Comment. Math. Helvetici 70 (1995) 1–9 0010-2571/95/010001-09$1.50 + 0.20/0

Harmonic cohomology classes of symplectic manifolds

OLIVIER MATHIEU

Introduction

First recall some definitions and some results of Hodge theory. Let X be an oriented riemannian manifold and let $d^* : A_k(X) \to A_{k-1}(X)$ be the associated operator which is dual to the de Rham operator d (where $A_*(X)$ denotes the space of smooth complex valued forms). A form α is called harmonic if it satisfies $d\alpha = d^*\alpha = 0$. One of the main results of Hodge theory states that when X is compact any cohomology class contains exactly one harmonic form.

The aim of this paper is to investigate similar questions for symplectic manifolds (as opposed to riemannian onés).

Let us assume that we are given a symplectic manifold (X, ω) of dimension $2m$. According to J. L. Kozsul [11] and J. L. Brylinski [4], one can similarily define the operator d^* and the notion of harmonic form (however d^* is denoted Δ or δ in loc. cit.). Define the harmonic cohomology $H^*_{har}(X)$ to be the space of all cohomology classes which contain at least one harmonic form. Our result is the following characterization of $H^*_{har}(X)$ as a subspace of $H^*(X)$. Let $G = SL(2)$ and let B be the subgroup of all upper triangular matrices. For a rational B-module M, there exists a unique maximal submodule $\mathcal{D}M$ which is a quotient of a rational G-module (an explicit construction of it will be given in section 2). In fact $H^*(X)$ has a canonical structure of B-module. The corresponding infinitesimal action is generated by the cup-product by $[\omega]$ and the operator $deg - m$, where deg is the degree operator. We then prove.

THEOREM 1. *We have* $H^*_{har}(X) = \mathcal{D}H^*(X)$.

Roughly speaking, theorem 1 means that we can characterize the harmonic cohomology classes in terms of $[\omega]$-divisibility. The proof of the result is an easy consequence of a classification result for representations of the Lie super-algebra $\mathbf{sl}(2) \times \mathbf{C}^2$. As corollary of the theorem we get.

COROLLARY 2. *Assume that X is compact. Then the following two assertions are equivalent.*

(2.1) *Any cohomology class contains at least one harmonic form.*

(2.2) *For any $k \leq m$ the cup-product $[\omega]^k : H^{m-k}(X) \to H^{m+k}(X)$ is an isomor phism.*

Actually assertion 2.2 is often satisfied. When X is a projective algebraic variety assertion 2.2 is nothing but the strong Lefschetz theorem. Assertion 2.1 has been proved for compact Kaehler manifolds and conjectured for general compact symplectic manifolds by J. L. Brylinski in [4] (see introduction and section 2.2 of [4]). Therefore in order to disprove Brylinski conjecture it suffices to give an example of a compact symplectic manifold which does not satisfy the strong Lefschetz theorem. Then we check that a some four-dimensional symplectic nilmanifolds X do not satisfy the statement of the strong Lefschetz theorem (see example, 10, 12). This example has been kindly communicated to us by Y. Benoist. Actually nilmanifolds have been already extensively used by various authors to give examples of symplectic manifolds not satisfying various properties of algebraic or complex varieties (see [1], [3], [5], [6], [7], [9], [10], [13], [14], [16] and [18]).

Remark. In [17] Dong Yan found a simpler proof of Corollary 7.

1. Indecomposable representations of the Lie super-algebra sl(2) $\times$ $\mathbf{K}^2$

Set $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. In order to describe the indecomposable representations of the Lie super-algebra $\mathbf{sl(2, K)} \times \mathbf{K}^2$, we will first describe the representation theory of a certain quiver Q. The vertices of Q will be indexed by symbols n_+ and n_-, where n runs over the set of all non-negative integers. Any vertex n_+ with $n > 0$ is the origin of exactly two arrows, with targets $(n-1)_-$ and $(n+1)_-$. The vertex 0_+ is the origin of an arrow with target 1_-. The quiver has two infinite connected components and its picture is as follows.

$$\circ_{0+} \longrightarrow \circ_{1-} \longleftarrow \circ_{2+} \longrightarrow \circ_{3-} \cdots$$

$$\circ_{0-} \longleftarrow \circ_{1+} \longrightarrow \circ_{2-} \longleftarrow \circ_{3+} \cdots$$

By definition the support of a representation $E = \oplus_{\gamma \in Q} E_\gamma$ of the quiver Q is the set $\{\gamma \in Q \mid E_\gamma \neq 0\}$.

