Acta Mathematica Sinica, New Series
1986, Vol. 2, No. 4, pp. 299—308

p–adic Continued Fractions III*

Wang Lianxiang (王连祥) Mo Deze (莫德泽)**
Institute of Mathematics, Academia Sinica

Received September 20, 1985

§I. Introduction

In order to discuss the irrationality, the transcendence and the algebraic independence for p–adic numbers, the first author introduced in two previous papers [1, 2] a simple form for p–adic continued fraction which is called p–adic simple continued fraction by making use of the algebraic theory of continued fraction in the real field mentioned by Schmidt[3], and gave a sufficient condition for certain p–adic integers which and whose sum, defference, product and quotient are all p–adic transcendental numbers.

In the present paper we shall apply a technique used in transcendental continued fraction in the real field by Bundschuh[4] and a criterion of algebraic independence for general p–adic numbers due to Wylegala[5] to generalize the results in [2] to the general case. Furthermore, we also obtain a sufficient condition of algebraic independence for a system of p–adic simple continued fractions, and establish the transcendence for the value ξ^η of power exponent function in p–adic numbers ξ and η under some condition by using a theorem on linear forms in logarithms in the p–adic case due to van der Poorten[6].

For the real case the second author have generalized the results of Bundschuh[4] in other paper [7].

§II. Notations, Definitions and Results

Let $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the fields of rational, real and complex numbers respectively. Let $\mathbb{Z}$ denote the domain of the rational integers, and $\mathbb{N}$ the set of the natural numbers. Let p be a fixed prime number, $\mathbb{Q}_p$ represent the field of p–adic numbers, and $\mathbb{Z}_p$ the ring of the integers of $\mathbb{Q}_p$, $\mathbb{C}_p$ the completion of the algebraic closure of $\mathbb{Q}_p$, $\mathbb{A}$ the set of the p–adic algebraic numbers of $\mathbb{C}_p$. Write

$$L_p = \{z \in \mathbb{C}_p \mid |z-1|_p < p^{-1/(p-1)}\},$$
$$U_p = \{z \in \mathbb{C}_p \mid |z|_p = 1\}.$$

The p–adic logarithm function $\log_p z$ is defined as an analytic function in the form

$$\log_p z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(z-1)^n}{n}$$

---

* Project Supported by the Science Fund of the Chinese Academy of Science.

** Now at Leizhou Teachers' College of Guangdong Province Zhanjiang.

which converges in the subdisc of those z in $\mathbb{C}_p$ with $|z-1|_p<1$. The p-adic exponential function $\exp z$ is defined as the series

$$\exp z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

which converges in $\{z\in\mathbb{C}_p \mid |z|_p<p^{-1/(p-1)}\}$. Finally, we define the power exponent function as follows

$$\xi^{\eta}=\exp(\eta\log_p\xi)=\sum_{n=0}^{\infty}\frac{1}{n!}(\eta\log_p\xi)^n$$

where $\xi\in L_p$ and $\eta\in\mathbb{Z}_p$.

For any $\zeta\in\mathbb{Q}_p\backslash\{0\}$ we can represent uniquely ζ as a p-adic simple continued fraction in the form (see [1,2])

$$\zeta=c_0+\cfrac{1}{c_1+\cfrac{1}{c_2+\ddots+c_{n-1}+\cfrac{1}{c_n+\ddots}}}=[c_0,c_1,c_2,\cdots,c_n,\cdots] \tag{1}$$

or

$$\zeta=c_0+\cfrac{1}{c_1+\cfrac{1}{c_2+\ddots+c_{n-1}+\cfrac{1}{\zeta_n}}}=[c_0,c_1,c_2,\cdots,c_{n-1},\zeta_n]$$

where c_n are all finite p-adic fractions with $0\leqslant c_0<p$, $0<c_n<p$, $n\geqslant 1$, and

$$|\zeta_n|_p=|c_n|_p=p^{v_n},\ v_n\in\mathbb{N},\ n\geqslant 0. \tag{2}$$

With these notations and definitions our results read as follows

Theorem 1. *Let* ζ, $\eta\in\mathbb{Q}_p\backslash\{0\}$ *with*

$$\xi=[a_0,a_1,a_2,\cdots] \text{ and } \eta=[b_0,b_1,b_2,\cdots].$$

If there exist a real number r *with* $r>1$ *and a sequence of real numbers* $\{s_n\}\in[1,\infty)^{\mathbb{N}}$ *with* $\lim\limits_{n\to\infty}s_n=\infty$ *such that*

$$|b_{n-1}|_p^{S_n}\leqslant|a_n|_p\leqslant r^{-1}|b_n|_p \tag{3}$$

for sufficiently large n, *then the numbers* ξ *and* η *are algebraically independent.*

