K] = H_1(\xi) \cap K.$$

Every support plane of $T(K)$ at a point of $\text{lin}\,\{e_1, y\} \cap \text{bd}\, T(K)$ is parallel to a certain line $\text{lin}\,\{d\}$ in L_{23}, by reason of the smoothness ensured by the choice of y. So there is a solid cylinder or pointed cone C which contains K, such that every plane which supports K at a point of $\text{lin}\,\{e_1, y\}$ is also a support plane of C. Then since 0 is an exposed point of K,

$$C \cap L_{23} \subset \text{lin}\,\{d\}.$$

Let Ψ_ξ be the unique direct homothety which satisfies

$$\Psi_\xi[H_2(\xi) \cap C] = H_1(\xi) \cap C$$

which exists for all small positive ξ. We find that

$$\Psi_\xi(x) = M_\xi x + \lambda_\xi d$$

for some real numbers $M_\xi > 0$ and λ_ξ; we shall suppose that $(d - v) \cdot v = 0$.
Write

$$\Phi_\xi \Psi_\xi^{-1}(x) = (1 + r(\xi))x + s(\xi).$$

LEMMA 4.10. *As* $\xi \to 0^+$, $r(\xi) = 0(k(\xi))$ *and* $s(\xi) = 0(k(\xi))$.

Proof. Let ρ be the Hausdorff metric on compact subsets of E^3, and write

$$K_j(\xi) = H_j(\xi) \cap K,\ C_j(\xi) = H_j(\xi) \cap C,\ \Lambda = \text{lin}\,\{e_1, y\}$$

for $j = 1, 2$. We first show that

$$\rho[K_j(\xi), C_j(\xi)] = 0(k(\xi)) \tag{3}$$

as $\xi \to 0^+$. Suppose this fails, so there exists $\varepsilon > 0$ and a sequence (ξ_n) of positive numbers tending to zero with

$$\rho[K_j(\xi_n), C_j(\xi_n)] > \varepsilon k(\xi_n) \tag{4}$$

for each n. Let l be a line which contains a relatively interior point of $\Lambda \cap K$, and which belongs to the pencil determined by the edges of C. For each n, we can by (4) choose a plane Π_n containing l so that

$$\rho[\Pi_n \cap K_j(\xi_n), \Pi_n \cap C_j(\xi_n)] < \varepsilon k(\xi_n).$$

Then we can choose corresponding end-points x_n, z_n of $\Pi_n \cap K_j(\xi_n)$, $\Pi_n \cap C_j(\xi_n)$ respectively such that

$$\|x_n - z_n\| > \varepsilon k(\xi_n) \tag{5}$$

for each n. Let w_n be the corresponding end-point of $\Pi_n \cap \Lambda \cap K$, and let $p_n = \operatorname{aff}\{x_n, w_n\}$, $q_n = \operatorname{aff}\{z_n, w_n\}$. Then q_n contains an edge of C, and

$$\inf\{\text{angle between } q_n \text{ and } \Lambda : n = 1, 2, \ldots\} > 0 \tag{6}$$

$$\|z_n - w_n\| = 0(k(\xi_n)). \tag{7}$$

The angle between $x_n - z_n$ and v tends to $\pi/2$ as $n \to \infty$, so using (5), (6) and (7), the angle between p_n and q_n is bounded away from 0 for large n.

Replace (ξ_n) by a subsequence so that x_n tends to a point x and Π_n tends to a plane Π containing $\{x\} \cup l$ as $n \to \infty$. Then p_n and q_n tend to support lines p and q respectively of $\Pi \cap K$ at x, using (6), and $p \neq q$. This is impossible since $\Pi \cap C$ has a unique support line at x. Hence (3) is established.

