Comment. Math. Helvetici 53 (1978) 385–394 Birkhäuser Verlag, Basel

Quadruple points of 3-manifolds in S^4

MICHAEL H. FREEDMAN*

A folk theorem (see Banchoff [B]) says that the number of normally triple points of a closed surface normally immersed in 3-space is congruent modulo two to its Euler characteristic. In general, a normal immersion of a compact n-manifold in an $n+1$-manifold will have a finite number, θ, of $(n+1)$-tuple points. θ, taken mod 2, is well defined under bordism of both the immersion and ambient manifold. An attractive place to try to evaluate θ is on the abelian group, "(oriented bordism of immersed n-manifolds in S^{n+1}, connected sum)" $= B_n$, since B_n is naturally isomorphic to the stable homotopy group π_n. Counting $(n+1)$-tuple points determines a homomorphism, $\theta_n : \pi_n \to Z_2$. The figure eight immersion of a circle shows that θ_1 is an isomorphism; Banchoff's proof shows that θ_2 is the zero map; the main result of this paper is that θ_3 is the unique epimorphism $\pi_3 \simeq Z_{24} \to Z_2$. Thus, we show that a (actually any) oriented 3-manifold may be generically immersed in S^4 with an odd number of quadruple points. Like Smale's inversion of S^2, our proof is abstract and does not yield an example.

A pleasing conjecture is that θ_n is the stable Hopf invariant for all n.

§1. B_n is the n^{th} Stable Stem

All terminology will be smooth; the spheres, S^i, are given a standard orientation. Let X be a compact oriented $n+1$-manifold with boundary components divided into $\partial^- X$ and $\partial^+ X$. $(X; \partial^- X, \partial^+ X) \overset{f}{\looparrowright} (S^{n+1} x[-1, +1];\ S^{n+1} x - 1, S^{n+1} x 1)$is called an immersed bordism between $f/\partial^- X$ and $f/\partial^+ X$ if f is a relative immersion. Let B_n be the set of immersions, g, of compact oriented n-manifolds, M, modulo the equivalence relation of immersed bordism. B_n is a group under connected sum of ambient spheres away from the immersions.

---

* The author is partially supported by an NSF grant.

Since $\nu M \xrightarrow{g} S^{n+1}$ is trivialized by the orientations, g determines a trivialization of $\tau(X)\oplus\varepsilon^1$. According to Smale-Hirsh theory immersions exist (and are unique up to regular homotopy) which induce arbitrary trivializations of $\tau(M)\oplus\varepsilon^1$ and $\tau(X)\oplus\varepsilon^1$. Consequently $B_n \cong$ {trivializations of $\tau(M)\oplus\varepsilon^1$}/{trivializations which extend to trivializations of $\tau(X)\oplus\varepsilon^1$, where $\partial X = M$}. The Pontryagin-Thom construction determines a homomorphism $i_n : B_n \to \pi_n$.

Since $\pi_i(S0, S0(n+1)) \cong 0$ $i \leq n$, a stable trivialization of ν_M determines a trivialization of $\tau(M)\oplus\varepsilon^1$; so i_n is epic. Since $\pi_i(S0, S0(n+2)) \cong 0$ $i \leq n+1$, a stable trivialization of ν_X determines a trivialization of $\tau(X)\oplus\varepsilon^1$; so i is monic.