Corollary 1. *Under the assumption of Theorem 1, all the six numbers* ξ, η, $\xi\pm\eta$ *and* $\xi\eta^{\pm1}$ *are p-adic transcendental numbers.*

Moreover we have

Theorem 2. *Let* $\xi_k\in\mathbb{Q}_p\backslash\{0\}$ *with*

$$\xi_k=[a_0^{(k)},a_1^{(k)},a_2^{(k)},\cdots],\ k=1,\cdots,q.$$

If there exist a real number r *with* $r>1$ *and a sequence of real numbers* $\{s_n\}\in[1,\infty)^{\mathbb{N}}$ *with* $\lim\limits_{n\to\infty}s_n=\infty$ *such that*

$$|a_n^{(k)}|_p\leqslant r^{-1}|a_n^{(k+1)}|_p,\ k=1,\cdots,q-1, \tag{4}$$

$$\left(\prod_{k=1}^{q}|a_n^{(k)}|_p\right)^{S_n}\leqslant|a_{n+1}^{(1)}|_p \tag{5}$$

for sufficiently large n, *then the numbers* $\xi_1,\cdots,\xi_q$ *are algebraically independent.*

Theorem 3. *Let $\xi \in U_p \cap L_p \setminus \{1\}$ and $\eta \in U_p$. Suppose that they have expansions in p-adic simple continued fractions*

$$\xi = [a_0, a_1, a_2, \cdots] \text{ and } \eta = [b_0, b_1, b_2, \cdots]$$

with

$$\log |a_n|_p \log |b_n|_p = o(\log \min(|a_{n+1}|_p, |b_{n+1}|_p)) \quad (n \to \infty). \tag{6}$$

where $\log x$ *denotes the real logarithm function of x, Then ξ^η is a p*-adic *transcendental number.*

§III. Lemmas

Let us regard $c_0, c_1, c_2, \cdots$ in (1) as variables. We define polynomials p_n, q_n, $n = -1, 0, 1, 2, \cdots$ by the following recurrence relations:

$$\begin{aligned} p_{-1} &= 1, \quad p_0 = c_0, \quad p_n = c_n p_{n-1} + p_{n-2}, \quad n \geqslant 1, \\ q_{-1} &= 0, \quad q_0 = 1, \quad q_n = c_n q_{n-1} + q_{n-2}, \quad n \geqslant 1. \end{aligned} \tag{7}$$

It is easy to verify that

$$\frac{p_n}{q_n} = [c_0, c_1, \cdots, c_n], \quad n \geqslant 0.$$

According to the paper [1] we have

Lemma 1. *Let $\zeta \in \mathbb{Q}_p \setminus \{0\}$ with $\zeta = [c_0, c_1, c_2, \cdots]$. Then*

$$\begin{aligned} &|q_n|_p = |c_1 c_2 \cdots c_n|_p, \quad n \geqslant 1; \\ &|p_n|_p = |c_0|_p |c_1 c_2 \cdots c_n|_p, \quad n \geqslant 1, \text{ if } c_0 \neq 0, \\ &|p_1|_p = 1, \quad |p_n|_p = |c_2 \cdots c_n|_p, \quad n \geqslant 2, \quad \text{if } c_0 = 0; \end{aligned}$$

and

$$\left|\zeta - \frac{p_n}{q_n}\right|_p = |q_n|_p^{-2} |c_{n+1}|_p^{-1}, \quad n \geqslant 1.$$

Now we write for $n \geqslant 1$

$$M_n = \max(|p_n|_p, |q_n|_p),$$
$$P_n' = p_n M_n, \quad Q_n' = q_n M_n.$$

Clearly, we see that M_n, P_n', Q_n' are all in $\mathbb{N}$ and that

$$M_{n+1} = M_n |c_{n+1}|_p.$$

Suppose that $P_n, Q_n \in \mathbb{N}$ with

$$\frac{P_n}{Q_n} = \frac{P_n'}{Q_n'} = \frac{p_n}{q_n} \text{ and } (P_n, Q_n) = 1. \tag{8}$$

Lemma 2. *Let $\zeta \in \mathbb{Q}_p \setminus \{0\}$ with $\zeta = [c_0, c_1, c_2, \cdots]$. Then we have for $n \geqslant 1$*

(i) $|\zeta|_p = \left|\frac{p_n}{q_n}\right|_p$.