We have

$$K_1(\xi) = \Phi_\xi(K_2(\xi))$$
$$C_1(\xi) = \Psi_\xi(C_2(\xi))$$

so

$$\Phi_\xi(C_2(\xi)) = \Phi_\xi \Psi_\xi^{-1}(C_1(\xi)) = (1 + r(\xi))C_1(\xi) + s(\xi). \tag{8}$$

Now

$$\rho[\Phi_\xi(C_2(\xi)), C_1(\xi)] \leq \rho[\Phi_\xi(C_2(\xi)), \Phi_\xi K_2(\xi))] + \rho[K_1(\xi), C_1(\xi)]$$
$$= t(\xi)\rho[C_2(\xi), K_2(\xi)] + \rho[K_1(\xi), C_1(\xi)]$$

where $t(\xi)$ is the ratio of Φ_ξ, and $t(\xi) \to 1$ as $\xi \to 0^+$, so

$$\rho[\Phi_\xi(C_2(\xi)), C_1(\xi)] = 0(k(\xi))$$

by (3). Combining this with (8) and writing it in terms of support functions, we obtain

$$r(\xi)h[C_1(\xi), g] + s(\xi) \cdot g = 0(k(\xi))$$

as $\xi \to 0^+$. By considering $g = \pm e_1$ we obtain $r(\xi) = 0(k(\xi))$, and taking $g = e_1, e_2, e_3$ we then find that $s(\xi) = 0(k(\xi))$.

Let $\sigma = (a_2 \cdot e_1)^{-1}(a_1 \cdot e_1)$, so by Lemma 4.2 $|\sigma| < 1$, and define

$$R_1(\xi) = \Phi_\xi(R(\xi))$$
$$R_2(\xi) = \Psi_\xi(R(\xi))$$
$$\bar{R}(\xi) = R(\xi) \cap K$$
$$\bar{R}_1(\xi) = \Phi_\xi(\bar{R}(\xi)) = R_1(\xi) \cap K$$
$$\bar{R}(\xi) = \Psi_\xi(\bar{R}(\xi)) \subset R_2(\xi).$$

LEMMA 4.11. *As* $\xi \to 0^+$, $k(\xi)^{-1}\bar{R}(\xi) \to R^*$ *where* $R^* = P \cap (\beta d + \text{lin}\{y\})$, $k(\xi)^{-1}\bar{R}_2(\xi) \to R_2^*$ *where* $R_2^* = R^* + (\sigma - 1)\beta d$, *and* $k(\xi)^{-1}\bar{R}_1(\xi) \to R_2^*$.

Proof. Since

$$T(\bar{R}(\xi)) = \xi e_1 + k(\xi)(P \cap (\beta v + \text{lin}\{y\})) = \xi e_1 + k(\xi)(P \cap (\beta d + \text{lin}\{y\}))$$

and $\xi = 0(k(\xi))$, we have

$$k(\xi)^{-1}T(\bar{R}(\xi)) \to R^* = P \cap (\beta d + \text{lin}\{y\})$$

as $\xi \to 0^+$. The map T^{-1} is differentiable, $DT^{-1}(0)$ is the identity map and the maximum distance of points of $T(\bar{R}(\xi))$ from 0 is $0(k(\xi))$, so $k(\xi)^{-1}\bar{R}(\xi)$ and $k(\xi)^{-1}T(\bar{R}(\xi))$ approach the same limit as $\xi \to 0^+$, hence $k(\xi)^{-1}\bar{R}(\xi) \to R^*$.

Since $T(H_2(\xi))$ contains the point $(a_2 \cdot e_1)(a_2 \cdot e_1 - \xi)^{-1}\beta k(\xi)d$ we find that $H_2(\xi)$ contains the point $(1+t(\xi))\beta k(\xi)d$ where $t(\xi) \to 0$ as $\xi \to 0^+$. For small positive ξ,

$$(H_2(\xi) \cap C) \cap L_{23} = \{(1+t(\xi))\beta k(\xi)d\}$$
$$(H_1(\xi) \cap C) \cap L_{23} = \{\sigma(1+t(\xi))\beta k(\xi)d\},$$

so $\sigma(1+t(\xi))\beta k(\xi)d = M_\xi(1+t(\xi))\beta k(\xi)d + \lambda_\xi d$ and since $M_\xi \to 1$ we have $\lambda_\xi \sim (\sigma-1)\beta k(\xi)$ as $\xi \to 0^+$. Then, since

$$\bar{R}_2(\xi) = M_\xi \bar{R}(\xi) + \lambda_\xi d$$

we have

$$k(\xi)^{-1}\bar{R}_2(\xi) \to R^* + (\sigma-1)\beta d = R_2^*$$

as $\xi \to 0^+$. Finally,

$$\bar{R}_1(\xi) = \Phi_\xi \Psi_\xi^{-1}(\bar{R}_2(\xi)) = (1+r(\xi))\bar{R}_2(\xi) + s(\xi)$$

so that $k(\xi)^{-1}\bar{R}_1(\xi) \to R_2^*$ by Lemma 4.10.