THEOREM 1. $B_n \overset{i_n}{\cong} \pi_n$

§2. *Generic immersions*

Let $G : M \to S^{n+1}$ be an immersion of a compact manifold. g determines maps $g_i : (\underbrace{Mx \cdots xM}_{i\text{-copies}} - \text{big diagonal}) \to (S^{n+1}x \cdots xS^{n+1})$. g_i^{-1} (small diagonal) $= M_i$ is the i-tuple set of g^{-1}. It is easy to see that the M_i are compact. An argument using the Thom-transversality theorem shows that g may be C^∞ approximated by an immersion $\bar{g}$ with $\bar{g}_i$ transverse to the small diagonal for all i; such immersions will be called *normal*. $\bar{M}_i = \bar{g}_i^{-1}$ (small diagonal) is an orientable submanifold of $\underbrace{Mx \cdots xM}_{i\text{-copies}}$ but does not have a prefered orientation since either $\underbrace{Mx \cdots xM}_{i\text{-copies}}$ or $\underbrace{S^{n+1}x \cdots xS^{n+1}}_{i\text{-copies}}$ will not inherit an orientation from its factors. Since an immersion is locally $1-1$ the symmetric group $S(i)$ acts freely on M_i; let N_i be the quotient manifold. When $i = n+1$ these considerations applied to $f : X \to S^{n+1}x[-1, 1]$ show that the number of $n+1$-tuple points of g determine a well defined homomorphism $\theta_n : B_n \to Z_2$.

The condition that g is a normal immersion has this equivalent form: every point in S^{n+1} should have a chart which intersects $g(M)$ in the l hyperplanes $x_{j_1} = 0, X_{j_2} = 0, \ldots, x_{j_l} = 0$, $1 \leq j_1, < \ldots, < j_l \leq n+1$. (For an open dense set of points l will be zero.)

§3. The computation of θ_3

Here is the program for computing θ_3. Starting with a generic immersion of an oriented 3-manifold, $g : M \to S^4$ we find N_2 naturally immersed in S^4 with a normal bundle having twisted (if N_2 is nonorientable) Euler class zero. Lemma 2 shows that the Hopf invariant of $[g] \in B_3 \cong \pi_3$, $H[g]$, is congruent to the Euler characteristic $\chi(N_2)$. In lemma 4 we replace N_2 by a surface $\bar{N}_2$ with the same Euler characteristic (mod 2) and also immersed in S^4 with twisted Euler class zero. When g has an even number of quadruple points, we show that the above immersion is regularly homotopic to a generic immersion with an even number of double points. It follows from a theorem of Whitney's [*W*] that a generically immersed surface in S^4 with an even number of double points and with twisted Euler class zero must have even Euler characteristic. So when $\theta_3[g]=0$, $\bar{N}_2$ admits an immersion with the above properties. Hence $\chi(N_2) \equiv \chi(\bar{N}_2) \equiv 0$ (mod 2). Now by Lemma 2 $\theta_3[g]=0$ implies $H[g]=0$, i.e. $\ker(H) \supset \ker(\theta_3)$. Since $H: \pi_3 \to Z_2$ is an epimorphism, so is $\theta_3: \pi_3 \to Z_2$. Knowing $\pi_3 \cong Z_{24}$ now completely determines θ_3.

Let $\pi: Mx \cdots xM \to M$ be the projection from the i-fold product of an oriented n-manifold to the first factor. The following commutative diagram shows that the restriction of π to M_i is an immersion.

$$\begin{array}{ccccc} 0 & \longrightarrow & \tau(M_i) & \longrightarrow & \tau(\Delta) \\ & & \downarrow{\scriptstyle (\pi/M_i)_*} & & \downarrow{\scriptstyle \alpha_*} \\ 0 & \longrightarrow & \tau(M) & \xrightarrow{g} & \tau(S^{n+1}) \end{array}$$

Δ is the small diagonal of $(S^{n+1})^i$. g_i is an immersion so $(g_i|)_*: \tau(M_i) \to (\Delta)$ is an injection. α is the restriction of projection to the first factor; α_* is an isomorphism and therefore $(\pi/M_i)_*$ is an injection as desired.

Let h be the map making the diagram:

$$\begin{array}{ccc} M_i & \xrightarrow{g o \pi} & S^{n+1} \\ & {\scriptstyle \text{proj.}} \searrow & \nearrow {\scriptstyle h} \\ & N_i & \end{array}$$

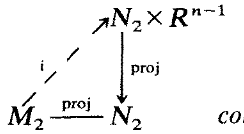
commute. $g0\pi$ is an immersion, so h is an immersion.

LEMMA 1. *The normal 2-plane bundle* $\nu_{N_2 \overset{h_2}{\hookrightarrow} S^{n+1}} = \nu_{h_2}$ *has a section.*

Proof. The normal bundle $\nu_{N_2 \hookrightarrow M}$ is trivialized by (say) the normal vector, v. $g_*(v)$ determines a linearly independent pair of vectors v_1 and v_2 in ν_{h_2}. v_1+v_2 defines the desired section.

