Local Triviality of the Restriction Map for Embeddings[1])

RICHARD S. PALAIS, Princeton, N.J. (USA)

Let M and W be differentiable manifolds, V a compact submanifold of W, and let $E(W, M)$ denote the space of differentiable embeddings of W in M in the C^r topology where $1 \leq r \leq \infty$ (see below for precise definitions). In a set of mimeographed notes R. THOM has stated that the restriction map $\pi: E(W, M) \to E(V, M)$ defined by $\pi(f) = f|V$, is a fiber space map in the sense of SERRE (i.e. has the covering homotopy property for polyhedra), however the proof indicated shows only that homotopies satisfying certain strong differentiability conditions in the parameter can be lifted. In this paper we will show that not only is the theorem stated by THOM correct but in fact something considerably stronger is true; namely the map π is locally trivial.

1. Notation

In all that follows differentiable means class C^∞ and manifold means differentiable manifold without boundary satisfying the second axiom of countability. If W and M are manifolds we write $M(W, M)$ for the set of differentiable maps of W into M. Let $f \in M(W, M)$ and suppose the following are given: a coordinate system $(y) = (y_1, \ldots y_n)$ for M with domain O, a compact subset K of the domain of a coordinate system $(x) = (x_1, \ldots x_k)$ for W such that $f(K) \subseteq O$, a positive number ϵ and a non-negative integer r. We define $N^r(f, (x), (y), K, O, \epsilon)$ to be the set of $g \in M(W, M)$ such that $g(K) \subseteq O$, $|y_i \circ f(p) - y_i \circ g(p)| < \epsilon$ for $p \in K$ and $1 \leq i \leq n$, and

$$\left| \frac{\partial^m (y_i \circ f)}{\partial x_{j_1} \ldots \partial x_{j_m})} (p) - \frac{\partial^m (y_i \circ g)}{\partial x_{j_1} \ldots \partial x_{j_m})} (p) \right| < \epsilon \quad \text{for}$$

$p \in K$, $m \leq r$, $1 \leq i \leq n$, and $1 \leq j_\alpha \leq k$. Fixing r the C^r-topology for $M(W, M)$ is defined by the condition that sets of the form $N^r(f, (x), (y), K, O, \epsilon)$ shall form a sub-base. We note that the C^0-topology is the usual compact-open topology and that the C^r-topologies get stronger with increasing r. The union of the C^r-topologies over all positive integers r is called the C^∞-topology for $M(W, M)$. In all that follows we shall assume that all spaces of differenti-

[1]) This paper was presented at the International Colloquium on Differential Geometry and Topology at Zurich in June 1960.

The author is a National Science Foundation postdoctoral fellow.

able maps are equipped with the C^r-topology for some fixed r with $1 \leq r \leq \infty$. If V, W, and M are three manifolds then by the *composition map* of $\mathsf{M}(V,W) \times \mathsf{M}(W, M)$ into $\mathsf{M}(V, M)$ we mean the map $(f, g) \to g \circ f$. If V is a submanifold of W then by the *restriction map* of $\mathsf{M}(W, M)$ into $\mathsf{M}(V, M)$ we mean the map $f \to f \mid V$. We leave to the reader the straightforward proof of the following proposition.

Proposition. *If V, W, and M are three manifolds then the composition map of $\mathsf{M}(V, W) \times \mathsf{M}(W, M)$ into $\mathsf{M}(V, M)$ is continuous.*

Note that if V is a submanifold of W amd i_V is the identity map of V into W, the the restriction map of $\mathsf{M}(W, M)$ into $\mathsf{M}(V, M)$ is the composite of the maps $f \to (i_V, f)$ of $\mathsf{M}(W, M)$ into $\mathsf{M}(V, W) \times \mathsf{M}(W, M)$ and the composition map of the latter into $\mathsf{M}(V, M)$, hence the restriction map is continuous. If B is a differentiable fiber bundle over a manifold W we write $\mathsf{X}(B)$ for the space of differentiable cross-sections of B, topologized as a subspace of $\mathsf{M}(W, B)$. In particular we write $T(W)$ for the tangent bundle to W so $\mathsf{X}(T(W))$ is the space of differentiable vector fields on W. We recall that $\mathsf{X}(T(W))$ is a locally convex topological vector space. If V is a submanifold of W we write $T(W) \mid V$ for the part of $T(W)$ over V, so $\mathsf{X}(T(W) \mid V)$ is the space of W-valued vector fields on V.