LEMMA 4.12. *Let $\kappa(\xi)$ be the distance of* 0 *from $R_1(\xi) \cap L_{23}$, or $+\infty$ if this intersection is empty. Then*

$$\liminf_{\xi \to 0^+} \kappa(\xi) > 0.$$

Proof. If $\operatorname{lin}\{y\}$ is parallel to both L_{23} and Z then $\kappa(\xi) = +\infty$ for $0 < \xi < 1$. We therefore need only consider the case when $Z \cap L_{23} \cap \operatorname{lin}\{y\} \neq \emptyset$, and we can assume that $y \in Z \cap L_{23}$.

Suppose the result fails, so there exists a sequence $(\xi(n))$ converging to 0^+ and $m_n \in R_1(\xi(n)) \cap L_{23}$ such that $m_n \to 0$ as $n \to \infty$. Write $\varphi_n = \Phi_{\xi(n)}$ and let $T(p_n)$ be the midpoint of $T(\bar{R}(\xi(n)))$, so that all the points p_n lie in a certain plane Π which contains L_1, since

$$k(\xi(n))^{-1}(T(p_n) - \xi(n)e_1) = k(\xi(q))^{-1}(T(p_q) - \xi(q)e_1)$$

for all n and q. For some α_n we have

$$m_n = \alpha_n \varphi_n(p_n) + (1-\alpha_n)\varphi_n(y).$$

Since $p_n \cdot e_1 > y \cdot e_1 = 0$, we have $\alpha_n < 1$, and since φ_n tends to the identity map as $n \to \infty$, we have $\alpha_n \to 1$, so we may assume $\alpha_n > 0$ for all n. Let the ray from a_2 through p_n intersect the boundary of K at a point b_n and intersect L_{23} at a point f_n.

Write

$$u_n = \alpha_n \varphi_n(p_n) + (1-\alpha_n)\varphi_n(f_n).$$

Then $[u_n, m_n]$ is parallel to $[y, f_n] \subset L_{23}$, so $u_n \in L_{23}$. Since $u_n \notin \operatorname{int} K$, $\varphi_n(p_n) \in K$, $\varphi_n(b_n) \in K$ and $b_n \in [f_n, p_n]$ we have

$$u_n \in [\varphi_n(b_n), \varphi_n(f_n)].$$

Then

$$\frac{\|m_n - \varphi_n(y)\|}{\|\varphi_n(p_n) - \varphi_n(y)\|} = \frac{\|u_n - \varphi_n(f_n)\|}{\|\varphi_n(p_n) - \varphi_n(f_n)\|} \le \frac{\|\varphi_n(b_n) - \varphi_n(f_n)\|}{\|\varphi_n(p_n) - \varphi_n(f_n)\|} \le 1.$$

As $n \to \infty$, m_n and $\varphi_n(p_n)$ both tend to 0 and $\varphi_n(y)$ tends to y, so

$$\frac{\|\varphi_n(b_n) - \varphi_n(f_n)\|}{\|\varphi_n(p_n) - \varphi_n(f_n)\|} \to 1.$$

Hence as $n \to \infty$,

$$\frac{\|b_n - f_n\|}{\|p_n - f_n\|} \to 1. \tag{9}$$

Write $\hat{b}_n = T(b_n)$, $\hat{f}_n = T(f_n)$, $\hat{p}_n = T(p_n)$, and observe that for each n, $\hat{b}_n, \hat{f}_n, \hat{p}_n$ and a_2 are collinear points of Π. Let w be the end of $\Pi \cap P$ with $\hat{p}_n \cdot w > 0$ for all n, so that $\chi = \|v\|^{-2} w \cdot v$ satisfies $1 > \chi^{-1}\beta > 0$. Let $K^* = \Pi \cap T(k)$. Then

$$\hat{p}_n = \xi(n)e_1 + \chi^{-1}\beta k(\xi(n))w$$

and the relative boundary of K^* contains the point

$$q_n = \xi(n)e_1 + k(\xi(n))w.$$

Let $[0, q_n]$ intersect $[b_n, a_2]$ at $r_n = \theta_n q_n$ where

$$\theta_n = (\chi a_2 \cdot e_1 + \beta\xi(n) - \chi\xi(n))^{-1}\beta a_2 \cdot e_1.$$