COROLLARY 1. *If* $n=3$ *then* $\chi(\nu_{h_2})=0\in H^2(N_2; Z_{\text{twisted}})$ *where the coefficients are twisted by* $w_1(\tau(N_2))$ *when* N_2 *is nonorientable.*

We need to ask the question: When is there an imbedding $i: M_2 \to N_2 \times R^{n-1}$ *making the diagram*



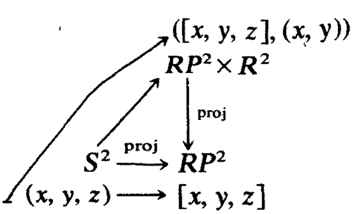
commute?

If ζ *is the line bundle associated to* $(M_2 \xrightarrow{\text{proj}} N_2)$, *i will exist if* ζ^{-1} *has geometric dimension* $\leq n-2$. *Since* $\dim(N_2)=n-1$ *this will happen if the Stiefel-Whitney class* $w_{n-1}(\zeta^{-1})=0$

From now on we consider the case $n=3$. Here $M_2 \xrightarrow{\text{proj}} N_2$ is a two fold covering of a possibly non-orientable surface by an orientable surface. If $w_1(\tau N_2)\neq 0$, $M_2 \xrightarrow{\text{Proj}} N_2$ is the orientation covering so $w_1(\zeta)=w_1(\tau N_2)$. In this case $\zeta\oplus\tau N_2$ is trivial since $w_1(\zeta\oplus\tau N_2)=w_1(\zeta)+w_1(\tau N_2)=0$ and $w_2(\zeta+\tau N_2)= w_1(\zeta)\cdot w_1(\tau N_2)+w_2(\tau N_2)=w_1(\zeta)\cdot w_1(\tau N_2)+(w_1(\tau N_2))^2=0$. As a result $\zeta^{-1}=\tau N_2$. If $w_1(\tau N_2)=0$, $w_1(\zeta\oplus\zeta\oplus\tau N_2)=w_1(\zeta)+w_1(\zeta)=0$, $w_2(\zeta\oplus\zeta\oplus\tau N_2)=w_1(\zeta)^2+ w_2(\tau N_2)=w_1(\zeta)^2+w_1(\tau N_2)^2=0+0=0$. So $\zeta^{-1}=\zeta+\tau N_2$. In both cases $w_2(\zeta^{-1})= w_2(\tau N_2)$, but $w_2(\tau N_2)[N_2]$ is congruent modulo 2 to the Euler characteristic $\chi(N_2)$ so $w_2(\zeta^{-1})[N_2]\equiv\chi(N_2) (\text{mod } 2)$. We now prove:

CLAIM. If i' is a generic immersion making the preceeding diagram commute, then $\#(\text{double points } (i'))\equiv\chi(N_2)$ (mod 2).

Proof. If the Euler characteristic of every component of N_2 is even then $w_{n-1}(\zeta^{-1})=0$ and, as stated above, i' may be chosen to be an imbedding. Any two choices for i' are regularly homotopic so $\#(\text{doublepoints } (i'))\equiv 0$ (mod 2) for any generic i'. For the general case we must consider the following example:



Note that $([0, 0, 1], (0, 0))$ is the only multiple value for i' and that i' is normal.

To remove a generic double point of an arbitrary i' one forms the connected sum $N_2 \# RP^2$ at $[0,0,1] \in RP^2$ and (the projection of the double point of i') $\in N_2$. Thus a generic double point of i' over a component of N_2 may be removed at the expense of lowering the Euler characteristic of that component by 1. This reduces the claim to the case first considered.

LEMMA 2. *The Hopf invariant* $H[g] \equiv \chi(N_2)$ (mod 2).