If $f \in \mathsf{M}(W, M)$ we write δf for the differential of f, a map of $T(W)$ into $T(M)$. If $p \in W$ then the restriction of δf to W_p (the tangent space to W at p) is a linear map δf_p of W_p into $M_{f(p)}$. If δf_p is non-singular for all $p \in W$ then f is called an *immersion* of W in M and if, in addition, f is one-to-one then f is called an *embedding* of W in M. We write $\mathsf{J}(W, M)$ and $\mathsf{E}(W, M)$ for the subspaces of $\mathsf{M}(W, M)$ consisting of immersions and embeddings respectively.

If M is a manifold we write $\mathrm{Diff}(M)$ for the group of diffeomorphisms of M into itself, $D(M)$ for the subgroup of $\mathrm{Diff}(M)$ consisting of diffeomorphisms with compact support (the support of a map $f: M \to M$ is the closure of $\{p \in M \mid f(p) \neq p\}$) and $D_0(M)$ for the arc component of the identity in $D(M)$. These groups are topologized of course as subspaces of $\mathsf{M}(M, M)$. That as such they are topological groups is well-known (see e.g. the footnote page 126 of [2]).

2. A general local triviality theorem[2])

Let G be a topological group. A G-space is a space X together with a fixed map $(g, x) \to gx$ of $G \times X$ into X such that

---

[2]) The author gratefully acknowledges the collaboration of Dr. MORRIS HIRSCH on this section.

1. for each $g \in G$ the map $g^* : x \to gx$ is a homeomorphism of X and
2. $g \to g^*$ is a homomorphism of G into the group of self homeomorphisms of X.

A map $f: X \to Y$ of one G-space into another is called equivariant if $f(gx) = gf(x)$ for all $(g, x) \in G \times X$. If x_o is an element of a G-space X then a local cross-section for X at x_0 is a map χ of a neighborhood U of x_0 into G such that $\chi(u)x_0 = u$ for all $u \in U$. If there exists a local cross-section for X at each point of X then we say X admits local cross-sections.

Theorem A. *If G is a topological group and X is a G-space admitting local cross-sections then any equivariant map of a G-space into X is locally trivial.*

Proof. Let Y be a G-space and $\pi\colon Y \to X$ an equivariant map. Given $x_0 \in X$ put $F = \pi^{-1}(x_0)$ and let $\chi\colon U \to G$ be a local cross-section for X at x_0. Define $h\colon U \times F \to \pi^{-1}(U)$ by $h(u, f) = \chi(u)f$. Since π is equivariant and χ is a local cross-section $\pi h(u, f) = \pi(\chi(u)f) = \chi(u)\pi(f) = \chi(u)x_0 = u$ so h has the formal properties of a local product representation of π. Clearly h is continuous and we leave it to the reader to verify that $y \to (\pi(y), \chi(\pi(y))^{-1}y)$ is a continuous map of $\pi^{-1}(U)$ into $U \times F$ which is a two sided inverse for h. q.e.d.

Remark. Under the hypotheses of the above theorem it is easily seen that if $x_0 \in X$ then $X_0 = Gx_0$, the orbit of x_0, is an open set in X and if G_{x_0} is the isotropy group at x_0 then $gG_{x_0} \to gx_0$ is a homeomorphism of G/G_{x_0} onto X_0. The existence of local cross-sections for X also implies that the fibering of G by left G_{x_0} cosets is a locally trivial principal fiber bundle with base space X_o and structural group G_{x_0}. If $\pi\colon Y \to X$ is an equivariant map then $F = \pi^{-1}(x_0)$ is invariant under G_{x_0} and so a G_{x_0}-space. If we put $Y_0 = \pi^{-1}(X_0)$ and $\pi_0 = \pi \mid Y_0$ then it is easily seen that (Y_0, X_0, π_0) is equivalent to the fiber bundle with fiber F associated with the above principal bundle. However this more precise form of the above theorem will not be needed for present purposes.

3. The action of diffeomorphisms on embeddings

Let W and M be manifolds. It follows from the proposition of Section 1 that $E(W, M)$ is a $\mathrm{Diff}(M)$-space if we define $gf = g \circ f$ for $g \in \mathrm{Diff}(M)$ and $f \in E(W, M)$. *A fortiori* $E(W, M)$ is a $D_0(M)$-space. If V is a submanifold of W then of course in the same way $E(V, M)$ is a $D_0(M)$-space. Moreover the restriction map $\pi\colon \mathsf{E}(W, M) \to \mathsf{E}(V, M)$ is clearly equivariant, for $\pi(gf) = (g \circ f) \circ i_V = g \circ (f \circ i_V) = g\pi(f)$ where i_V is the identity map of V. It will follow from Section 2 that π is locally trivial if we can show that the $D_0(M)$-space $\mathsf{E}(V, M)$ admits local cross-sections. In the next section we will give an explicit construction of local cross-sections in case V is compact.

