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The Kervaire Invariant of Hypersurfaces in Complex Projective Spaces

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1. Introduction

In [4], E. H. Brown and F. Peterson defined the Kervaire invariant for $(8k+2)$-dimensional spin manifolds. The purpose of this paper is to calculate it for certain manifolds. Precisely, let $V^n(d)$ be a non-singular hypersurface of degree d in complex projective $(n+1)$-space $\mathbf{C}P^{n+1}$. Assume that $n \equiv 1 \pmod 4$ $(n \neq 1)$ and d is odd. Then $V^n(d)$ is an $(8k+2)$-dimensional differentiable manifold with a spin structure. Moreover, since $V^n(d)$ is simply connected, spin structure is unique up to homotopy. Therefore we have a well defined Kervaire invariant $K(V^n(d)) \in \mathbf{Z}/2$. The result is

THEOREM (1.1).

$$K(V^n(d)) = \begin{cases} 0 & \text{if } \ d \equiv \pm 1 \pmod 8 \\ 1 & \text{if } \ d \equiv \pm 3 \pmod 8. \end{cases}$$

A motivation for this calculation arose when the author was trying to understand the topology of some well-known complex manifolds, such as the hypersurfaces in complex projective spaces. For example, if n is odd, then it can be shown that (cf. Remark (5.1)), there are closed simply connected almost smooth manifold (by an almost smooth manifold, we mean a PL manifold M with a smooth structure on M-pt.) $M^{2n}(d)$ and $(n-1)$ connected almost smooth manifold $N^{2n}(d)$ such that

$$H_*(M^{2n}(d); \mathbf{Z}) \cong H_*(\mathbf{C}P^n; \mathbf{Z}), \qquad H_*(N^{2n}(d); \mathbf{Z}) \triangleq H_*((b_n/2)\, S^n \times S^n; \mathbf{Z})$$

and

$$V^n(d) \underset{PL}{\cong} M^{2n}(d) \# N^{2n}(d) \tag{1.2}$$

where b_n is the nth Betti number of $V^n(d)$ and $\cong_{PL}$ denotes a PL homeomorphism. (The cohomology ring of $M^{2n}(d)$ is not isomorphic to that of $\mathbf{C}P^n$ if $d \neq 1$. $M^{2n}(d)$ is only a rational homotopy $\mathbf{C}P^n$.)

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It might be natural to ask whether (1.2) holds in the differentiable category or not. The answer to this question is given by

THEOREM (1.3) (i) *If* $n=1$, 3 or 7, *then there is a closed simply connected differentiable manifold* $M^{2n}(d)$ *such that*

$H_*(M^{2n}(d); \mathbf{Z}) \cong H_*(\mathbf{C}P^n; \mathbf{Z})$ *and* $V^n(d) \cong M^{2n}(d) \# (b_n/2) S^n \times S^n$
($\cong$ *stands for a diffeomorphism*).

(ii) *If n is odd* ($\neq 1, 3, 7$) *and* $d \not\equiv \pm 3 \pmod 8$, *then there is a closed simply connected differentiable manifold* $M^{2n}(d)$ *such that*

$H_*(M^{2n}(d); \mathbf{Z}) \cong H_*(\mathbf{C}P^n; \mathbf{Z})$ *and* $V^n(d) \cong M^{2n}(d) \# (b_n/2) S^n \times S^n$.

(iii) *If* $n \equiv 1 \pmod 4$ $(n \neq 1)$ and $d \equiv \pm 3 \pmod 8$, *then there is no such decomposition of* $V^n(d)$.

Remark (1.4). For the remaining case $n \equiv 3 \pmod 4$ $(n \neq 3, 7)$ and $d \equiv \pm 3 \pmod 8$, we can not say anything reflecting the mysterious part of the Kervaire invariant one problem.

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2. Preliminaries on the Topology of $V^n(d)$

Let $V^n(d)$ be a non-singular hypersurface of degree d in complex projective space $\mathbf{C}P^{n+1}$. Since any two non-singular hypersurfaces of the same degree are diffeomorphic, to study the topology of them, we may assume that $V^n(d)$ is defined by the equation $z_0^d + z_1^d + \cdots + z_{n+1}^d = 0$, where $[z_0, z_1, \ldots, z_{n+1}]$ is the homogeneous coordinate of $\mathbf{C}P^{n+1}$. Now let $W^n(d)$ be the non-singular affine hypersurface in $\mathbf{C}^{n+1}$ defined by $z_0^d + z_1^d + \cdots + z_n^d = 1$.

Then we can consider $W^n(d)$ as an open submanifold of $V^n(d)$ by considering $\mathbf{C}^{n+1}$ as affine part of $\mathbf{C}P^{n+1}$ defined by $z_{n+1} \neq 0$. $W^n(d)$ is a special type of so-called Brieskorn variety and by the works of Brieskorn [3] and Milnor [8] the topology of it is quite well understood. For example, it has the same homotopy type as the bouquet of $(d-1)^{n+1}$ copies of the n-sphere S^n.

Now let $i: W^n(d) \to V^n(d)$ be the inclusion. Then we have the following

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[1]) The main result of this paper has alse been proved by W. Browder and J. Wood [10]. (Added in proof.)

LEMMA (2.1). (i) *If n is odd, then*

$$i_*: H_n(W^n(d); A) \to H_n(V^n(d); A)$$

is surjective.

(ii) *If n is even, then*

$$\operatorname{Cok}(i_*: H_n(W^n(d); A) \to H_n(V^n(d); A)) \cong A$$

where A is either $\mathbf{Z}$ *or* $\mathbf{Z}/2$.