For any non-negative integers $a \leq b$ of the same parity we set $[a_+, b_+] = \{a_+, (a+1)_-, (a+2)_+, \ldots, b_+\}$ and $[a_-, b_-] = \{a_-, (a+1)_+, (a+2)_-, \ldots, b_-\}$.

Similarily when $a < b$ have different parities we set $[a_+, b_-] = \{a_+, (a+1)_-, (a+2)_+, \ldots, b_-\}$ and $[a_-, b_+] = \{a_-, (a+1)_+, (a+2)_-, \ldots, b_+\}$. These sets $[a_\pm, b_\pm]$ are called the finite intervals of Q (where we have assumed that the parities and the signs of $a_\pm$ and $b_\pm$ are simultaneously equal or different). Clearly the finite intervals are exactly all the finite connected subsets of the quiver Q. For any finite interval I of Q, let $E = E(I)$ be the representation of Q defined as follows. As a vector space we have $E_\gamma = \mathbf{K}$ if $\gamma \in I$ and $E_\gamma = 0$ if not. Moreover any arrow $\varepsilon : n_+ \to m_-$ between two vertices in I acts from E_{n_+} to E_{m_-} as 1 and the other arrows act (necessarily) as zero. Let $\mathscr{C}$ be the category of all representations of Q with finite support.

LEMMA 3. (3.1) *Any representation* $E \in \mathscr{C}$ *is a direct sum of indecomposable representations.*

(3.2) *Any indecomposable representation* $E \in \mathscr{C}$ *is one of the* $E(I)$.

Proof. The first statement is a general non-sense statement. Let E be any indecomposable representation in $\mathscr{C}$ and let I be its support. Then I is connected and E can be seen as an indecomposable representation of the subquiver I. As I is a quiver of Dynkin type, the staement follows from Gabriel theorem [8]. Q.E.D.

Let $\mathfrak{a} = \mathfrak{g} \oplus V$ be the Lie super-algebra over $\mathbf{K}$ defined as follows. Its degree 0 part is the subspace $\mathfrak{g}$ with basis $\{e, f, h\}$ and Lie brackets $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. As Lie algebra it is isomorphic with $\mathbf{sl}(2)$. The degree one part V has basis $\{d, d^*\}$ and is an abelian Lie super-algebra. The remaining brackets $[e, d] = 0$, $[h, d] = d$, $[f, d] = d^*$, $[e, d^*] = d$, $[h, d^*] = -d^*$, $[f, d^*] = 0$ correspond with the natural action of $\mathbf{sl}(2)$ over the two-dimensional space. Thus $\mathfrak{a}$ is the Lie super-algebra $\mathbf{sl(2)} \times \mathbf{K}^2$.

Let $\mathscr{V}$ be the category of all $\mathfrak{a}$-modules M on which h acts diagonally with only finitely many different eigenvalues (the multiplicity of each eigenvalue could be infinite). In order to simplify the statements, we do not require that the $\mathfrak{a}$-modules are $\mathbf{Z}/2\mathbf{Z}$-graded. We will now define two families of $\mathfrak{a}$-modules.

Definition of the $\mathfrak{a}$*-modules* $I(n)$. For any non-negative integer n, let $L(n)$ be the unique simple $\mathfrak{g}$-module of dimension $n + 1$. Let $I(n) = Ind(\mathfrak{g}, \mathfrak{a})L(n)$ be the induced module (actually $I(n)$ is also coinduced from $L(n)$). As a vector space, we have $I(n) = L(n) \otimes \bigwedge V$. Let M be any $\mathfrak{a}$-module in the category $\mathscr{V}$. The hypothesis on the eigenvalues of h implies that as $\mathfrak{g}$-module M is a direct sum of finite dimensional simple representations. (In particular M is a rational representation of the corresponding group $G = SL(2)$.) Thus the module $I(n)$ is projective in the category $\mathscr{V}$. As $I(n)$ is self dual, $I(n)$ is injective as well.