Proof. We use the following definition of the Hopf invariant of $\alpha \in \pi_n$. By the Freudenthal suspension theorem there is an $\alpha' \in \pi_{2n+1}(S^{n+1})$ which stablizes to α. Let $a: S^{2n+1} \to S^{n+1}$ represent α' and be transverse to $* \in S^{n+1}$. $a^{-1}(*)$ is a framed submanifold of dimension n in S^{2n+1}. Any frame vector determines a self-linking number $L(a^{-1}(*), a^{-1}(*'))$ which, modulo 2, is the Hopf invariant.

The composition $g': M \overset{g}{\looparrowright} S^{n+1} \hookrightarrow S^{n+1} \times R^{n-1}$, $s \longmapsto s \times 0$, is a framed immersion. The number of double points of a generic immersion, $\tilde{g}$, approximating g' is easily seen to be congruent modulo 2 to the self-linking number of a generic framed imbedding approximating $g'': M \overset{g}{\looparrowright} S^{n+1} \hookrightarrow S^{n+1} \times R^n$, $s \longmapsto s \times 0$. By our definition this self-linking number modulo 2 is $H[g]$. We will show #(double points $\tilde{g}) \equiv \chi(N_2)$ (mod 2).

$\tilde{g}$ can be chosen so that the diagram

$$\begin{array}{ccc} & & S^4 \times R^2 \\ & \overset{\tilde{g}}{\nearrow} & \downarrow \text{proj} \\ M & \xrightarrow{g} & S^4 \end{array}$$

commutes. The douple points of $\tilde{g}$ are the double points of $\tilde{g}/: \pi(M_2) \to g \circ \pi(M_2) \times R^2$. There is a generic immersion $j: M_2 \to N_2 \times R^2$ making

$$\begin{array}{ccc} \pi(M_2) & \xrightarrow{\tilde{g}/} & g \circ \pi(M_2) \times R^2 \\ \uparrow \pi/ & & \uparrow h_2 \times \mathrm{id} \\ M_2 & \longrightarrow & N_2 \times R^2 \end{array}$$

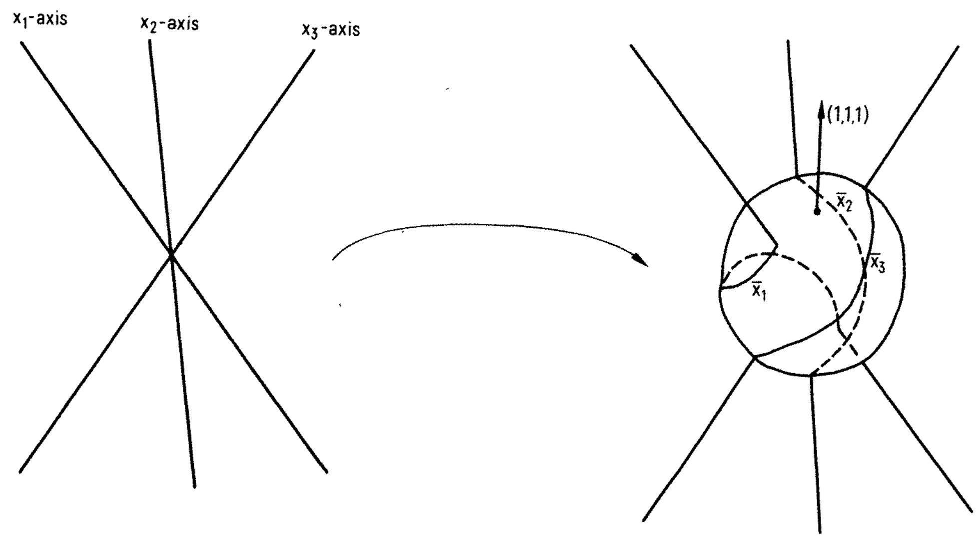
commute. Our characterization of g being generic implies that h_2 only identifies 0 and 1-simplexes of N_2. So the number of double points of j is equal to the number of double points of $\tilde{g}$. Lemma 2 now follows by setting $j = i'$ in the discussion immediately preceding its statement. ∎

If $g: M \looparrowright S^4$ is a generic immersion of an oriented 3-manifold, $h_2: N_2 \looparrowright S^4$ though not usually generic does have singularities of a special kind. As an analogy it is helpful to imagine the singularities of the double point set of a generically immersed surface in 3-space. The next lemma considers the case: q has no quadruple points. We analyse the singularities of h_2 to show that h_2 is regularly homotopic to a normal immersion with an even number of double points.