(ii) $|\zeta|_p = |Q_n|_p^{-1}$, *if* $c_0 \neq 0$.

Proof. By (1) and (2) we see that

$$|\zeta|_p = \begin{cases} |\zeta_0|_p = |c_0|_0, & \text{if } c_0 \neq 0, \\ |\zeta_1|_p^{-1} = |c_1|_p^{-1}, & \text{if } c_0 = 0. \end{cases}$$

On the other hand, it follows from Lemma 1 that

$$\left|\frac{p_n}{q_n}\right|_p = \begin{cases} |c_0|_p, & \text{if } c_0 \neq 0, \\ |c_1|_p^{-1}, & \text{if } c_0 = 0. \end{cases}$$

Thus the property (i) is proved. To show (ii) it is sufficient to note that

$$|Q_n|_p = |Q_n'|_p = |c_0|_p^{-1}, \text{ if } c_0 \neq 0$$

which is from the equality (12) in [1]. □

Lemma 3. *Let $\zeta\in\mathbb{Q}_p\setminus\{0\}$ with $\zeta=[c_0,c_1,c_2,\cdots]$. If $1\leqslant c_0<p$, then*

$$Q_n<P_n \quad for\ n\geqslant 1.$$

Proof. First, we show by induction that

$$q_n<p_n, \quad for \quad n\geqslant 1.$$

In fact, if $n=1$, we have by (7) that

$$q_1=c_1q_0+q_{-1}=c_1,$$
$$p_1=c_1p_0+p_{-1}=c_1c_0+1$$

which implies $q_1<p_1$ since $c_0\geqslant 1$. If $n=2$, then we have $q_2<p_2$ by noting that

$$q_2=c_2c_1+1<c_2c_1c_0+c_2+c_0=p_2$$

since $c_0\geqslant 1$. Now we assume that $n\geqslant 2$ and $q_{n-1}<p_{n-1}$ and $q_n<p_n$. Then it is easily seen that

$$q_{n+1}=c_{n+1}q_n+q_{n-1}<c_{n+1}p_n+p_{n-1}=p_{n+1}.$$

Thus we complete the step of induction. In view of (8) we deduce that

$$\frac{P_n}{Q_n}=\frac{p_n}{q_n}>1 \text{ for } n\geqslant 1$$

which completes the proof of Lemma 3.

Lemma 4. *Let $\zeta\in\mathbb{Q}_p\setminus\{0\}$ with $\zeta=[c_0,c_1,c_2,\cdots]$ and let s be any real number with $s>1$. If*

$$|c_{n-1}|_p^s<|c_n|_p, \ for\ sufficiently\ large\ n,$$

then

$$\max(P_n,Q_n)<|c_n|_p^{s'} \qquad for\ sufficiently\ large\ n,$$

where

$$s'=\max\left(3,\frac{2s}{s-1}\right).$$

Proof. Noting that $s>1$ and (2) we see that the assumption of Lemma 4 implies that $\lim_{n\to\infty}|c_n|_p=\infty$. Hence there exists a natural number $n_0>1$ such that

$$P_{n_0}\leqslant P'_{n_0}<|c_{n_0}|_p^{s''}|c_n|_p,\ P_{n_0+1}\leqslant P'_{n_0+1}<|c_{n_0+1}|_p^{s''}|c_n|_p$$

for $n\geqslant n_0$, where

$$s''=\max\left(2,\frac{s+1}{s-1}\right).$$

In general, we shall prove by induction on m that

$$P_m\leqslant P'_m<|c_n|_p|c_m|_p^{s''} \text{ for } m\geqslant n_0. \tag{9}$$

Assume that the above inequalities hold for m and $m-1$. Then we have

$$\begin{aligned}P'_{m+1}&=p_{m+1}M_{m+1}\\&=(c_{m+1}p_m+p_{m-1})M_m|c_{m+1}|_p\\&=(c_{m+1}p_mM_m+p_{m-1}M_{m-1}|c_m|_p)|c_{m+1}|_p\\&<(p|c_n|_p|c_m|_p^{s''}+|c_n|_p|c_{m-1}|_p^{s''}|c_m|_p)|c_{m+1}|_p\\&<|c_n|_p|c_m|_p^{s''}(p+|c_m|_p^{s''/s+1-s''})|c_{m+1}|_p\\&<|c_n|_p|c_{m+1}|_p^{(s''+1)/s+1}\\&\leqslant|c_n|_p|c_{m+1}|_p^{s''}\end{aligned}$$