Definition of the $\mathfrak{a}$*-modules* $V(I)$. Recall that $L(n) \otimes V = L(n-1) \oplus L(n+1)$ for any $n > 0$. So there is an arrow in Q from n_+ to m_- exactly when there is an intertwining operator from $V \otimes L(n)$ to $L(m)$. Moreover such an intertwining operator is unique up to scalar. Thus for any arrow $\varepsilon : n_+ \to m_-$ we will denote by $\Theta_\varepsilon : V \otimes L(n) \to L(m)$ the corresponding operator.

For any interval I of Q we define a representation $V(I)$ of $\mathfrak{a}$ as follows. As $\mathfrak{g}$-module, we have $V(I) = \oplus_{\gamma \in I} L(|\gamma|)$, where for any $n_\pm \in Q$ we set $|n_\pm| = n$. In order to define the action $\mu : V \otimes V(I) \to V(I)$ of V on $V(I)$ one only needs to define its components $\mu_{\gamma,\gamma'} : V \otimes L(|\gamma|) \to L(|\gamma'|)$, for $\gamma, \gamma' \in I$. It is given by $\mu_{\gamma,\gamma'} = \Theta_\varepsilon$ if $\varepsilon : \gamma - \gamma'$ is an arrow and by $\mu_{\gamma,\gamma'} = 0$ otherwise.

Note that the representations $V(I)$ and $I(n)$ are indecomposable and non-isomorphic up to the case $V([a_+, a_+]) \simeq V([a_-, a_-])$, i.e. when V acts trivially. This representation will be simply denoted by $L(a)$.

PROPOSITION 4. (4.1) *Any $\mathfrak{a}$-module in $\mathscr{V}$ is a direct sum of indecomposable representations.*

(4.2) *Any indecomposable $\mathfrak{a}$-module in $\mathscr{V}$ is isomorphic to some $V(I)$ or to some $I(n)$.*

Proof. As previously the first statement is an obvious categorical statement. We will now prove the second statement. Let M be an indecomposable representation in $\mathscr{V}$. Set $C = d.d^*$.

(1) First assume that $C.M \neq 0$. Choose a simple $\mathfrak{g}$-submodule $L \subset M$ with $\Delta.L \neq 0$ and set $n + 1 = dim(L)$. Let M' be the $\mathfrak{a}$-submodule generated by L. Clearly we have $M' \simeq I(n)$. As $I(n)$ is an injective module, we have $M = M' \simeq I(n)$.

(2) Assume now that we have $C.M = 0$. Let L be the space of V-invariant vectors in M and let L' be some $\mathfrak{g}$-invariant complement. For any integer $n \geq 0$, let L^n and L'^n be the isotypical component of type $L(n)$ in L and L'. Note that $V.L' \subset L$. Then define a representation E of the quiver Q as follows. As vector space set $E_{n_+} = Hom_{\mathfrak{g}}(L(n), L')$ and $E_{m_-} = Hom_{\mathfrak{g}}(L(n), L)$. For any arrow $\varepsilon : n_+ \to m_-$ of Γ, let $\Psi_\varepsilon : L'^n \to L^m$ be the corresponding component of the action $\Psi : V \otimes L' \to L$ of V on M. Identify $L^n \simeq E_{n_+} \otimes L(n)$ and $L'^m \simeq E_{m_-} \otimes L(m)$. Then define the action of ε as the map $\rho(\varepsilon) : E_{n_+} \to E_{m_-}$ by the formula $\Psi_\varepsilon = \rho(\varepsilon) \otimes \Theta_\varepsilon$. Note that we call recover the $\mathfrak{a}$-module M from the quiver representation E. Hence E is an indecomposable representation of Γ. By Lemma 3, E is isomorphic to $E(I)$ for some finite interval I. Hence M is isomorphic to $V(I)$. Q.E.D.