Proof. Consider the following exact sequence

$$0 \to H_{n+1}(V) \to H_{n+1}(V, W) \to H_n(W) \to H_n(V) \to H_n(V, W) \to 0, \tag{2.2}$$

where the coefficient A is either $\mathbf{Z}$ or $\mathbf{Z}/2$ and V (resp. W) stands for $V^n(d)$ (resp. $W^n(d)$).

We have only to show that

$$H_n(V, W) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ A & \text{if } n \text{ is even}. \end{cases}$$

Let $V' = \{[z_0, \ldots, z_{n+1}] \in V^n(d);\ z_{n+1} = 0\}$. Then we have $W = V - V'$. Note also that

$$V' = V^{n-1}(d) \subset \mathbf{C}P^n = \{[z_0, \ldots, z_{n+1}] \in \mathbf{C}P^{n+1};\ z_{n+1} = 0\}.$$

Let T be the tubular neighborhood of V' in V. Then, by the excision $H_n(V, W) \cong H_n(T, \partial T)$. By the Lefschetz duality $H_n(T, \partial T) \cong H^n(T)$. Since T is homotopy equivalent to V', we have $H^n(T) \cong H^n(V')$. But, it is well-known, by the Lefschetz hyperplane section theorem (cf. [1]), that

$$H^n(V') = \begin{cases} 0 & \text{if } n \text{ is odd} \\ A & \text{if } n \text{ is even}. \end{cases}$$

Therefore we have

$$H_n(V, W) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ A & \text{if } n \text{ is even}. \end{cases}$$

This proves Lemma (2.1).

Let $K_n(A) = \operatorname{Ker}(i_*: H_n(W; A) \to H_n(V; A))$. Then we have

LEMMA (2.3). *The natural map* $K_n(\mathbf{Z}) \to K_n(\mathbf{Z}/2)$ *is surjective.*

Proof. This follows from the exact sequence (2.2) and the fact that $H_*(W; \mathbf{Z})$, $H_*(V, \mathbf{Z})$, $H_*(V, W; \mathbf{Z})$ have no torsion.

LEMMA (2.4).

(i) $$\operatorname{rank} H_n(V^n(d)) = \begin{cases} \frac{1}{d}\{(d-1)^{n+2}-(d-1)\} & n\text{: odd} \\ \frac{1}{d}\{(d-1)^{n+2}+(d-1)\}+1 & n\text{: even} \end{cases}$$

(ii) $$\operatorname{rank} K_n(\mathbf{Z}) = \frac{1}{d}\{(d-1)^{n+1}+(-1)^{n+1}(d-1)\}.$$

Proof. (i) follows from the Lefschetz hyperplane section theorem ([1]) and the formula for the Euler number of $V^n(d)$. (ii) follows from (i) and the exact sequence (2.2).

As we mentioned before, topology of $W^n(d)$ is well-understood. We quote some of the results from Hirzebruch and Mayer [6].

Let $\mathbf{Z}/d$ be the cyclic group of order d and let $G=\mathbf{Z}/d\oplus\cdots\oplus\mathbf{Z}/d$ ($(n+1)$ copies). Let $w_j \in G$ ($j=0,\dots,n$) be the element corresponding to the generator for the jth factor. G acts on $W^n(d)$ as follows. Let $w_0^{k_0}\dots w_n^{k_n}\in G$ and $(z_0,\dots,z_n)\in W^n(d)$. Then

$$w_0^{k_0}\dots w_n^{k_n}(z_0,\dots,z_n)=(\zeta^{k_0}z_0,\dots,\zeta^{k_n}z_n)$$

where $\zeta=\exp[2\pi i/d]$.

There is a homology class $h\in H_n(W^n(d);\mathbf{Z})$ such that h can be represented by an imbedded sphere $S^n\subset W^n(d)$ whose normal bundle is isomorphic to the tangent bundle $\tau(S^n)$. Moreover we have

THEOREM (2.5). ([6]).

$$H_n(W^n(d);\mathbf{Z})\cong\mathbf{Z}(G)\,h.$$

Here $\mathbf{Z}(G)$ is the group ring of G and

$$\mathbf{Z}(G)\,h\cong\mathbf{Z}(G)/I(G).$$

$I(G)$ is the ideal of $\mathbf{Z}(G)$ generated by $\{1+w_j+\cdots+w_j^{d-1}\}$ $j=0,\dots,n$.

The intersection numbers can be given as follows. Let

$$\varepsilon:\mathbf{Z}(G)\to\mathbf{Z}$$

be an additive homomorphism defined by

$$\varepsilon(1) = -\varepsilon(w_0 \dots w_n) = (-1)^n (-1)^{n(n-1)/2}$$
$$\varepsilon(g) = 0 \quad \text{for} \quad g \in G, \quad g \neq 1, w_0 \dots w_n$$

and let $-: \mathbf{Z}(G) \to \mathbf{Z}(G)$ be the ring automorphism defined by $g \to g^{-1}$, $g \in G$. Let $\eta = (1 - w_0) \dots (1 - w_n) \in \mathbf{Z}(G)$. Then we have

THEOREM (2.6). ([6]). *The intersection number of two elements xh, $yh \in H_n(W^n(d))$ is given by*

$$xh \circ yh = \varepsilon(\bar{y} x \eta).$$

Here we identify the group $H_n(W^n(d))$ with $\mathbf{Z}(G)\,h$ by Theorem (2.5).