We have

$$\frac{\|\hat{b}_n-\hat{f}_n\|}{\|\hat{p}_n-\hat{f}_n\|}<\frac{\|r_n-\hat{f}_n\|}{\|\hat{p}_n-\hat{f}_n\|}=\frac{r_n\cdot e_1-\hat{f}_n\cdot e_1}{\hat{p}_n\cdot e_1-\hat{f}_n\cdot e_1}=\frac{\theta_n\xi(n)}{\xi(n)}=\theta_n\to\chi^{-1}\beta \tag{10}$$

as $n\to\infty$.

Using the projective invariance of the cross-ratio $[b_n, p_n; f_n, a_2]$ we find that as $n\to\infty$,

$$\frac{\|\hat{b}_n-\hat{f}_n\|}{\|\hat{p}_n-\hat{f}_n\|}\sim\frac{\|b_n-f_n\|}{\|p_n-f_n\|}\to 1$$

by (9). This contradicts (10), which proves the Lemma.

LEMMA 4.13. *There are sequences $(\xi_n), (\xi'_n)$ of positive numbers tending to zero such that $k(\xi'_n)\bar{R}_1(\xi_n)$ converges to a chord R_1^* of P, such that $R_1^*=\mu R_2^*$ for some $\mu>0$.*

Proof. For $0<\xi<1$ let the end-points of $T(\bar{R}_1(\xi))$ lie in the planes $\xi' e_1+L_{23}$ and $\xi'' e_1+L_{23}$ with $\xi''\geqslant\xi'$. By Lemma 4.12 there is an $\eta>0$ such that $T(R_1(\xi))$ contains no point of L_{23} within distance η of 0 when ξ is small. Then for small positive ξ the angle between $T(R_1(\xi))$ and its orthogonal projection on L_{23} is less than $\tan^{-1}(2\xi'/\eta)$, so

$$0\leq\xi''-\xi'<(2\xi'/\eta)k(\xi'')W$$

Where W is the diameter of P. Since k is concave and $k(0)=0$ we have

$$1\leq\frac{k(\xi'')}{k(\xi')}\leq\frac{\xi''}{\xi'}<1+\frac{2Wk(\xi'')}{\eta}$$

so that

$$\frac{k(\xi'')}{k(\xi')}\to 1 \tag{11}$$

as $\xi\to 0^+$.

Since the ends of $T(\bar{R}_1(\xi))$ belong to the relative boundaries of $\xi' e_1+k(\xi')P$ and of $\xi'' e_1+k(\xi'')P$, we can choose a sequence (ξ_n) tending to 0 from above, such that $k(\xi'_n)^{-1}T(\bar{R}_1(\xi_n))$ tends to a line-segment R_1^* whose ends will, by (11), be in relbd P. The differentiability of T^{-1} now ensures that $k(\xi_n')^{-1}\bar{R}_1(\xi_n)$ tends to R_1^*. By Lemma 4.11, $k(\xi_n)^{-1}\bar{R}_1(\xi_n)\to R_2^*$, so we conclude that $R_1^*=\mu R_2^*$ for some $\mu>0$.

LEMMA 4.14. *P is an ellipse.*

Proof. We now allow β to vary, and introduce β as an argument for R^*, R_1^* and R_2^*. Then

$$R^*(\beta)=P\cap(\beta d+\operatorname{lin}\{y\})$$
$$R_2^*(\beta)=R^*(\beta)+\beta(\sigma-1)d\subset\sigma\beta d+\operatorname{lin}\{y\}$$
$$R_1^*(\beta)=\mu(\beta)R_2^*(\beta)=P\cap(\mu(\beta)\sigma\beta d+\operatorname{lin}\{y\}).$$

If $\mu(\beta)<1$, then $R_1^*(\beta)$ is closer to 0 than $R^*(\beta)$ and has shorter length; since $R_1^*(\beta)$ is a chord of P, we must therefore have $\mu(\beta)\geq 1$. If $\mu(\beta)>1$, then the length of $R_1^*(\beta)$ is greater than that of $R^*(\beta)$, so $|\mu(\beta)\sigma\beta|<|\beta|$, while if $\mu(\beta)=1$ then $|\mu(\beta)\sigma\beta|=|\sigma\beta|<|\beta|$.