4. Construction of local cross-sections for $\mathsf{E}(V, M)$

Let M be a differentiable manifold. By a well-known theorem of H. WHITNEY M can be embedded as a closed submanifold of some EUCLIDean space, and it follows that there exists a complete RIEMANNian metric for M which we now fix. We write Exp for the corresponding exponential map of $T(M)$ into M. If X is a differentiable vector field on M we define $E(X)\colon M \to M$ by $E(X)(p) = \operatorname{Exp}(X_p)$. Since E is the composite of the maps $X \to (X, \operatorname{Exp})$ of $\mathsf{X}(T(M))$ into $\mathsf{M}(M, T(M)) \times \mathsf{M}(T(M), M)$ and the composition map of the latter into $\mathsf{M}(M, M)$ it follows from the proposition of Section 1 that

Lemma a. *E is a continuous map of $\mathsf{X}(T(M))$ into $\mathsf{M}(M, M)$.*

If K is a subset of M we write $\mathsf{M}_K(M, M)$ for the set of differentiable maps of M into M with support in K, and $\mathsf{X}_K(T(M))$ for the set of differentiable vector fields on M with support in K. Clearly E maps $\mathsf{X}_K(T(M))$ into $\mathsf{M}_K(M, M)$. If K is a compact subset of M then it is easily seen that $\mathsf{M}_K(M, M)$ consists of maps for which the pre-images of compact sets are compact, and that a neighborhood of the identity in $\mathsf{M}_K(M, M)$ consists of maps f for which δf_p is a linear isomorphism of M_p onto $M_{f(p)}$ for all $p \in M$ and which are therefore, by the implicit function theorem, local diffeomorphisms of M into M. By Theorem 4.2 of [2] it follows that a neighborhood of the identity in $\mathsf{M}_K(M, M)$ consists of covering maps of finitely many sheets. Now if $f\colon M \to M$ is a covering map, $p, q \in M$, and there is an arc σ joining p to q such that $f(\sigma)$ lies entirely within a disc, then $f(p) = f(q)$. It follows that if $\epsilon > 0$ is chosen so that every point of distance less than ϵ from K is contained in a geodesically convex disc of radius 2ϵ, then a covering map f in $M_K(M, M)$ which satisfies $\varrho(f(x), x) < \epsilon$ for $x \in K$ must be one-to-one. Thus a neighborhood of the identity in $\mathsf{M}_K(M, M)$ is included in $D(M)$. Since E maps 0, the zero vector field on M, into i_M, the identity map of M, we can find a neighborhood U of 0 in $\mathsf{X}_K(T(M))$ such that $E(X) \in D(M)$ for $X \in U$. Since $\mathsf{X}(T(M))$ is locally convex we can suppose U is convex. Then if $X \in U$ the map $t \to E(tX)$ of $[0, 1]$ into $D(M)$ is an arc from i_M to $E(X)$, so $E(X) \in D_0(M)$. We have proved

Lemma b. *If K is a compact subset of M then E maps a neighborhood of the zero vector field in $\mathcal{X}_K(T(M))$ into $D_0(M)$.*

Lemma c. *Let V be a closed submanifold of M and K a neighborhood of V in M. Then there is a continuous linear map $k\colon \mathcal{X}(T(M) \mid V) \to \mathcal{X}_K(T(M))$ such that $k(X) \mid V = X$.*

Proof. It is well known that if $N(V, M)$ is the normal bundle to V in M the there is a neighborhood O of the zero cross-section of $N(V, M)$ which is mapped diffeomorphically by Exp onto a neighborhood of V in M. Without loss of generality we can assume that $K = \operatorname{Exp}(O)$ is such a "tubular" neighborhood of V and we write $J\colon K \to O$ for the inverse diffeomorphism. Let $T(M) \oplus T(M)$ be the Whitney sum of $T(M)$ with itself, i.e. the differentiable fiber bundle over M associated with $T(M)$ whose fiber at p is $M_p \oplus M_p$. We define a map $\mathcal{T}\colon T(M) \oplus T(M) \to T(M)$ as follows: $\mathcal{T}(u, v)$ is the vector at $\operatorname{Exp}(v)$ that results from parallel translating u along the path $t \to \operatorname{Exp}(tv)$. In coordinates $\mathcal{T}(u, v)$ is the solution of a system of first order linear ordinary differential equations whose coefficients depend linearly on the components of v and the Christoffel symbols, and whose initial conditions are the components of u. The theorem that the solutions of ordinary differential equations with coefficients which are differentiable functions of the variables and parameters depend differentiably on these parameters and the initial condition insures that $\mathcal{T}$ is a differentiable map. We note that $\mathcal{T}$ clearly satisfies