LEMMA 3. *If g has no quadruple points then* $h_2: N_2 \looparrowright S^4$ is regularly homotopic to a generic immersion with an even number of double points.

Proof. Let T be the subset of S^4 in the image of three distinct points under g. T is a finite family of circles. $h_2/: N - h_2^{-1}(1) \rightarrow S^4$ is an imbedding since $g o \pi/: M_2 \rightarrow M$ is $2-1$ on $M_2 \cap (g \times g)^{-1}(T \times T)$. From our characterization of generic maps, we see that some normal open $3-$disk $(= d^3)$ to T in S^4 may be parametrized to meet $h_2(N_2)$ in a $\{x_1\text{-axis} \cup x_3\text{-axis}\} \subset R^3$. Consider the distortion depicted below as a standard model for separating the sheets of $h_2(N_2)$ in a neighborhood of a point on T. h_2 is moved slightly in the normal directions to T.

Specifically if the x_1, x_2 and x_3-axes are generated by the vectors $x_1 = (1, 0, 0)$,



$x_2=(0,1,0)$ and $x_3=(0,0,1)$ the curves in diagram 1 are geodesic arcs $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, on the unit sphere determined by the condition that their midpoints be $(0,-\sqrt{2}/2,\sqrt{2}/2)$, $(\sqrt{2}/2,0,-\sqrt{2}/2)$ and $(-\sqrt{2}/2,\sqrt{2}/2,0)$ respectively. Let θ be the 3×3 matrix with these vectors as its rows.

If the model on the left for $h_2(N_2)\cap d^3$ is transported around a circle, c, of T the resulting monodromy of the axes may be represented by a 3×3 – orthogonal matrix, M, with the property that two entries in each row are zero and the remaining entry is ± 1. The i-th row indicates to which axis (and with which orientation) the i-th axis is transported. (We note that $\nu_{T\hookrightarrow S^4}$ is orientable so $\operatorname{Det}(M)=+1$). If the model on the right is invariant under the linear transformation (also denoted by M) defined by right multiplication by M, then our model may be used to separate the sheets of $h_2(N_2)$ along all of C. In general, though, separating these sheets along C will result in a finite number of generic double points; our present purpose is to calculate this number in terms of M. Put $\overline{x_iM}=\bar{x}_1$, $\bar{x}_2$, or $\bar{x}_3$ as $x_iM=\pm x_1$, $\pm x_2$, or $\pm x_3$. The model on the right is invariant under M iff $\bar{x}_iM=\overline{x_iM}$ for $i=1$, 2, and 3; if the above equality fails to hold we will see that $D(M)=\sum_{i=1}^{3}(1-(x_i\theta M)\cdot(x_iM\theta))(\text{mod } 2)$ ($\cdot$ denotes vector dot product) measures the failure. Note that $x_i\theta\perp x_i$ and $x_iM\theta\perp \times_i M$. Since M is orthogonal $x_1\theta M\perp \times_i M$, as a result $x_i\theta M$ and $x_iM\theta$ both lie in the plane P_i perpendicular to x_iM and must have one of four possible coordinates (restricting our coordinate system to this plane) in that plane: $(\pm\sqrt{2}/2,\pm\sqrt{2}/2)$. The number, $(1-(x_i\theta M)\cdot(x_iM\theta))$, is equal (mod 2) to the number of times a transverse arc, γ_i, in P_i from $x_i\theta M$ to $x_iM\theta$ must cross the coordinate axes. The arc γ_i determines a homotopy from $\bar{x}_iM$ to $\overline{x_iM}$ through geodesic arcs. Using the model on the right for most of C and then "splicing in" this homotopy at the end we may separate the sheets of $h_2(N_2)$ along all of C with generic double points resulting from transverse crossings of the coordinates axes by γ_i. It follows that h_2 is regularly homotopic to a general immersion with $\sum D(M)$ double points, where the sum is taken over each circle component to T.