Fix β_0, and let β_1 be the number with least absolute value which satisfies

$$R^*(\beta_1)=\alpha R^*(\beta_0)+\lambda d$$

for real numbers α,λ; interpret $R^*(0)$ as $P\cap\operatorname{lin}\{y\}$. Suppose that $\beta_1\neq 0$. Write $\beta_2=\mu(\beta_1)\sigma\beta_1$, so that $|\beta_2|<|\beta_1|$. Then

$$\begin{aligned}R^*(\beta_2)=R_1^*(\beta_1)=\mu(\beta_1)R_2^*(\beta_1)&=\mu(\beta_1)(R^*(\beta_1)+(\beta_1(\sigma-1)d)\\&=\mu(\beta_1)\alpha R^*(\beta_0)+\mu(\beta_1)(\beta_1(\sigma-1)+\lambda)d.\end{aligned}$$

This is impossible by choice of β_1, so we conclude that $\beta_1=0$. Thus the midpoint of $R^*(\beta_0)$ lies on $\operatorname{lin}\{d\}$. Since β_0 was chosen arbitrarily, it follows that the chords of P parallel to $\operatorname{lin}\{y\}$ have collinear midpoints. Varying y over the smooth points of P and taking limits, we find that the chords of P parallel to any given line have collinear midpoints. Then P is an ellipse, by a standard result given in Busemann [6], page 92.

After an affine transformation, we can suppose that P is the unit circle, and then

$$bd\,T(K)=\{(x,y,z):y^2+z^2=(k(x))^2,\ 0\leq x\leq 1\}.$$

The remaining Lemmas prove that K is an ellipsoid.

LEMMA 4.15. *Suppose L_{23} is not parallel to Z. Then K is an ellipsoid.*

Proof. In view of Lemma 4.9 and the Remark, we have

$$0 \leq a_1 \cdot e_1 < a_2 \cdot e_1 \leq 1.$$

If $a_1 \in bd\, K$, then the support plane H of K at a_1 would satisfy $H \cap K = \{a_1\}$, while $a_2 - a_1 + H$ would intersect K in a proper section. So

$$0 < a_1 \cdot e_1 < a_2 \cdot e_1 < 1.$$

We also find that T is the projective transformation

$$T(x, y, z) = (1 + \delta z)^{-1}(x, y, z), \quad \text{for some } \delta \neq 0.$$

Notice that by rotational symmetry, $T(K)$ is preserved by the reflection R in L_{12}.

Consider a plane H which contains $a_1 + L_2$, and form the sequence of sections

$$\begin{aligned}
&H_0 \cap T(K) \text{ where } H_0 = H,\ H_1 \cap T(K) \text{ where } H_1 = R(H_0),\\
&H_2 \cap K \text{ where } H_2 = T^{-1}(H_1),\ H_3 \cap K \text{ where } H_3 = a_2 - a_1 + H_2,\\
&H_4 \cap T(K) \text{ where } H_4 = T(H_3),\ H_5 \cap T(K) \text{ where } H_5 = R(H_4),\\
&H_6 \cap K \text{ where } H_6 = T^{-1}(H_5),\ H_7 \cap K \text{ where } H_7 = a_1 - a_2 + H_6,\\
&H_8 \cap T(K) \text{ where } H_8 = T(H_7);
\end{aligned}$$

all of these sections are projectively equivalent. Write $a_1 \cdot e_1 = \beta$, $a_2 \cdot e_1 = \alpha$, and consider a point $(\beta + x, y, z) \in H$, with $z \neq \pm 1/\delta$. Then

$$\begin{aligned}
&H_0 \text{ contains } (\beta + x, y, z),\ H_1 \text{ contains } (\beta + x, y, -z),\\
&H_2 \text{ contains } (1 + \delta z)^{-1}(\beta + x, y, -z),\\
&H_3 \text{ contains } (1 + \delta z)^{-1}(\alpha + x + \delta z(\alpha - \beta), y, -z),\\
&H_4 \text{ contains } (\alpha + x + \delta z(\alpha - \beta), y, -z),\\
&H_5 \text{ contains } (\alpha + x + \delta z(\alpha - \beta), y, z),\\
&H_6 \text{ contains } (1 - \delta z)^{-1}(\alpha + x + \delta z(\alpha - \beta), y, z),\\
&H_7 \text{ contains } (1 - \delta z)^{-1}(\beta + x + 2\delta z(\alpha - \beta), y, z),\\
&H_8 \text{ contains } (\beta + x + 2\delta z(\alpha - \beta), y, z);
\end{aligned}$$