since $|c_m|_p\geqslant 2p$. Taking $m=n$ in (9) we obtain

$$P_n\leqslant P'_n<|c_n|_p|c_n|_p^{s''}=|c_n|_p^{s'} \text{ for sufficiently large } n.$$

Similarly, we have

$$Q_n \leqslant Q_n' < |c_n|_p^{s'} \text{ for sufficiently large } n.$$

This completes the proof of Lemma 4. □

Remark. If $n_0 = 1$ in the proof of Lemma 4, then it is easy to verify that

$$P_i' < |c_i|_p^{s'},\ i = 1, 2.$$

Moreover, we obtain the inequalities

$$\max(P_n, Q_n) < |c_n|_p^{s'}, \text{ for all } n \geqslant 1.$$

Suppose that $\alpha \in \mathbb{A}$ and $p(x)$ is its minimal polynomial with degree d. Let $K = \mathbb{Q}(\theta)$ be the splitting field in $\mathbb{C}$ of $p(x)$ with $\theta \in \mathbb{C}$. As usual, a place of K corresponds a valuation of K. Let V denote the set of all places of K. Write V_∞ and V_0 for the set of infinite and finite places of K which correspond all Archimedean and non–Archimedean valuations, respectively. Write $n_v = [K_v : \mathbb{Q}_v]$ for every place v of K. If $v \in V_0$ and v lies p we write $v|p$. We normalize the v–adic valuation $|\cdot|_v$ so that

(i) if $v \in V_\infty$ and if v is real, then $|\theta|_v = |\theta|$,

(ii) if $v \in V_\infty$ and if v is complex, then $|\theta|_v = |\theta|^2$,

where $|\cdot|$ represents the ordinary absolute value in $\mathbb{R}$ or $\mathbb{C}$.

(iii) if $v|p$, then $|\theta|_v = (\phi_v(\theta))^{n_v}$, where $\phi_v(\theta)$ is a corresponding non–Archimedean valuation. In particular,

$$|p|_v = p^{-n_v}.$$

We define the absolute height of the algebraic number θ by the formula

$$H(\theta) = \prod_{v \in V} \max(1, |\theta|_v),$$

and the absolute logarithmic height of θ by

$$h(\theta) = [K : \mathbb{Q}]^{-1} \log H(\theta).$$

In this paper we denote the ordinary height of θ (that is the maximum of the absolute values of coefficients of the minimal polynomial of θ) by $\bar{H}(\theta)$. We define the absolute logarithmic height of p–adic number $\alpha \in \mathbb{A}$ by $h(\alpha) = h(\theta)$. □

Lemma 5. *(i) $h(\alpha^m) = mh(\alpha)$ for any $\alpha \in \mathbb{A}$ and $m \in \mathbb{N}$. (ii) $h(a/b) = \log \max(|a|, |b|) = \log \bar{H}(a/b)$ for $a, b \in \mathbb{Z}$ with $b \neq 0$ and $(a, b) = 1$.*

Proof. See [8].

Lemma 6. *Let $\alpha_1, \cdots, \alpha_q \in \mathbb{C}_p$ be the limits (p–adic convergence) of a sequences of algebraic numbers $\{a_1^{(n)}\}, \cdots, \{a_q^{(n)}\}$, respectively. If there are sequences $\{g_n\}$ and $\{h_n\} \in [1, \infty)^{\mathbb{N}}$ with $\lim_{n\to\infty} g_n = \lim_{n\to\infty} h_n = \infty$ such that for sufficiently large n*

$$0 < |\alpha_{k+1} - a_{k+1}^{(n)}|_p \leqslant g_n^{-1} |\alpha_k - a_k^{(n)}|_p,\ k = 1, \cdots, q-1, \tag{10}$$

$$0 < |\alpha_1 - a_1^{(n)}|_p \leqslant \exp\left(-h_n D^{(n)} \sum_{k=1}^{q} h(a_k^{(n)})\right) \tag{11}$$

where $D^{(n)} = [\mathbb{Q}(a_1^{(n)}, \cdots, a_q^{(n)}) : \mathbb{Q}]$, then the numbers $\alpha_1, \cdots, \alpha_q$ are algebraically independent.