LEMMA (2.7).

$$K_n(\mathbf{Z}) = \{xh \in H_n(W^n(d));\ wxh = xh\,(w = w_0 \dots w_n)\}.$$

Proof. Let us define a $\mathbf{Z}/d$ action on $W^n(d)$ by

$$\zeta(z_0, \dots, z_n) = (\zeta z_0, \dots, \zeta z_n), \qquad \zeta = \exp[2\pi i/d].$$

Then obviously we have $wxh = xh \leftrightarrow \zeta_*(xh) = xh$ where ζ_* is the homomorphism on the homology induced from the action of ζ. Now the action of $\mathbf{Z}/d$ on $W^n(d)$ can be extended to that on $V^n(d)$ by

$$\zeta[z_0, \dots, z_{n+1}] = [\zeta z_0, \dots, \zeta z_n, z_{n+1}].$$

The quotient space of $V^n(d)$ with respect to this action can be shown to be $\mathbf{C}P^n$. Therefore we have, by a well known theorem (see [2])

$$H_n(V^n(d)\colon \mathbf{Q})^{\mathbf{Z}/d} = H_n(\mathbf{C}P^n; \mathbf{Q}) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \mathbf{Q} & \text{if } n \text{ is even}. \end{cases} \tag{2.8}$$

Here the left hand side is the group of invariant homology classes. If n is even $(n = 2m)$, then $H_n(V^n(d); \mathbf{Q})^{\mathbf{Z}/d}$ is generated by $[V^m(d)] \in H_n(V^n(d))$, $V^m(d) = \{[z_0, \dots, z_{n+1}] \in V^n(d),\ z_i = 0 \text{ for } i > m\}$.

Now let us assume that $wxh = xh$ for an element $xh \in H_n(W)$. Then we have $\zeta_*(i_*(xh)) = i_*(xh)$ where $i\colon W \to V$ is the inclusion. Therefore $i_*(xh)$ is an invariant homology class. By (2.8), we have

$$i_*(xh)=0 \quad \text{if} \quad n \text{ is odd } i_*(xh)=a[V^m(d)] \quad \text{for some} \quad a\in\mathbf{Q} \quad \text{if} \quad n \text{ is even.}$$

But if $i_*(xh)=a[V^m(d)]$ for $a\neq 0$, the it would follow that

$$i_*(xh)\circ[V^m(d)]=a[V^m(d)]\circ[V^m(d)]=ad\neq 0.$$

This is a contradiction, since clearly we have

$$i_*(xh)\circ[V^m(d)]=0 \quad \text{for any} \quad xh\in H_n(W).$$

Therefore we have $i_*(xh)=0\in H_n(V^n(d);\mathbf{Q})$. But since $H_n(V^n(d);\mathbf{Z})$ has no torsion, it follows that $i_*(xh)=0$. Thus we obtain $\{xh\in H_n(W);\ wxh=xh\}\subset K_n(\mathbf{Z})$.

Now since both $K_n(\mathbf{Z})$ and $\{xh\in H_n(W);\ wxh=xh\}$ are direct summands of $H_n(W)$, we have only to show that the ranks of them coincide. Now the action of $\mathbf{Z}/d$ on W is free and the quotient manifold can naturally be identified with $\mathbf{C}P^n-V^{n-1}(d)$. Therefore we have

$$\operatorname{rank}\{xh\in H_n(W);\ wxh=xh\}=\operatorname{rank} H_n(\mathbf{C}P^n-V^{n-1}(d)).$$

The homology exact sequence of the pair $(\mathbf{C}P^n, \mathbf{C}P^n-V^{n-1}(d))$ yields,

$$\operatorname{rank} H_n(\mathbf{C}P^n-V^{n-1}(d))=\begin{cases}\operatorname{rank} H^{n-1}(V^{n-1}(d)) & \text{if} \quad n \text{ is odd}\\ \operatorname{rank} H^{n-1}(V^{n-1}(d))-1 & \text{if} \quad n \text{ is even}.\end{cases}$$

But this is the same formula for rank $K_n(\mathbf{Z})$ (cf. Lemma (2.4)).

3. The Kervaire-Milnor Map

Since $W^n(d)$ is a parallelizable $(n-1)$ connected $2n$-manifold, if n is odd ($n\neq 1$, 3, 7), we have the Kervaire-Milnor homomorphism (see [7])

$$\varphi: H_n(W)\to\mathbf{Z}/2$$

which is defined as follows. Let $xh\in H_n(W)$ be an element. Then xh can be represented by an imbedded sphere $S^n\subset W$. The normal bundle ν of this imbedding is either trivial or the tangent bundle of the sphere, $\tau(S^n)$. We put

$$\varphi(x)=\begin{cases}0 & \text{if} \quad \nu \text{ is trivial}\\ 1 & \text{if} \quad \nu=\tau(S^n).\end{cases}$$

It is known that the map φ is quadratic with respect to the intersection pairing; $\varphi(x+y)=\varphi(x)+\varphi(y)+x\circ y \pmod 2$.

Obviously, φ can be considered as a homomorphism from $H_n(W; \mathbf{Z}/2)$ to $\mathbf{Z}/2$. Now recall that we have a special element $h\in H_n(W)$. By a property of h, we have $\varphi(h)=1$.

According to Theorem (2.5), we have $H_n(W)\cong \mathbf{Z}(G)\,h$.

Let $g\in G$ be any element. Since g acts on W as a diffeomorphism, we should have $\varphi(gh)=1$ for any $g\in G$.