in particular, $a_1 + L_2 \subset H_8$. If we repeatedly apply this process to the plane $a_1 + L_{23}$, which contains the point $(\beta, 0, 2\delta)$, we find that the sections

$$T(K) \cap (a_1 + \text{lin}\,\{e_2, 2\delta e_3 + 4n\delta^2(\alpha - \beta)e_1\})$$

are ellipses for $n = 0, 1, 2, \ldots$ .Taking limits, we find that $L_{12} \cap T(K)$ is an ellipse whose perimeter has equation

$$4(x - \tfrac{1}{2})^2 + A^2 y^2 = 1$$

for some $A \neq 0$ by, symmetry in L_1. We can now determine the function k, and we find that $T(K)$ is the ellipsoid whose surface has equation

$$4(x - \tfrac{1}{2})^2 + A^2 y^2 + A^2 z^2 = 1,$$

so K is then an ellipsoid as claimed.

LEMMA 4.16. *Suppose that Z is parallel to L_{23}. Then K is an ellipsoid.*

Proof. In this case, T is the identity map. Write $a_2 \cdot e_1 = \alpha$, $a_1 \cdot e_1 = \beta$, and for small positive λ let

$$H_1(\lambda) = \{(x, y, z) : y = \lambda(x - \beta)\}$$
$$H_2(\lambda) = \{(x, y, z) : y = \lambda(x - \alpha)\}$$

which contain a_1 and a_2 respectively. The relative boundaries of $\pi_{13}(H_1(\lambda) \cap K)$ and $\pi_{13}(H_2(\lambda) \cap K)$ have equations

$$\lambda^2(x - \beta)^2 + z^2 = R(x) \tag{12}$$
$$\lambda^2(x - \alpha)^2 + z^2 = R(x) \tag{13}$$

respectively, where $R(x) = (k(x))^2$. There are numbers $\Phi_\lambda > 0$, t_λ such that

$$\pi_{13}(H_1(\lambda) \cap K) = \Phi_\lambda \pi_{13}(H_2(\lambda) \cap K + (t_\lambda, 0, 0),$$

since these regions are symmetric about L_1. For $i = 1, 2$ let $J_i(\lambda) = [\xi_i(\lambda), \eta_i(\lambda)]$ be the interval

$$J_1(\lambda) = \{x \cdot e_1 : x \in H_i(\lambda) \cap K\}$$

so that $\Phi_\lambda J_2(\lambda) + t_\lambda = J_1(\lambda)$.

Using (12) the equation of the relative boundary of $\pi_{13}(H_2(\lambda) \cap K)$ can also be written

$$\lambda^2(\Phi_\lambda x + t_\lambda - \beta)^2 + \Phi_\lambda^2 z^2 = R(\Phi_\lambda x + t_\lambda). \tag{14}$$

From (13) and (14) we deduce

$$\Phi_\lambda^2\lambda^2(x-\alpha)^2-\lambda^2(\Phi_\lambda x+t_\lambda-\beta)^2=\Phi_\lambda^2R(x)-R(\Phi_\lambda x+t_\lambda) \tag{15}$$

for all $x\in J_2(\lambda)$. Notice that $\Phi_\lambda\to 1$, $t_\lambda\to 0$ as $\lambda\to 0$.

We next show that R is twice differentiable on $(0,1)$. First let us show that $x>\Phi_\lambda x+t_\lambda$ for all $x\in J_2(\lambda)$ when λ is small and positive.