Proof. See Chapter III in [5].

Lemma 7. *Let $\xi, \eta, \zeta \in \mathbb{Q}_p \cap L_p$. Then*

(i) $\log_p(\xi\eta) = \log_p \xi + \log_p \eta$,

(ii) $|\log_p \zeta|_p = |\zeta - 1|_p$,

(iii) $\log_p \xi = \log_p \eta \Rightarrow \xi = \eta$.

Proof. See [9] (or [5], [10]).

Lemma 8. *Let* $\alpha_1, \cdots, \alpha_q \in \mathbb{A} \cap L_p$ *with*

$$D = [\mathbb{Q}(\alpha_1, \cdots, \alpha_q) : \mathbb{Q}],$$
$$\max(e^e, H(\alpha_k)) \leqslant A_k,\ k = 1, \cdots, q.$$

Put

$$\Omega = \log A_1 \cdots \log A_{q-1}.$$

Then the inequalities

$$0 < \left| \sum_{k=1}^{q} b_k \log_p \alpha_k \right|_p < \exp(-C\ \Omega\ \log \Omega \log A_q\ \log B) \tag{12}$$

have no solutions in rational integers $b_1, \cdots, b_q$ *with*

$$b_q \not\equiv 0 \pmod p \text{ and } B = \max(|b_1|, \cdots, |b_q|)$$

where C *is a constant depending only on* p, q, D.

Proof. Note that

$$|\alpha_1^{b_1} \cdots \alpha_q^{b_q} - 1|_p = \left| \sum_{k=1}^{q} b_k \log_p \alpha_k \right|_p$$

according to (ii) in Lemma 7 and see Theorem 1 in [6] and Chapter II in [5].

§IV. Proofs of Theorems

Obviously, Theorem 1 and Corollary 1 are special cases, so we give here the proof of Theorem 2 only.

Proof of Theorem 2. By means of Lemma 1 and the condition (4) in Theorem 2 we see that

$$0 < \left| \xi_{k+1} - \frac{p_n(\xi_{k+1})}{q_n(\xi_{k+1})} \right|_p = |q_n(\xi_{k+1})|_p^{-2} |a_{n+1}^{(k+1)}|_p^{-1}$$
$$\leqslant r^{-(2n+1)} |q_n(\xi_k)|_p^{-2} |a_{n+1}^{(k)}|_p^{-1} = r^{-(2n+1)} \left| \xi_k - \frac{p_n(\xi_k)}{q_n(\xi_k)} \right|_p.$$

Take $g_n = r^{2n+1}$. Then the condition (10) in Lemma 6 is satisfied. To show that the condition (11) is satisfied we shall prove the following inequalities

$$\max(P_n(\xi_k), Q_n(\xi_k)) < |a_n^{(k)}|_p^{s'}, \qquad k = 1, \cdots, q, \tag{13}$$

for sufficiently large n, where s' is a constant determined below. According to the hypotheses (4) and (5) of Theorem 2, we see that for sufficiently large n

$$|a_n^{(k)}|_p^s < \left(\prod_{k=1}^{q} |a_n^{(k)}|_p \right)^{S_n} \leqslant |a_{n+1}^{(1)}|_p < |a_{n+1}^{(k)}|_p \tag{14}$$

where s is a given constant with $s > 1$. Therefore (13) hold Lemma 4. Clearly, we have

$$0 < \left| \xi_1 - \frac{p_n(\xi_1)}{q_n(\xi_1)} \right|_p = |q_n(\xi_1)|_p^{-2} |a_{n+1}^{(1)}|_p^{-1} < |a_{n+1}^{(1)}|_p^{-1} \leqslant \left(\prod_{k=1}^{q} |a_n^{(k)}|_p \right)^{-S_n} \tag{15}$$

by (14). On the other hand, by use of Lemma 5 (ii) and (13) with $s' = \max(3, 2s/(s-1))$ and by noting that $D^{(n)} = 1$, we obtain

$$\exp\left(-h_n D^{(n)} \sum_{k=1}^{q} h\left(\frac{p_n(\xi_k)}{q_n(\xi_k)} \right) \right) = \prod_{k=1}^{q} (\max(P_n(\xi_k), Q_n(\xi_k)))^{-h_n} > \prod_{k=1}^{q} (|a_n^{(k)}|_p)^{-S' h_n}. \tag{16}$$