Let us introduce some notations. We will denote by G the group $SL(2)$ and by B its subgroup of upper triangular matrices. Let $\mathfrak{g}$ and $\mathfrak{b}$ be the corresponding Lie algebras. We can choose a basis $\{e, f, h\}$ of $\mathfrak{g}$ in a such way that

(1) it satisfies $[h, e] = 2e, [h, f] = -2f, [e, f] = h$.
(2) $\{e, h\}$ is a basis of $\mathfrak{b}$.

By G-module (or B-module) we mean rational G (or B-module). Any B-module M admits a weightspace decomposition $M = \oplus_{n \in \mathbf{Z}} M_n$, where $M_n = \{m \in M \mid h.m = n.m\}$. The weights of M are the integers n such that $M_n \neq 0$. Denote by $\mathscr{V}_B$ (respectively $\mathscr{V}_G$) be the category of all B-modules (respectively G-modules) with finitely many weights. Note that a G-module M belongs to $\mathscr{V}_G$ if its decomposition into isotypical components is finite (however the dimension of each isotypical component could be infinite).

Let M be a G-module, and let $\rho : \mathfrak{g} \to End(M)$ be the corresponding infinitesimal representation. For any interger n denote by $M[n]$ the B-module with the same underlying space and whose infinitesimal action is given by $e.x = \rho(e).x$ and $h.x = (\rho(h) + n).x$ for any $x \in M[n]$ (in other words we get this new representation of B by twisting by the character $n\omega$ where ω is the fundamental character of B). For any B-module M let $\mathscr{D}M$ be the maximal submodule which is a quotient of a rational G-module.

LEMMA 5. *Let* $M \in \mathscr{V}_B$.

(5.1) *There are some* $M(n) \in \mathscr{V}_G$ *such that* $M \simeq \oplus_{n \in \mathbf{Z}} M(n)[n]$. *Moreover we have* $M(n) = 0$ *for almost all* $n \in \mathbf{Z}$.

(5.2) Set $\mathscr{F}_n M = \oplus_{m \geq n} M(m)[m]$. *This gives rise to a filtration* $\cdots \mathscr{F}_n M \subset \mathscr{F}_{n-1} M \cdots$ *of* M. *This filtration is independant of the decomposition* (5.1).

(5.3) *We have* $\mathscr{D}M = \mathscr{F}_0 M$. *In particular we have* $\mathscr{D}M[n] = M[n]$ *if* $n \leq 0$ *and* $\mathscr{D}M[n] = 0$ *when* $n > 0$ *for any* $M \in \mathscr{V}_G$.

(5.4) *We have* $M = \mathscr{F}_0 M$ *if and only if for any integer* $k \geq 0$ *the map* $e^k : M_{-k} \to M_k$ *is onto.*

Proof. A B-module isomorphic to $L[n]$ where L is a simple G-module is called a string module. It is easy to show that any $M \in \mathscr{V}_B$ is a direct sum of string modules. Thus the existence of the decomposition (5.1) follows easily.

A one dimensional B-modules is isomorphic to $\mathbf{C}[n]$, for some integer n. Clearly $\mathbf{C}[n]$ is a B-factor of a G-module if and only if $n \leq 0$. As we have $\mathscr{D}L[n] = L \otimes \mathscr{D}(\mathbf{C}[n])$ for any $L \in \mathscr{V}_B$, the formula (5.3) follows. Then we deduce the assertion (5.2) from the formula $\mathscr{F}_n M = (\mathscr{D}L[n])[-n]$. Assertion (5.4) is easy. Q.E.D.

For any $\mathfrak{a}$-module $M \in \mathscr{V}$, set $h(M) = (Ker\, d\,|_M)/(Im\, d\,|_M)$ and $M_{har} = Ker\, d\,|_M \cap Ker\, d^*\,|_M$. Moreover let $h_{har}(M)$ be the image of M_{har} in $h(M)$. By definition the spaces $h(M)$ and $h_{har}(M)$ are called (respectively) the cohomology of M and the harmonic cohomology of M. As B normalizes $\mathbf{C}d$, these spaces are actually B-modules.