Now it is clear that this property together with the quadraticity determine φ uniquely on $H_n(W)$. Note that one can also define φ for the cases $n=1$, 3 or 7 by the above characterization. We have

LEMMA (3.1). *If d is even, then there is an element $xh_2\in K_n(\mathbf{Z}/2)$ such that $\varphi(xh_2)=1$ ($h_2=h \pmod 2$).*

Proof. Put $n=2k-1$ and

$$xh=\prod_{l=0}^{k-1}\left(\sum_{d-2\geqq i\geqq j\geqq 0} w_{2l}^{i}w_{2l+1}^{j}\right)h.$$

We claim that $xh_2\in K_n(\mathbf{Z}/2)$ and $\varphi(xh_2)=1$. To prove $xh_2\in K_n(\mathbf{Z}/2)$, it suffices to show that $xh\in K_n(\mathbf{Z})$. Now we calculate;

$$wxh=\prod_{l=0}^{k-1}\left(\sum_{d-2\geqq i\geqq j\geqq 0} w_{2l}^{i+1}w_{2l+1}^{j+1}\right)h.$$

But

$$\begin{aligned}\sum_{d-2\geqq i\geqq j\geqq 0} w_{2l}^{i+1}w_{2l+1}^{j+1} &= \sum_{d-2\geqq i\geqq j\geqq 1} w_{2l}^{i}w_{2l+1}^{j}+\sum_{j=1} w_{2l}^{d-1}w_{2l+1}^{j}\\ &\equiv \sum_{d-2\geqq i\geqq j\geqq 1} w_{2l}^{i}w_{2l+1}^{j}+(1+w_{2l}+\cdots+w_{2l}^{d-2})(1+w_{2l+1}+\cdots+w_{2l+1}^{d-2})\\ &\quad -(1+w_{2l}+\cdots+w_{2l}^{d-2})(w_{2l+1}+\cdots+w_{2l+1}^{d-2})=\sum_{d-2\geqq i\geqq j\geqq 0} w_{2l}^{i}w_{2l+1}^{j}.\end{aligned}$$

Here $\equiv$ denotes the congruence modulo the ideal $I(G)$ which is generated by $\{1+w_j+\cdots+w_j^{d-1}\}\ j=0,\dots,n$.

Thus we have $wxh=xh$.

By Lemma (2.7), this proves $xh\in K_n(\mathbf{Z})$. Next we calculate $\varphi(xh)$.
We have

$$xh=\sum_{\substack{d-2\geqq i_m\geqq j_m\geqq 0\\ m=0,\dots,k-1}} w_0^{i_0}w_1^{j_0}\dots w_{n-1}^{i_{k-1}}w_n^{j_{k-1}}h.$$

Therefore the number of the monomials in the above expression of xh is $\{\frac{1}{2}d(d-1)\}^k$. On the other hand

$$\begin{aligned} & w_0^{i_0}w_1^{j_0}\dots w_{n-1}^{i_{k-1}}w_n^{j_{k-1}}h \circ w_0^{i'_0}w_1^{j'_0}\dots w_{n-1}^{i'_{k-1}}w_n^{j'_{k-1}}h \\ & \quad = \varepsilon(w_0^{i_0-i'_0}w_1^{j_0-j'_0}\dots w_n^{j_{k-1}-j'_{k-1}}(1-w_0)\dots(1-w_n)) \\ & \quad = \begin{cases} \pm 1 & \text{if } i_m-i'_m, j_m-j'_m=0 \text{ or } \pm 1 \text{ for all } m \\ & \text{and for at least one } m, i_m\neq i'_m \text{ or } j_m\neq j'_m. \\ 0 & \text{otherwise}. \end{cases} \end{aligned}$$

Therefore the numbers of unordered pair $(w_0^{i_0}w_0^{j_0}\dots w_n^{j_{k-1}}h, w_0^{i'_0}\dots w_n^{j'_{k-1}}h)$ for which the intersection number is equal to ± 1 is

$$\{\tfrac{3}{2}(d-1)(d-2)\}^k.$$

Therefore we have

$$\varphi(xh)=\{\tfrac{1}{2}d(d-1)\}+\{\tfrac{3}{2}(d-1)(d-2)\}^k \bmod 2=(d-1)^k\{((d/2)^k+(\tfrac{3}{2}(d-2))^k\} \bmod 2=1.$$

This proves Lemma (3.1).

To study W, it is convenient to study the "suspension" of W, denoted by W', which is defined by $z_0^d+z_1^d+\dots+z_n^d+z_{n+1}^2=1$.

We have natural isomorphisms (cf. [6]).

$$H_n(W;\mathbf{Z})\cong H_{n+1}(W';\mathbf{Z}) \qquad H_n(W;\mathbf{Z}/2)\cong H_{n+1}(W';\mathbf{Z}/2).$$

Under this isomorphism, the class $h\in H_n(W;\mathbf{Z})$ (resp. $h_2\in H_n(W;\mathbf{Z}/2)$) corresponds to a class $h'\in H_{n+1}(W';\mathbf{Z})$ (resp. $h'_2\in H_{n+1}(W';\mathbf{Z}/2)$).

LEMMA (3.2) ([6]). *The isomorphism* $H_n(W;\mathbf{Z}/2)\cong H_{n+1}(W';\mathbf{Z}/2)$ *respects the bilinear pairing defined by the intersection number* mod 2 *and therefore induces a quadratic function* $\varphi':H_{n+1}(W';\mathbf{Z}/2)\to\mathbf{Z}/2$. *Moreover* φ' *is defined by* $\varphi'(xh'_2)==\frac{1}{2}xh'\circ xh'$ (mod 2) *where* $xh'_2=xh'$ (mod 2).

Proof. Calculation shows

$$xh\circ yh\equiv xh'\circ yh' \pmod 2$$

for any $xh, yh\in H_n(W;\mathbf{Z})$. This proves the former part of the lemma. The latter part follows from this and the fact that $xh'\circ xh'=\pm 2$ for any $x=w_0^{k_0}\dots w_0^{k_n}$.