Choosing $h_n = s_n/s$ and comparing (15) with (16) we have

$$0 < \left|\xi_1 - \frac{p_n(\xi_1)}{q_n(\xi_1)}\right|_p < \exp\left(-h_n D^{(n)} \sum_{k=1}^{q} h\left(\frac{p_n(\xi_k)}{q_n(\xi_k)}\right)\right)$$

for sufficiently large n. Hence the condition (11) in Lemma 6 is satisfied. And so we established the algebraic independence of the system of p-adic numbers $\xi_1, \cdots, \xi_q$.

Now we give some examples of p-adic simple continued fractions which are algebraically independent.

Example 1. In Theorem 1 put $r = p$, $s_n = n$, $|a_0|_p = p$, $|b_0|_p = p^2$ and

$$|b_{n-1}|_p^n = |a_n|_p = p^{-1}|b_n|_p, \quad n \geqslant 1.$$

This implies that

$$|a_1|_p = p^2, \quad |b_1|_p = p^3, \quad |a_2|_p = p^6, \quad |b_2|_p = p^7,$$
$$|a_3|_p = p^{21}, \quad |b_3|_p = p^{22}, \quad \cdots.$$

By Theorem 1 the p-adic continued fractions

$$\xi = [p^{-1}, p^{-2}, p^{-6}, p^{-21}, \cdots]$$

and

$$\eta = [p^{-2}, p^{-3}, p^{-7}, p^{-22}, \cdots]$$

are algebraically independent. The other two numbers

$$\xi = [(p+1)p^{-1}, (p+1)p^{-2}, (p+1)p^{-6}, (p+1)p^{-21}, \cdots]$$

and

$$\eta = [(p+1)p^{-2}, (p+1)p^{-3}, (p+1)p^{-7}, (p+1)p^{-22}, \cdots]$$

are also algebraically independent. □

Example 2. In Theorem 2 take $r = p$, $s_n = 3n$, and $|a_0^{(1)}|_p = p$, $|a_0^{(k)}|_p^3 = |a_1^{(k)}|_p$, $k = 1, \cdots, q$, and

$$|a_n^{(k)}|_p = p^{-1}|a_n^{(k+1)}|_p, \ k = 1, \cdots, q-1,$$
$$|a_{n+1}^{(1)}|_p = \left(\prod_{k=1}^{q} |a_n^{(k)}|_p\right)^{3n}.$$

This implies that

$$|a_0^{(1)}|_p = p, \ |a_0^{(2)}|_p = p^2, \cdots, |a_0^{(q)}|_p = p^q,$$
$$|a_1^{(1)}|_p = p^3, \ |a_1^{(2)}|_p = p^6, \cdots, |a_1^{(q)}|_p = p^{3q},$$
$$|a_2^{(1)}|_p = p^{9q(q+1)/2}, \ |a_2^{(2)}|_p = p^{9q(q+1)/2+1}, \cdots,$$
$$|a_2^{(q)}|_p = p^{9q(q+1)/2+q-1}, \cdots.$$

According to Theorem 2, p-adic continued fractions

$$\xi_1 = [p^{-1}, p^{-3}, p^{-9q(q+1)/2+q-1}, \cdots],$$
$$\xi_2 = [p^{-2}, p^{-6}, p^{-9q(q+1)/2-1}, \cdots],$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$\xi_q = [p^{-q}, p^{-3q}, p^{-9q(q+1)/2-q+1}, \cdots],$$

are algebraically independent.