$$\mathcal{T}(u, o) = u \tag{1}$$

$$\mathcal{T}(\alpha u + \beta u', v) = \alpha \mathcal{T}(u, v) + \beta \mathcal{T}(u', v) \tag{2}$$

Let α be a differentiable real valued function on M which is identically zero on a neighborhood of $M - K$ and is identically one on V and let π denote the projection of $T(M)$ into M. We define a map $k\colon \mathcal{X}(T(M) \mid V) \to \mathcal{M}(M, T(M))$ by $k(X)(p) = \alpha(p)\mathcal{T}(X_{\pi J(p)}, J(p))$ if $p \in K$ and $k(X)(p) =$ zero vector at p if $p \in M - K$. We note that $k(X)(p)$ is a vector at $\operatorname{Exp} J(p) = p$, so $k(X)$ is a vector field on M. Moreover the support of $k(X)$ is included in the support of α which is in turn included in K, hence k maps $\mathcal{X}(T(M) \mid V)$ into $\mathcal{X}_K(T(M))$. If $p \in V$ then $J(p) =$ the zero vector at p and $\alpha(p) = 1$ so $k(X)(p) = X_p$ by property (1) of $\mathcal{T}$. That k is a linear map follows from property (2) of $\mathcal{T}$. Finally k is continuous because it is the composite of the following continuous maps: (a) the map $X \to ((X \circ \pi \circ J) \times J,\ \mathcal{T})$ of $\mathcal{X}(T(M) \mid V)$ into $\mathcal{M}(K, T(M) \oplus T(M)) \times \mathcal{M}(T(M) \oplus T(M),\ T(M))$, (b) the composition map of the latter into $M(K, T(M))$, (c) the map h of $X(T(M)|K)$ into $X_K(T(M))$ defined by $h(X)_p = \alpha(p)X_p$ for $p \in K, h(X)(p) = 0$ for $p \in M - K$. q.e.d.

Lemma d. *Let V be a compact submanifold of M, i_V the identity embedding of V in M and Z the zero vector field on V. Then if U is any sufficiently small neighborhood of i_V in $E(V, M)$ there is a continuous map $X: U \to X(T(M)|V)$ such that $X(i_V) = Z$ and* $\operatorname{Exp}(X(f)_v) = f(v)$ *for $f \in U$ and $v \in V$.*

Proof. Let $|Z|$ denote the image of Z, i.e. the zero cross-section of $T(M)|V$. Then the map $h: T(M)|V \to V \times M$ defined by $h(v) = (p(v), \operatorname{Exp}(v))$ (where $p: T(M) \to M$ is projection map) is non-singular on $|Z|$ and embeds $|Z|$ onto the diagonal of $V \times V$ in $V \times M$. Since $|Z|$ is compact and $\dim (T(M)|V) = \dim V \times M$ it follows that h maps a neighborhood of $|Z|$ in $T(M)|V$ diffeomorphically onto a neighborhood O of the diagonal of $V \times V$ in $V \times M$. Let $j: O \to T(M)|V$ be the inverse diffeomorphism and let $U = \{f \in E(V, M) \mid (v, f(v)) \in O$ for all $v \in V\}$. Clearly U is a neighborhood of i_V in $E(V, M)$ (even in the C^0-topology). Define $X(f)$ for $f \in U$ by $X(f)(v) = j(v, f(v))$. Note that X is the composite of the maps $f \to (i_V \times f, j)$ of U into $M(V, O) \times M(O, T(M)|V)$ and the composition map of the latter into $M(V, T(M)|V)$ so X is continuous by the proposition of Section 1. Now $(v, f(v)) = hj(v, f(v)) = h(X(f)(v)) = (p(X(f)(v)), \operatorname{Exp}(X(f)(v)))$. Thus $p(X(f)(v)) = v$, so $X(f) \in X(T(M)|V)$, and $\operatorname{Exp}(X(f)(v)) = f(v)$. q.e.d.