PROPOSITION 6. *For any* $M \in \mathscr{V}$, *we have* $h_{har}(M) = \mathscr{D}h(M)$.

Proof. First note that M_{har} is already a G-module. So we have $h_{har}(M) \subset \mathscr{D}h(M)$. It suffices to check the proposition when M is indecomposable. We will check the proposition by a case-by-case analysis by using the classification of indecomposable modules given in Proposition 4. Note that when $\mathscr{D}h(M) = 0$ the assertion is obvious.

First case: $M = I(n)$. In that case we have $h(M) = 0$.

Second case: $M = L(a)$. In that case we have $h(M) = h_{har}(M) = L(a)$.

Third case: $M = V([a_+, b_-])$. In this case $Ker\, d = M_{har}$, and $h(M) = h_{har}(M) \simeq L((b-a-1)/2)[-(a+b+1)/2]$.

Fourth case: $M = V[a_-, b_-])$. In that case $Ker\, d = M_{har}$, and $h(M) = h_{har}(M) \simeq L((a+b)/2)[(a-b)/2]$.

Fifth case: $M = V([a_-, b_+])$. In that case $h(M) \simeq L((b-a-1)/2)[(a+b+1)/2]$. In particular $\mathscr{D}h(M) = 0$.

Sixth case: $M = V([a_+, b_+])$. In that case $h(M) \simeq L((a+b)/2)[(b-a)/2]$. In particular $\mathscr{D}h(M) = 0$.

In the last four cases $a < b$ are non-negative integers with the correct parity. The proof in the first two cases are obvious. The third and fourth cases can be easily proved by induction. We then deduce the proof for the last two cases by duality.

2. A characterization of harmonic cohomology classes

Set $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. For a manifold X, denote by $A_*(X)$ the space of $\mathbf{K}$-valued forms. By symplectic form we mean a K-valued closed 2-form ω such that ω^m never vanishes, where $m = 1/2\, dim\, X$ (usually one requires that ω is $\mathbf{R}$-valued).

Proof of Theorem 1. Let (X, ω) be a symplectic manifold of dimension $2m$. Let v be the corresponding 2-vector. Let h be the endomorphism of $A_*(X)$ which acts over $A_k(X)$ as $k - m$. Set $e = e(\omega), f = i(v)$. Then following operators [2], [15], e, f and h spans a Lie algebra isomorphic to $\mathbf{sl}(2)$. Moreover the operators d and d^* spans a two-dimensional $\mathbf{sl}(2)$-module. As we have $d^2 = d^{*2} = d.d^* + d^*.d = 0$, the span of e, f, h, d, d^* is precisely the Lie super-algebra $\mathfrak{a}$. The space of forms, viewed as $\mathfrak{a}$-module belongs to class $\mathscr{V}$. Moreover the cohomology and the harmonic cohomology of M as defined in section 1 coincides with the cohomology and the harmonic cohomology of X. Thus Theorem 1 follows from Proposition 6.

COROLLARY 7. *Let X be a (not necessarily compact) symplectic manifold of dimension $2m$. Then we have $H^*(X) = H^*_{har}(X)$ if and only if for any $k \leq m$ the cup-product $[\omega]^k : H^{m-k}(X) \to H^{m+k}(X)$ is onto.*

Proof. Corollary 7 is an easy consequence of the theorem together with Lemma 5.4.

Proof of Corollary 2. Corollary 2 follows from Corollary 7 and Poincaré duality.

COROLLARY 8. *(Assume $K = \mathbf{R}$) Let X be a symplectic manifold of dimension $2m$. Any cohomology class of degree 2 contains a harmonic form.*

Proof of Corollary 8. Let E be the kernel of the map $[\omega]^{m-1} : H^2(X) \to H^{2m}(X)$, $[\alpha] \mapsto [\alpha].[\omega]^{m-1}$. We claim that the following equality: $H^2(X) = E + \mathbf{K}.[\omega]$ holds. Actually when X is non-compact we already have: $H^2(X) = E$. In the compact case we have $[\omega^m] \neq 0$. So we have $H^2(X) = E \oplus \mathbf{K} \,.\, [\omega]$. Clearly we have $K \subset \mathscr{D}H^*(X)$ and $[\omega]$ is a harmonic form. Thus Corollary 8 comes from the theorem.