Proof of Theorem 3. First, given a real number $s > 1$, we shall show under the assumption of Theorem 3 that for any real number $C_1 > 0$ there exists a natural number $n_1 = n_1(s, C_1)$ such that

$$\max\left(\left|\xi - \frac{p_n(\xi)}{q_n(\xi)}\right|_p, \left|\eta - \frac{p_n(\eta)}{q_n(\eta)}\right|_p\right) \leqslant \exp(-C_1 \log H_1 \log H_2) \tag{17}$$

where

$$\left(\frac{p_n(\xi)}{q_n(\xi)}, \frac{p_n(\eta)}{q_n(\eta)}, H_1, H_2\right) \in \mathbb{Q}^2 \times \mathbb{N}^2,$$

$$H_1 = H\left(\frac{p_n(\xi)}{q_n(\xi)}\right) = \max(P_n(\xi),\ Q_n(\xi)),$$

$$H_2 = H\left(\frac{p_n(\eta)}{q_n(\eta)}\right) = \max(P_n(\eta),\ Q_n(\eta)).$$

Indeed, from Lemma 1 we have

$$\left|\xi - \frac{p_n(\xi)}{q_n(\xi)}\right|_p = |q_n(\xi)|_p^{-2} |a_{n+1}|_p^{-1} < |a_{n+1}|_p^{-1},\ n \geqslant 1. \tag{18}$$

In view of (6) we see that

$$\log|a_n|_p = o(\log|a_{n+1}|_p),\ n \to \infty,$$
$$\log|b_n|_p = o(\log|b_{n+1}|_p),\ n \to \infty,$$

which imply that for any real number $s > 1$,

$$|a_n|_p^s < |a_{n+1}|_p,\ |b_n|_p^s < |b_{n+1}|_p$$

provided that n is large enough. Hence for sufficiently large n we have

$$\max(P_n(\xi),\ Q_n(\xi)) < |a_n|_p^{s'},$$
$$\max(P_n(\eta),\ Q_n(\eta)) < |b_n|_p^{s'},$$

according to Lemma 4. Again by (6) we see that for any $C_1 > 0$,

$$\begin{aligned}\log H_1 \log H_2 &= \log\ \max(P_n(\xi), Q_n(\xi))\ \log\ \max(P_n(\eta), Q_n(\eta)) \\ &\leqslant s'^2 \log|a_n|_p \log|b_n|_p \leqslant C_1^{-1} \log|a_{n+1}|_p,\end{aligned}$$

i.e.

$$|a_{n+1}|_p^{-1} < \exp(-C_1 \log H_1 \log H_2)$$

for sufficiently large n. This together with (18) implies that

$$\left|\xi - \frac{p_n(\xi)}{q_n(\xi)}\right|_p < \exp(-C_1 \log H_1 \log H_2)$$

for sufficiently large n. Similarly, we can obtain

$$\left|\eta - \frac{p_n(\eta)}{q_n(\eta)}\right|_p < \exp(-C_1 \log H_1 \log H_2)$$

for sufficiently large n. So we complete the proof of (17).

On the other hand, by Lemma 1 we see that

$$\begin{aligned}\left|\xi^{-1}\frac{p_n(\xi)}{q_n(\xi)} - 1\right|_p &= \left|\xi^{-1}\left(\frac{p_n(\xi)}{q_n(\xi)} - \xi\right)\right|_p = \left|\frac{p_n(\xi)}{q_n(\xi)} - \xi\right|_p \\ &= |q_n(\xi)|_p^{-2} |a_{n+1}|_p^{-1} < |a_{n+1}|_p^{-1} \leqslant p^{-1} \leqslant p^{-1/(p-1)}.\end{aligned}$$

Therefore $\xi^{-1}\frac{p_n(\xi)}{q_n(\xi)} \in L_p$. It follows from Lemma 7 (i) that

$$\log_p\left(\frac{p_n(\xi)}{q_n(\xi)}\right) = \log_p\left(\xi^{-1}\frac{p_n(\xi)}{q_n(\xi)}\right) + \log_p \xi.$$

By means of (17) and Lemma 7 (ii) we obtain

$$\left|\log_p\left(\frac{p_n(\xi)}{q_n(\xi)}\right) - \log_p\xi\right|_p = \left|\log_p\left(\xi^{-1}\frac{p_n(\xi)}{q_n(\xi)}\right)\right|_p$$

$$= \left|\xi^{-1}\frac{p_n(\xi)}{q_n(\xi)} - 1\right|_p = \left|\xi - \frac{p_n(\xi)}{q_n(\xi)}\right|_p$$

$$< \exp(-C_{\text{.}}\log H_1 \log H_2), \text{ for sufficiently large } n. \tag{19}$$