We have

$$\pi_{12}(H_i(\lambda)\cap K)=[b_i(\lambda),c_i(\lambda)]$$

where $(c_i(\lambda)-b_i(\lambda))\cdot e_i>0$ for $i=1,2$. Since $\pi_{12}(K)$ is smooth at 0 and e_1, we have

$$c_i(\lambda)\cdot e_2=\lambda(1-a_i\cdot e_1)+0(\lambda)$$
$$b_i(\lambda)\cdot e_2=-\lambda(a_i\cdot e_1)+0(\lambda)$$

as $\lambda\to 0^+$. From Lemma 4.2 it follows that $|\alpha|>|\beta|$ and $|1-\alpha|<|1-\beta|$, so

$$|c_2(\lambda)\cdot e_2|<|c_1(\lambda)\cdot e_2|$$
$$|b_2(\lambda)\cdot e_2|>|b_1(\lambda)\cdot e_2|$$

for all small positive λ. Since $\pi_{12}(K)$ is symmetric in L_1, has no edges parallel to L_2, but has support lines parallel to L_2 at 0 and e_1, we find that

$$c_2(\lambda)\cdot e_1>c_1(\lambda)\cdot e_1$$
$$b_2(\lambda)\cdot e_1>b_1(\lambda)\cdot e_1$$

for all small positive λ. That is,

$$\eta_2(\lambda)>\eta_1(\lambda),\ \xi_2(\lambda)>\xi_1(\lambda)$$

whenever $\lambda\in I=(0,\mu)$, say. We can also suppose that ξ_2 and η_2 are monotonic on I.

Defining

$$\zeta(\lambda)=\min\{\eta_2(\lambda)-\eta_1(\lambda),\ \xi_2(\lambda)-\xi_1(\lambda)\},$$

we find $x-(\Phi_\lambda x+t_\lambda)\geq\zeta(\lambda)$ for $x\in J_2(\lambda)$, and ζ is a positive continuous function on I. Let $x \in (0, 1)$, choose $\lambda' \in I$ so that $x \in \operatorname{int} J_2(\lambda')$ and choose, by the concavity of $k, y \in (x - \zeta(\lambda'), x) \cap \operatorname{int} J_2(\lambda')$ such that R is twice differentiable at y. Choose $\lambda \in (0, \lambda')$ such that $y = \Phi_\lambda x + t_\lambda$. From (15) it now follows that R is twice differentiable at x.

Differentiating (15) *twice with respect to* x, we obtain

$$R''(x) = R''(\Phi_\lambda x + t_\lambda) \tag{16}$$

for $x \in \operatorname{int} J_2(\lambda)$. If $\lambda' \in I$ and $x, y \in \operatorname{int} J_2(\lambda')$ satisfy $x - \zeta(\lambda') < y < x$, we can, as above, choose $\lambda \in (0, \lambda')$ such that

$$y = \Phi_\lambda x + t_\lambda.$$

By (16) we then have $R''(x) = R''(y)$. It follows that R'' is constant on $\operatorname{int} J_2(\lambda')$, and so R'' is constant on $(0, 1)$. Therefore R is a quadratic form. Since $R(0) = R(1) = 0$ and R is positive on $(0, 1)$, we have

$$R(x) = A^2(x - x^2)$$

for some $A \neq 0$, and the surface of K is the ellipsoid with equation

$$A^2(x - \tfrac{1}{2})^2 + y^2 + z^2 = \tfrac{1}{4}A^2.$$

Lemmas 4.15 and 4.16 now show that K is an ellipsoid. This completes the proof of Theorem 1.

REFERENCES.

[1] P. W. Aitchison, A characterisation of the ellipsoid, *J. Austral. Math. Soc. 11* (1970), 385–394.
[2] ——, The determination of convex bodies by some local conditions, *Duke Math. J, 41* (1974), 193–209.
[3] ——, C. M. Petty and C. A. Rogers, A convex body with a false centre is an ellipsoid, *Mathematika 18* (1971), 50–59.
[4] T. Bonneson and W. Finchel, Theorie der konvexen Körper (Springer, Berlin, 1934).
[5] G. R. Burton, Sections of convex bodies, *J. London Math. Soc.* (2) *12* (1976), 331–336.
[6] H. Busemann, The geometry of geodesics (Academic Press, New York, 1955).
[7] D. G. Larman, A note on the false centre problem, *Mathematika 21* (1974), 216–227.
[8] C. A. Rogers, Sections and projections of convex bodies, *Portugaliae Math. 24* (1965), 99–103.

Mathematics Department, University College London.
Mathematisches Institut, Universität Bern.

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