We complete the proof of Lemma 3 by showing that for every admissible M, $D(M)\equiv 0 \pmod 2$. $D(M)\equiv 1-\sum_{i=1}^{3}(x_i\theta M)\cdot(x_iM\theta)\equiv 1-\sum_{i,j=1}^{3}(\theta M)_{ij} \pmod 2$. Put $(\bar{M})_{ij}=|(M)_{ij}|$. All the non-zero terms in the last sum are $\pm 1/2$, replacing M by $\bar{M}$ reverses an even number of these signs so we have $D(M)\equiv 1-\sum_{i,j=1}(\theta\bar{M})_{ij}(\bar{M}\theta)_{ij} \pmod 2$. If $\bar{M}$ is a simple transposition $\theta\bar{M}=(\theta\bar{M})^T=\bar{M}^T\theta^T=-\bar{M}\theta$ so $D(M)\equiv 1+\sum_{i,j=1}^{3}(\theta\bar{M})_{ij}^2=1+\sum_{i,j=1}^{3}(\theta)_{ij}^2=1+3\equiv 0$. If $\bar{M}$ is a cycle of order 3, one checks that $\theta\bar{M}=\bar{M}\theta$ so again $D(M)\equiv 0 \pmod 2$. The lemma follows. ∎

When g has an even $\neq 0$ number of quadruple points, we perform some oriented 0-surgeries to enlarge our ambient manifold S^4 to $\underset{k\text{-copies}}{\#}(S^1\times S^3)$. We note that if one chose to, this freedom could be built in from the start; our bordism group, B_n, is isomorphic to "bordism of immersions of oriented 3-manifolds in stably framed 4-manifolds". An oriented 0-surgery is the operation of removing an imbedded $S^0 \times D^n$ from an oriented n-manifold and gluing back $D^1 \times S^{n-1}$ in a standard manner so as to obtain a new oriented manifold. The notion is often generalized to an operation on a pair, (oriented n-manifold, oriented $(n$-1) dimensional submanifold). Below we will perform oriented 0-surgery with $S^0 \times 0$ imbedded on a pair of generic quadruple points of a immersed 3-manifold in S^4; for this an additional but obvious extension of the notion is required. Rather than give an abstract definition, we have written out the results of our 0-surgery on $(S^4, g(M))$.

Let $q, q', \ldots, q_k, q_k'$ be the quadruple points of g arbitrarily paired. For each pair (q_i, q_i') we perform an oriented 0-surgery on S^4 and a corresponding modification of g. In terms of the image of g the result of a single surgery is: (S^4, image(g) $-(S^0 \times D^4, S^0 \times (\cup_{\text{hyperplanes}} x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0)$

$$\cup (D^1 \times S^3, D^1 \times \left(S^3 \cap \bigcup_{\text{hyperplane}} x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0\right)$$

Call the new immersion $\bar{g} : \bar{M} \to \#_k S^1 \times S^3$.

If within each chart, D^4, about a quadruple point of g the positive direction along the 4 axes is consistently determined by the difference of the orientations on S^4 and M, the new manifold $\bar{M}$ will be oriented, and in fact diffeomorphic to $M \#_{j=1}^{4k} (S^1 \times S^2)_j$. Let $\bar{M}_2$ and $\bar{N}_2$ correspond to M_2 and N_2. As proved for N_2, $\bar{N}_2$ is immersed (by $\bar{h}_2$) in $\#_k (S^1 \times S^3)$ with $\chi(\nu_{\bar{h}_2}) = 0$. $\bar{N}_2$ abstractly is the result of $\binom{4}{2} k = 6k$ 0-surgeries on N_2. Since a 0-surgery does not change the Euler characteristic modulo 2, $\chi(\bar{N}_2) \equiv \chi(N_2)$ (mod 2). We are ready to prove:

LEMMA 4. *If g has an even number of quadruple points, there is a surface $\bar{N}_2$ satisfying*:

1) $\chi(\bar{N}_2) \equiv \chi(N_2)$ (mod 2)
2) *$\bar{N}_2$ is generically immersed in S^4 with an even number of double points; call its normal bundle ν.*
3) $\chi(\nu) = 0 \in H^2(\bar{N}_2; Z_{\text{twisted}})$.