LEMMA (3.3). *If d is odd, then $\varphi(xh_2)=0$ for any $xh_2\in K_n(\mathbf{Z}/2)$.*

Proof. Since the natural map $K_n(\mathbf{Z})\to K_n(\mathbf{Z}/2)$ is surjective (Lemma (2.3)), we have only to show that $\varphi=0$ on $K_n(\mathbf{Z})$. Thus let xh be an element of $K_n(\mathbf{Z})$. We have $wxh=xh$. By induction, we obtain $w^j xh=xh$ for any j. Therefore

$$(1+w+\cdots+w^{d-1})\,xh=dxh.$$

Since d is odd by the assumption

$$\varphi(xh)=\varphi(dxh)=\varphi((1+w+\cdots+w^{d-1})\,xh).$$

Now we claim that $\varphi((1+w+\cdots+w^{d-1})\,xh)=0$ for any $xh\in H_n(W)$. To prove this, by Lemma (3.2), it suffices to show that

$$(1+w+\cdots+w^{d-1})\,xh'\circ(1+w+\cdots+w^{d-1})\,xh'\equiv 0 \pmod 4$$

for any $xh'\in H_{n+1}(W';\mathbf{Z})$. Now let us write

$$xh'=\sum_{\mathbf{K}} a_{\mathbf{K}}w^{\mathbf{K}}h',\qquad \mathbf{K}=(k_0,\dots,k_n)\quad 0\leqq k_j\leqq d-2,\qquad w^{\mathbf{K}}=w_0^{k_0}\dots w_n^{k_n}.$$

Then we have

$$\begin{aligned}(1+w+\cdots+w^{d-1})\,&xh'\circ(1+w+\cdots+w^{d-1})\,xh'\\ &=\varepsilon((1+w+\cdots+w^{d-1})^2\,x\bar{x}(1-w_0)\dots(1-w_{n+1}))\\ &=\varepsilon(d(1+w+\cdots+w^{d-1})\,x\bar{x}(1-w_0)\dots(1-w_{n+1})),\end{aligned}$$

and

$$x\bar{x}=\sum_{\mathbf{K},\mathbf{K}'} a_{\mathbf{K}}a_{\mathbf{K}'}w^{\mathbf{K}-\mathbf{K}'}.$$

Therefore we have only to prove the following.

(i) $\varepsilon((1+w+\cdots+w^{d-1})\,(1-w_0)\dots(1-w_{n+1}))=\pm 4$

(ii) $\varepsilon((1+w+\cdots+w^{d-1})\,(g+g^{-1})\,(1-w_0)\dots(1-w_{n+1}))\equiv 0 \pmod 4$ for any $g\in G$.

But these two can be checked by a direct calculation. This proves Lemma (3.3).

In view of this lemma, if d is odd, then $\varphi:H_n(W;\mathbf{Z}/2)\to\mathbf{Z}/2$ induces a well-defined quadratic function $\varphi:H_n(V;\mathbf{Z}/2)\to\mathbf{Z}/2$. On the other hand, if $n\equiv 1 \pmod 4$, then E. H. Brown and F. Peterson [4] defined a quadratic function $\psi:H^n(V;\mathbf{Z}/2)\to\mathbf{Z}/2$

(with respect to the bilinear pairing defined by the cup product evaluated on the fundamental cycle). We have

PROPOSITION (3.3). *φ and ψ above are dual to each other under the Poincaré duality.*

Proof. It will be indicated in §5 that there is an almost smooth $(n-1)$-connected $2n$-manifold $N^{2n}(d)$ and a map $f: V^n(d) \to N^{2n}(d)$ such that

$$f^*: H^n(N^{2n}(d); \mathbf{Z}/2) \cong H^n(V^n(d); \mathbf{Z}/2).$$

Then the proposition follows from the naturality of Brown-Peterson's ψ and the fact that for almost smooth $(n-1)$ connected $2n$-manifolds, the Kervaire-Milnor map φ and Brown-Peterson's ψ are dual to each other.

4. Proof of Theorem (1.1)

In this section, we prove Theorem (4.1), which is the main result of this paper. By virtue of Proposition (3.3), Theorem (1.1) is an immediate consequence of it.

THEOREM (4.1). *Assume that both n and d are odd. Then the Art-Kervaire invariant of the well-defined quadratic function $\varphi: H_n(V^n(d); \mathbf{Z}/2) \to \mathbf{Z}/2$ is given by*

$$K(V^n(d)) = \begin{cases} 0 & \text{if } d \equiv \pm 1 \pmod 8 \\ 1 & \text{if } d \equiv \pm 3 \pmod 8). \end{cases}$$

To prove this theorem, we have to investigate the manifold W' more carefully. Let $\mathbf{Z}/d$ be the cyclic group of order d. Then $\mathbf{Z}/d$ acts on W' by

$$\zeta(z_0, \dots, z_{n+1}) = (\zeta z_0, \dots, \zeta z_n, z_{n+1}), \qquad \zeta = \exp[2\pi i/d].$$

Let $\mathbf{Z}[1/d]$ be the subring of $\mathbf{Q}$ consisting of all the rational numbers of the form e/d^k, $e, k \in \mathbf{Z}$. Then $H_{n+1}(W'; \mathbf{Z}[1/d])$ is a free $\mathbf{Z}[1/d]$ module of rank $(d-1)^{n+1}$. Let

$$H_{n+1}(W'; \mathbf{Z}[1/d])^{\mathbf{Z}/d} = \{x; \zeta_* x = x\}.$$

and let $v: H_{n+1}(W'; \mathbf{Z}[1/d]) \to H_{n+1}(W'; \mathbf{Z}[1/d])$ be defined by

$$v(x) = x + \zeta_* x + \cdots + \zeta_*^{d-1} x.$$

It is easy to see that $v^2 = dv$ and $\zeta_* v = v$. Let

$$\operatorname{Ker} v = \{x \in H_{n+1}(W'; \mathbf{Z}[1/d]); vx = 0\}.$$

Then we have

LEMMA (4.2).