Theorem B. *If f_0 is an embedding of a compact manifold V in a manifold M then there is a neighborhood U of f_0 in $E(V, M)$ and a map $\chi: U \to D_0(M)$ such that $f = \chi(f)f_0$ for all $f \in U$. In other words the $D_0(M)$-space $E(V, M)$ admits local cross-sections.*

Proof. We can assume that V is a submanifold of M and that $f_0 = i_V$ the identity embedding of V. Let $\chi = E \circ k \circ X$ where $X: U \to X(T(M)|V)$ is in lemma d and $k: X(T(M)|V) \to X_K(T(M))$ is as in lemma c, K being any compact neighborhood of V. We note that since $X(i_V)$ is the zero vector field on V and k is linear it follows from lemma b that if U is chosen sufficiently small then $\chi(U) \subseteq D_0(M)$. If $f \in U$ and $v \in V$ then by lemmas c and d $\chi(f)(f_0(v)) = \chi(f)(v) = \operatorname{Exp}(k(X(f))(v)) = \operatorname{Exp}(X(f)(v)) = f(v)$. q.e.d.

As noted in Section 3 it follows from Theorems A and B that

Theorem C. *If W and M are manifolds and V is a compact submanifold of W then the restriction map of $E(W, M)$ into $E(V, M)$ is locally trivial.*[3])

[3]) Added in proof. A similar theorem has been proved independently by J. CERF in his thesis which should appear soon.

5. Applications

If M is a manifold and V a compact manifold than an arc in $E(V, M)$ is called an isotopy and two elements in the same arc component are called isotopic. It is an immediate consequence of Theorem B that if f_0 and f_1 are isotopic elements of $E(V, M)$ then there exists $g_1 \in D_0(M)$ such that $f_1 = g_1 f_0$ and in fact if f_t is an isotopy from f_0 to f_1 then there is an arc g_t in $D_0(M)$ starting from the identity such that $f_t = g_t f_0$. Then if V is a submanifold of W and F_0 is an embedding of W in M such that $f_0 = F_0 | V$, then $F_t = g_t \circ F_0$ is an isotopy in $E(W, M)$ such that $f_t = F_t | V$. This might well be called the isotopy extension theorem. It implies of course that whether or not an embedding f of V in M can be extended to an embedding of W in M depends only on the isotopy class of f. In particular if W is a submanifold of M then any embedding of V isotopic to i_V can be extended to an embedding of W in M isotopic to i_W. As another special case we note that if V is a compact submanifold of M then every element of $D_0(V)$ can be extended to an element of $D_0(M)$. If W is a manifold with boundary and if the boundary V of W is compact let M be the double of V. Then any diffeomorphism of M which is isotopic to the identity and maps V into itself must map W into itself. It follows that the restriction homomorphism $\pi\colon D(W) \to D(V)$ maps $D_0(W)$ onto $D_0(V)$, i.e. every diffeomorphism of V isotopic to the identity can be extended to diffeomorphism of W isotopic to the identity. This can be proved by other means and is a known result.

6. Generalizations

The author and Dr. Morris Hirsch have shown that if W is a manifold with boundary, V a compact submanifold with boundary and M a manifold without boundary then the restriction map of $E(W, M)$ into $E(V, M)$ is locally trivial. We have also shown that if B is a differentiable fiber bundle over W than the restriction map of $X(B)$ into $X(B | V)$ is locally trivial so that as a special case (taking $B = M \times W$) the restriction map of $M(W, M)$ into $M(V, M)$ is locally trivial. We cannot show that the restriction map of $J(W, M)$ into $J(V, M)$ is locally trivial, and in fact this is in general false. However, if $\dim M > \dim W$ then we can show that the restriction map of $J(W, M)$ into $J(V, M)$ is a fiber bundle in the sense of Hurewicz, i.e. it has the covering homotopy property for arbitrary spaces. This is so even when V is not compact. Using this Hirsch and the author have given a simplified proof of a somewhat strengthened version of the classification theory of immersions of a manifold W in a manifold M of higher dimension which was proved by HIRSCH in his thesis [1]. These results will appear in a joint paper now in preparation.

REFERENCES

1. M. W. HIRSCH, *Immersions of Manifolds*, Trans. Amer. Math. Soc., Vol. 93 (1959) pp. 242–276.
2. R. S. PALAIS, *Natural operations on differential forms*, Trans. Amer. Math. Soc., Vol. 92 (1959) pp. 125–141.

The Institute for Advanced Study

(Received May 23, 1960)