Remarks. (1) *By Corollary* 7, *we have* $H^*(X) = H^*_{har}(X)$ whenever $H^i(X) = 0$ for $i > m$ (compare with Proposition 2.2.12 and Corollary 2.2.13 of [4]).

(2) Any closed form or degree 1 is harmonic (see Brylinski [4]; actually its proof works also for $\mathbf{K} = \mathbf{C}$). However some cohomology classes of degree 3 are not harmonic (see section 3).

3. Counter-examples to Brylinski conjecture

Let $\mathfrak{n}$ be the real nilpotent Lie algebra with basis e_1, e_2, e_3, e_4 and with Lie brackets $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$ and $[e_i, e_j] = 0$ for $i + j \geq 5$. Let $(\varepsilon_i)_{1 \leq i \leq 4}$ be the dual basis. Set $\omega_1 = \varepsilon_1 \wedge \varepsilon_4$, $\omega_2 = \varepsilon_2 \wedge \varepsilon_3$ and $\omega = \omega_1 + \omega_2$. Note that ω_1 and ω_2 are cocycles.

LEMMA 9. *The cohomology groups $H^1(\mathfrak{n})$ and $H^3(\mathfrak{n})$ have dimension two. However the multiplication map $[\alpha] \in H^1(\mathfrak{n}) \mapsto [\alpha].[\omega] \in (\mathfrak{n})$ is zero.*

Proof. Note that $H^1(\mathfrak{n}) \simeq \mathfrak{n}/([\mathfrak{n}, \mathfrak{n}])^*$ has dimension two and is generated by $[\varepsilon_1]$ and $[\varepsilon_2]$. Hence by Poincaré duality, $H^3(\mathfrak{n})$ also has dimension two.

We have $\omega_1 \wedge \varepsilon_2 = d(\varepsilon_3 \wedge \varepsilon_4)$, $\omega_2 \wedge \varepsilon_2 = d(\varepsilon_2 \wedge \varepsilon_4)$ and $\omega_1 \wedge \varepsilon_1 = \omega_2 \wedge \varepsilon_2 = 0$. Hence we have $[\omega] \,.\, H^1(\mathfrak{n}) = 0$. Q.E.D.

Let N be the connected simply connected real Lie group with Lie algebra $\mathfrak{n}$. Actually N is a semi-product $\mathbf{R} \times \mathbf{R}^3$. Thus N contains some cocompact discrete subgroup Γ (the natural structure constant are rational). Set $X = N/\Gamma$ and identify $\mathfrak{n}$ with the space of all right invariant vector fields on N. In particular ω is Γ right-invariant. Thus it defines a symplectic form over X. We will still denote this form by ω.

EXAMPLE 10. For the compact symplectic nilmanifold (X, ω) we have $H^*(X) \neq H^*_{har}(X)$, i.e. it does not satisfy Brylinski conjecture.

Proof. There is a canonical isomorphism $H^*(X) \simeq H^*(\mathfrak{n})$ (Nomizu, see e.g. [12]). Following Lemma 9, we get $[\omega].H^1(X) = 0$ and $H^3(X) \neq 0$. Using Corollary 2, we get $H^*_{har} \neq H^*(X)$. Q.E.D.

Actually there is another way to give counter-examples to Brylinski conjecture. For a manifold X, set $b_i = \dim H^i(X)$.

LEMMA 11. *Let (X, ω) be a compact symplectic manifold of dimension $2m$. If X satisfy Brylinski conjecture, then its odd degree Betti numbers b_{2i+1} are even.*

Proof. Let i be an integer, $0 \leq i \leq m$. Assume that X satisfy Brylinsky conjecture. By Corollary 2.2 and by Poincaré duality the bilinear map $[\alpha], [\beta] \in H^i(X) \mapsto \langle [\alpha].[\beta].[\omega]^{m-i} \mid X \rangle$ is non-degenerated. When i is odd, this bilinear map is skew-symmetric. It follows easily that all odd degree Betti numbers are even.