Proof. The $\bar{N}_2$ constructed above is immersed in $\nabla_k (S^1 \times S^3)$ with the above normal bundle condition. The proof of Lemma 3 shows how to regularly homotop this immersion to satisfy condition 2. $\bar{N}_2 \to \nabla_k S^1 \times S^3$. Framed surgery on k

circles in ($\nabla_k(S^1\times S^3)-$image ($\bar{N}_2$)) returns the ambient manifold to S^4 without affecting the normal bundle of $\bar{N}_2$. ∎

A theorem of Whitney's [W] says that if a compact surface, Q, is imbedded in S^4 with normal bundle ν and $\chi(\nu)=m\cdot$ generator $\in H^2(Q;Z_{\text{twisted}})$ then $m\equiv 2\chi(Q)\pmod 4$. The introduction of a double point changes the twisted Euler class $\chi(\nu)$ by $\pm 2\cdot$ generator. As a result, Whitney's theorem stated for immersions of Q in S^4 becomes: $m\equiv 2\chi(Q)\pm 2(\#\text{double points of } Q)\pmod 4$. If g has an even number of quadruple points Whitney's theorem for immersions and Lemma 4 show that $\chi(\bar{N}_2)$ and therefore $\chi(N_2)$ is even. Lemma 2 now says that $H[g]=0$. Thus we have $\theta_3[g]=0$ implies $H[g]=0$, i.e. $\ker(H)\supset\ker(\theta_3)$. Since $H:\pi_3\to Z_2$ is well known to be an epimorphism, θ_3: $\pi_3\to Z_2$ is also epic. Since $\pi_3\cong Z_{24}$, θ_3 is completely determined, we have proved:

THEOREM. $\theta_3:\pi_3\to Z_2$ *is the unique epimorphism.*

§4. *Remarks and problems*

Remark 1. Since the J_3-homomorphism: $\pi_3(SO)\to\pi_3$ is onto, every element of B_3 is realized by an immersed 3-sphere. In particular there is a generic immersion of S^3 in S^4 with an odd number of quadruple points.

Remark 2. There is no local argument for converting quadruple points of $M\looparrowright S^4$ to double points of $M\looparrowright S^4\times R^2$ as inspection of the immersion $4(T^3)\looparrowright T^4$ obtained by omitting successive circle factors will show. It seems to be necessary to work down through the strata to prove our theorem, so analogous computations for $n>3$ are likely to be more difficult.

Remark 3. In this paper we have gone to great trouble to express the Hopf invariant in terms of the lowest dimensional strata of a generic immersion $g:M^3\to S^4$, and our arguments have been special to the dimensions involved. There is, however, a simple way in every dimension of reading off the Hopf invariant from the highest dimensional strata, the double point set. If ξ is the line bundle associated to $M_2\xrightarrow{\text{Proj}}N_2$, $H(g)=0$ iff $w_{n-1}(\varepsilon^{-1})=0$ on all but an even number of path components of N_2. This is easily seen by comparing our definition of Hopf invariant with our solution to the "question" preceding corollary 2.

PROBLEM 1. Is there a generic immersion of S^3 in S^4 with a single quadruple point?

PROBLEM 2. Explicitly construct a generic immersion of S^3 in S^4 with an odd number of quadruple points.

PROBLEM 3. Compute θ_n for $n > 3$.

Conjecture. θ_n is the stable Hopf invariant.

REFERENCES

[B] THOMAS F. BANCHOOF. Triple points and surgery of immersed surfaces, Proc. A.M.S. Vol. 46 No. 3 Dec 1974

[W] H. WHITNEY. On the Topology of Differentiable Manifolds, Lectures in Topology, Mich. Univ. Press, 1940

Mathematics, University of California
San Diego, La Jolla, Calif. 92093 U.S.A.

Received April 25/September 26, 1977