(i) $H_{n+1}(W'; \mathbf{Z}[1/d]) \cong H_{n+1}(W'; \mathbf{Z}[1/d])^{\mathbf{Z}/d} \oplus \operatorname{Ker} v$.
(ii) *If* $x \in H_{n+1}(W'; \mathbf{Z}[1/d])^{\mathbf{Z}/d}$, $\quad y \in \operatorname{Ker} v$,

then $x \circ y = 0$.

Proof. (i) Let $x \in H_{n+1}(W'; \mathbf{Z}[1/d])$ be any element. Then we have

$$x = (1/d)\, vx + (x - (1/d)\, vx).$$

But

$$\zeta_*(1/d)\, vx = (1/d)\, \zeta_* vx = (1/d)\, vx \quad \text{and} \quad v(x - (1/d)\, vx) = vx - (1/d)\, v^2 x = 0.$$

Thus we have

$$(1/d)\, vx \in H_{n+1}(W'; \mathbf{Z}[1/d])^{\mathbf{Z}/d} \quad \text{and} \quad x - (1/d)\, vx \in \operatorname{Ker} v.$$

Now assume $x \in H_{n+1}(W'; \mathbf{Z}[1/d])^{\mathbf{Z}/d} \cap \operatorname{Ker} v$. Then $\zeta_* x = x$ and $vx = 0$.

Therefore $dx = (1 + \zeta_* + \cdots + \zeta_*^{d-1})\, x = vx = 0$.

But since $H_{n+1}(W'\ \mathbf{Z}[1/d])$ has no d-torsion, it follows that $x = 0$.

(ii) If $x \in H_{n+1}(W'; \mathbf{Z}[1/d])$ and $y \in \operatorname{Ker} v$, then

$$dx \circ y = vx \circ y = x \circ vy = 0.$$

Hence $x \circ y = 0$. This proves Lemma (4.2).

Now let $\{e_1, \ldots, e_s\}$ be a basis for free $\mathbf{Z}[1/d]$-module $\operatorname{Ker} v$. Let $A = (a_{ij})$ be the matrix defined by $a_{ij} = e_i \circ e_j$. Then we claim

LEMMA (4.3).

$$K(V^n(d)) = \begin{cases} 0 & \text{if} \quad \det A \equiv \pm 1 \pmod 8 \\ 1 & \text{if} \quad \det A \equiv \pm 3 \pmod 8). \end{cases}$$

Proof. First note that the bilinear form on $\operatorname{Ker} v$ defined by the intersection number is even. Namely $x \circ x$ is divisible by 2 in $\mathbf{Z}[1/d]$ for any $x \in \operatorname{Ker} v$. Therefore we can construct a quadratic function q on

$$\begin{aligned} \operatorname{Ker} v \otimes \mathbf{Z}/2 &\cong (H_{n+1}(W')/H_{n+1}(W')^{\mathbf{Z}/d}) \otimes \mathbf{Z}/2 \\ &\cong (H_n(W)/H_n(W)^{\mathbf{Z}/d}) \otimes \mathbf{Z}/2 \cong (H_n(W)/K_n(\mathbf{Z})) \otimes \mathbf{Z}/2 \\ &\cong H_n(V) \otimes \mathbf{Z}/2 \cong H_n(V; \mathbf{Z}/2). \end{aligned}$$

by $q(x) = \frac{1}{2} x \circ x \bmod 2$ for $x \in \operatorname{Ker} v$.

By Lemma (3.2), this is the same as the quadratic function $\varphi: H_n(V; \mathbf{Z}/2) \to \mathbf{Z}/2$ defined in §3. Then the lemma follows from [6] §9.

Now let us extend the basis $e_1, \dots, e_s$ by adding elements $f_1, \dots, f_t$ ($f_i \in H_{n+1}(W'; \mathbf{Z}[1/d])^{\mathbf{Z}/d}$) to obtain a basis for $H_{n+1}(W'; \mathbf{Z}[1/d])$. This is possible by Lemma (4.2) (i). We know also by Lemma (4.2) that $e_i \circ f_j = 0$ for any i and j. Let $B = (b_{ij})$ be the matrix defined by

$$b_{ij} = f_i \circ f_j .$$

Then the intersection matrix of W' with respect to the basis $e_1, \dots, e_s, f_1, \dots, f_t$ is given by

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Let $\det W'$ be the determinant of the bilinear form on $H_{n+1}(W'; \mathbf{Z})$ defined by the intersection numbers. Then since $e_1, \dots, e_s, f_1, \dots, f_t$ is a basis for $H_{n+1}(W'; \mathbf{Z}[1/d])$, we have

$$|\det W'| = |\det A| \cdot |\det B| \cdot d^{2a} \tag{4.5}$$

for some $a \in \mathbf{Z}$. Now let us calculate $|\det W'|$ and $|\det B|$. First $|\det W'|$;

LEMMA (4.6).

$$|\det W'| = 2^{\operatorname{rank} K_n(\mathbf{Z})}.$$

Proof. First we recall the following fact.