EXAMPLE 12. Any four-dimensional nilmanifold whose first Betti number b_1 is 3 does not satisfy Brylinski conjecture (see e.g. [7] for such an example of nilmanifolds).

Remark. As the referee kindly pointed out, one can prove that the manifold X of example 10 does not satisfy Brylinski conjecture without using Theorem 1. Actually if a cohomology class is harmonic, then it contains an harmonic and N-invariant cocycle. Thus one can disprove Brylinski conjecture by easy computations in the (finite dimensional) complex $\bigwedge \mathfrak{n}^*$.

Acknowledgements

We thank Y. Benoist which provides us an example of compact symplectic variety not satisfying the strong Lefschetz theorem. We also thank J. L. Brylinski, M. Duflo, E. Ghys, A. Weinstein and the referee for useful and helpful comments or discussions. We thank the Sloan foundation for its generous support.

REFERENCES

[1] E. ABBENA, *An example of an almost Kaehler manifold which is not Kaehlerian*, Boll. Un. Mat. Ital. 6 (1984) 383–392.
[2] J. N. BERNSTEIN, *The Lie super-algebra osp (1,2) over symplectic manifolds and representations of Poisson algebras*, Seminar on super-manifold, ed Leites, Stockholm University (1977).
[3] N. BROTHERTON, *Some parallelizable four manifolds not admitting a complex structure*, Bull. London Math. Soc. *10* (1978) 303–304.
[4] J. L. BRYLINSKI, *A differential complex for Poisson manifold*, J. of Diff. Geo. *28* (1988) 93–114.
[5] L. Cordero, M. Fernández and A. Gray, *Symplectic manifolds without Kaehler structures*, Topology *25* (1986), 375–380.
[6] L. Cordero, M. Fernández and M. de León *Examples of compact non-Kaehler almost Kaehler manifolds*, Proc. Am. Math. Soc. *95* (1985), 280–286.
[7] M. Fernández, M. Gotay and A. Gray, *Compact parallelizable four-dimensional symplectic and complex manifolds*, Proc. Amer. Math. Soc. *103* (1988) 1209–1212.
[8] P. Gabriel, *Unzerleqbare darstellungen I*, Manuscripta Math. *6* (1972) 71–103.
[9] M. J. GOTAY, *A class non-polarizable symplectic manifolds*, Monatsch. Math. *103* (1987) 27–30.
[10] K. KODAIRA, *On the structure of compact complex analytic surfaces I*, Amer. J. Math. *86* (1964) 751–798.
[11] J. L. KOSZUL, *Crochet de Shouten-Nijenjuis et cohomologie* in "Elie Cartan et les mathèmatique d'aujourd'huis", Asterisque (1985) 251–271.
[12] M. S. RAGHUNATHAN, *Discrete subgroups in Lie groups*, Springer-Verlag, Berlin. Ergebnisse der Math. *68* (1972).
[13] W. P. THURSTON, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. 55 (1976) 467–468.
[14] A. VAN DE VEN, *On the Chern numbers of certain complex and almost complex manifolds*, Proc. Nat. Sci. U.S.A. *55* (1966), 1624–1627.
[15] A. WEIL, *Varietes Kahleriennes*, Hermann, Paris. Publ. Inst. Math. Univ. Nancago, *6* (1958).
[16] A. WEINSTEIN, *Lectures on symplectic manifolds*, CBMS Regional Conf. Ser. in Math., *29*. Amer. Math. Soc., Providence, R.I. (1977).
[17] D. YAN, *Hodge Theory on Symplectic Manifolds*, Preprint.
[18] S. T. YAU, *Parallelizable manifolds without complex structures*, Topology *15* (1976) 51–53.

Rutgers University
Hill Center
New Brunswick
NJ 09803
USA

and

U.A. 1 du CNRS
7, rue Rene Descartes
67000 Strasbourg
France

Received April 22, 1992; revised September 8, 1994