Let M^{4k} be a $4k$-dimensional oriented compact manifold with boundary. Let $\det M$ be defined by the determinant of bilinear forms on $H_{2k}(M; \mathbf{Z})/\mathrm{Tor}$ defined by the intersection numbers. Then

$$|\det M| = \#[\mathrm{Cok}: H_{2k}(M; \mathbf{Z})/\mathrm{Tor} \to H_{2k}(M, \partial M; \mathbf{Z})/\mathrm{Tor}]. \tag{4.7}$$

Here $\#$ denotes the order of a group if it is finite and zero if it is infinite.

In our case, we know by [8], that $|\det W'| = \# H_n(K; \mathbf{Z})$ where $K = \{z \in \mathbf{C}^{n+2};\ z_0^d + \cdots + z_n^d + z_{n+1}^2 = 0\} \cap S^{2n+3}$. By [6], we have

$$\# H_n(K; \mathbf{Z}) = \prod_{1 \leq k_j \leq d-1} (1 + \zeta^{k_0} \dots \zeta^{k_n}), \quad \zeta = \exp[2\pi i/d], \quad j = 0, \dots, n.$$

Now we show

$$\prod_{1 \leqq k_j \leqq d-1} (1+\zeta^{k_0} \dots \zeta^{k_n}) = 2^{\operatorname{rank} K_n(\mathbf{Z})}, \quad j=0, \dots, n. \tag{4.8}$$

To prove this, we use the induction on n. For simplicity, we write $\beta_{n,d}$ for the left hand side of (4.8). If $n=0$, we have $\beta_{0,d}=(1+\zeta)\dots(1+\zeta^{d-1})$.

But we have

$$x^{d-1}+x^{d-2}+\dots+1=(x-\zeta)\dots(x-\zeta^{d-1}).$$

Substituting $x=-1$, we obtain

$$(-1)^{d-1}(1+\zeta)\dots(1+\zeta^{d-1})=(-1)^{d-1}+\dots+1.$$

Since d is odd, we obtain $\beta_{0,d}=1$. This checks the case $n=0$, for $K_0(\mathbf{Z})=\{0\}$. Now assume that (4.8) holds for $n<k$, $k \geqq 1$. Let us write $\beta_{k-1,d}$ formally as

$$\beta_{k-1,d}=(1+\zeta)^{a_1}\dots(1+\zeta^d)^{a_d}$$

where

$$a_j = \#\{(k_0, \dots, k_n); \quad \textstyle\sum k_j \equiv j \pmod{d}\}.$$

Then by the definition of $\beta_{k,d}$, we have

$$\beta_{k,d} \cdot \beta_{k-1,d} = \{(1+\zeta)\dots(1+\zeta^d)\}^{a_1+\dots+a_d}.$$

But clearly

$$a_1+\dots+a_d=(d-1)^{n+1} \quad \text{and} \quad (1+\zeta)\dots(1+\zeta^d)=2.$$

Therefore

$$\beta_{k,d}\beta_{k-1,d}=2^{(d-1)^{n+1}}.$$

By the induction hypothesis and Lemma (2.4) (ii), we obtain the required result. This proves (4.8) and hence Lemma (4.6). Next we calculate $\det B$.

LEMMA (4.9).

$$\det B = 2^{\operatorname{rank} K_n(\mathbf{Z})} d^{2b+1}.$$

for some $b \in \mathbf{Z}$.

Proof. The action of $\mathbf{Z}/d$ on W' has two fixed points $(0, 0, \ldots, 0, \pm 1)$. Let W_0 be the compact manifold obtained from W' by subtracting an equivariant open tubular neighborhood of $(0, 0, \ldots, 0, \pm 1)$ and ∞ (here ∞ is the "point at infinity"; if $W' \cup \{\infty\}$ is the one point compactitication of W', the action of $\mathbf{Z}/d$ extends to $W' \cup \{\infty\}$). Then $\mathbf{Z}/d$ acts on W_0 freely and $\partial W_0 = K \cup S^{2n+1} \cup S^{2n+1}$. The boundary of the quotient manifold $\bar{W}_0 = W_0/\mathbf{Z}/d$ is

$$\partial(\bar{W}_0) = K/\mathbf{Z}/d \cup L_1 \cup L_2$$

where L_1 and L_2 are lens spaces of type $(d; 1, \ldots, 1)$. By a standard argument of homology for covering spaces, we have an isomorphism

$$H_{n+1}(W_0; \mathbf{Z}[1/d])^{\mathbf{Z}/d} \cong H_{n+1}(\bar{W}_0; \mathbf{Z}[1/d]).$$

Now since degree of the map $W_0 \to \bar{W}_0$ is d and rank $H_{n+1}(W_0; \mathbf{Z})^{\mathbf{Z}/d}$ is even, we have

$$|\det B| = |\det \bar{W}_0|\, d^{2b} \quad \text{for some} \quad b \in \mathbf{Z}.$$

Now the Cartan-Leray spectral sequence yields

(i) $H_{n+1}(\partial \bar{W}_0; \mathbf{Z})$ is a torsion group
(ii) $H_{n+1}(\bar{W}_0; \mathbf{Z})$ is a free abelian group.
(iii) $H_{n+1}(\bar{W}_0, \partial \bar{W}_0; \mathbf{Z})$ is isomorphic to the direct sum of a free abelian group of

the same rank as $H_{n+1}(\bar{W}_0; \mathbf{Z})$ and $\mathbf{Z}/d$.
(iv) $H_n(\partial \bar{W}_0; \mathbf{Z}) \cong H_n(K; \mathbf{Z}) \oplus \mathbf{Z}/d \oplus \mathbf{Z}/d \oplus \mathbf{Z}/d$.
(v) $H_n(\bar{W}_0; \mathbf{Z}) \cong \mathbf{Z}/d$.
(vi) The natural map $H_n(\partial \bar{W}_0; \mathbf{Z}) \to H_n(\bar{W}_0; \mathbf{Z})$ is surjective.

From the above data and (4.7), we obtain $|\det \bar{W}_0| = 2^{\operatorname{rank} K_n(\mathbf{Z})} d$. This proves Lemma (4.9).

Proof of Theorem (1.1).

By Lemma (4.9), we have

$$|\det B| = 2^{\operatorname{rank} K_n(\mathbf{Z})} d^{2b+1}. \tag{4.10}$$

By (4.5) and Lemma (4.6), we have

$$|\det A|\, |\det B|\, d^{2a} = 2^{\operatorname{rank} K_n(\mathbf{Z})}. \tag{4.11}$$

Combining (4.10) and (4.11), we obtain

$$|\det A| = d^{-2(a+b)-1}.$$

But since $d^2 = 1 \pmod 8$ (recall that d is odd), we have $|\det A| \equiv d \pmod 8$. Theorem (4.1) now follows from this by Lemma (4.3).

5. Proof of Theorem (1.3)

Let $e_1, \dots, e_r, f_1, \dots, f_r$ be a symplectic basis for $H_n(V^n(d); \mathbf{Z})$. Thus

$$e_i \circ e_j = f_i \circ f_j = 0$$
$$e_i \circ f_j = \delta_{ij}.$$

By Lemma (2.11), the map $i_*: H_n(W) \to H_n(V)$ is surjective. Therefore, we can choose elements $e'_1, \dots, e'_r, f'_1, \dots, f'_r$ such that $i_*(e'_i) = e_i$ and $i_*(f'_i) = f_i$. Now the Kervaire-Milnor map φ restricted to the submodule of $H_n(W)$ generated by $\{e'_i, f'_i\}_{i=1,\dots,r}$ gives rise to a well defined Art-Kervaire invariant K defined by

$$K = \sum_{i=1}^{r} \varphi(e'_i)\, \varphi(f'_i) \bmod 2.$$

By Haefliger's imbedding theorem [5] and Whitney's technique [9], we can imbed a plumbed manifold U into W to realize the homology classes e'_1, f'_i. The boundary of U is the standard sphere or the Kervaire sphere according as (i) $n = 1$, 3, 7 or n is odd ($\neq 1, 3, 7$) and $K = 0$ or (ii) n is odd ($\neq 1, 3, 7$) and $K = 1$ respectively. Now assume the former. Then ∂U is diffeomorphic to the standard sphere. Moreover U is diffeomorphic to $rS^n \times S^n - \mathring{D}^{2n}$. Look at the complement $V - \mathring{U}$. Since the boundary of this manifold is diffeomorphic to the standard sphere, we can attach a disc D^{2n} along the boundary to obtain a closed differentiable manifold $M^{2n}(d)$. By the construction, clearly M is simply connected and $H_*(M^{2n}(d); \mathbf{Z}) \cong H_*(\mathbf{C}P^n; \mathbf{Z})$.

The above argument proves (i) of Theorem (1.3). (The case $n = 1$ is more or less trivial.) We now prove (ii). First assume that d is even. Then according to Lemma (3.1), there is an element $xh \in K_n(\mathbf{Z})$ such that $\varphi(xh) = 1$. We change the elements e'_i, f'_i as follows:

if $\varphi(e'_i) = 0$, then $e''_i = e'_i$, if $\varphi(e'_i) = 1$ then
$e''_i = e'_i + xh$, the same for f'_i.

Then clearly we have $i_*(e''_i) = e_i$, $i_*(f''_i) = f_i$ and the Kervaire invariant corresponding to e_i'', f_i'' is zero. Now if $d \equiv \pm 1 \pmod 8$, then the Kervaire invariant is zero by Theorem (4.1). Then the same argument as before proves (ii).

Next we prove (iii). Assume the contrary. Then as elements of $\Omega_{8k+2}^{\mathrm{spin}}$, we have

$$[V] = [M] + \frac{b_n}{2}[S^n \times S^n].$$

Therefore

$$K(V) = K(M) + \frac{b_n}{2} K(S^n \times S^n) = K(M).$$

But since $H^n(M; \mathbf{Z}/2) = 0$, we have $K(M) = 0$ and hence $K(V) = 0$. This contradicts Theorem (4.1).

Remark (5.1). The above argument and the generalized Poincaré conjecture show that there are almost smooth manifold $M^{2n}(d)$ and $(n-1)$ connected almost smooth manifold $N^{2n}(d)$ such that

$$H_*(M^{2n}(d); \mathbf{Z}) \cong H_*(\mathbf{C}P^n; \mathbf{Z}), \qquad H_*(N^{2n}(d); \mathbf{Z}) \cong H_*\left(\frac{b_n}{2} S^n \times S^n; \mathbf{Z}\right)$$

and

$$V^n(d) \underset{PL}{\cong} M^{2n}(d) \# N^{2n}(d).$$

Remark (5.2). Let Σ^{2^k-3} be the Kervaire sphere of dimension $2^k - 3$. Then the above argument shows that there is a compact differentiable manifold M^{2^k-2} such that

(i) $\Sigma = \partial M$

(ii) $H_*(M; \mathbf{Z}) \cong H_*(\mathbf{C}P^{2^{k-1}-1} - D; \mathbf{Z})$, in particular $H_{2^{k-1}-1}(M; \mathbf{Z}/2) = 0$.

(iii) all the Stiefel Whitney classes of M vanish.

This follows from considering the variety $V^{2^{k-1}-1}(d)$ with $d \equiv \pm 3 \pmod 8$ and the fact that all the Stiefel Whitney classes of $V^{2^{k-1}-1}(d)$ vanish.

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