Assume now that the assrtion of Theorem 3 is fales, and that ξ^η is algebraic over $\mathbb{Q}$. Then we apply Lemma 8 with $q = 2$, $\alpha_1 = \gamma = \xi^\eta$, $\alpha_2 = \frac{P_n(\xi)}{Q_n(\xi)}$, $b_1 = Q_n(\eta)$, $b_2 = -P_n(\eta)$. Without loss of generality, we may assume that $(P_n(\eta), p) = 1$ (see [1]). Clearly, we see from (19) and Lemma 2 that

$$|b_1 \log_p \alpha_1 + b_2 \log_p \alpha_2|_p$$

$$= |Q_n(\eta)|_p \left|\log_p\gamma - \frac{p_n(\eta)}{q_n(\eta)}\log_p\left(\frac{P_n(\xi)}{Q_n(\xi)}\right)\right|_p$$

$$= |Q_n(\eta)|_p \left|\log_p\xi\left(\eta - \frac{p_n(\eta)}{q_n(\eta)}\right) + \frac{p_n(\eta)}{q_n(\eta)}\left(\log_p\xi - \log_p\left(\frac{p_n(\xi)}{q_n(\xi)}\right)\right)\right|_p$$

$$\leqslant |\eta|_p^{-1}\max\left(|\log_p\xi|_p\left|\eta - \frac{p_n(\eta)}{q_n(\eta)}\right|_p, \ |\eta|_p\left|\xi - \frac{p_n(\xi)}{q_n(\xi)}\right|_p\right)$$

$$\leqslant \exp(-C_1 \log H_1 \log H_2) \tag{20}$$

since $|\log_p\xi|_p = |\xi - 1|_p < p^{-1/(p-1)} < 1$ and $|\eta|_p = 1$. If

$$|b_1 \log_p\alpha_1 + b_2 \log_p\alpha_2|_p \neq 0,$$

according to Lemma 8 we have

$$|b_1 \log_p\alpha_1 + b_2 \log_p\alpha_2|_p$$

$$> \exp\left(-C\log H(\gamma)\log\log H(\gamma)\log H\left(\frac{P_n(\xi)}{Q_n(\xi)}\right)\log\max(P_n(\eta), Q_n(\eta))\right)$$

$$= \exp(-C_2 \log H_1 \log H_2) \tag{21}$$

where

$$C_2 = C\log A_1 \log\log A_1, A_1 = \max(e^e, H(\gamma)).$$

Taking $C_1 = 2C_2$, the inequalities (20) and (21) give a contradiction. Therefore we obtain

$$\frac{P_n(\eta)}{Q_n(\eta)}\log_p\left(\frac{P_n(\xi)}{Q_n(\xi)}\right) = \log_p\gamma,$$

and so

$$\left(\frac{P_n(\xi)}{Q_n(\xi)}\right)^{P_n(\eta)} = \gamma^{Q_n(\eta)}$$

by Lemma 7. According to Lemma 5 (i) we have

$$P_n(\eta)h\left(\frac{P_n(\xi)}{Q_n(\xi)}\right) = Q_n(\eta)h(\gamma).$$

Noting that $Q_n(\eta) < P_n(\eta), n \geqslant 1$ by Lemma 3, we see that

$$\log H_1 = \log H\left(\frac{P_n(\xi)}{Q_n(\xi)}\right) = h\left(\frac{P_n(\xi)}{Q_n(\xi)}\right) < h(\gamma).$$

This means that $\log H_1$ is bounded by the constant $h(\gamma)$. However, this is impossible since $\{P_n(\xi)\}$, $\{Q_n(\xi)\}$ are all unbounded sequences of natural numbers. Consequently the number ξ^η must be transcendental.

Finally, we give

Example 3. In Theorem 3, put $a_0 = 1$, $|a_1|_p = p^2$, and $|a_{n+1}|_p = p^{|a_n|_p}$, $a_n = b_n$, $n \geqslant 1$. It is easy to verify that $|\xi - 1|_p = p^{-2} < p^{-1/(p-1)}$ and so $\xi \in U_p \bigcap L_p \setminus \{1\}$, $\eta \in U_p$, and that $\log |a_n|_p \log |a_n|_p = o(\log |a_{n+1}|_p)$ $(n \to \infty)$. Thus ξ^ξ is a p-adic transcendental number by Theorem 3.

References

[1] Wang Lianxiang, *Scientia Sinica*, Series A, **28**(1985), 1009—1017.

[2] ————, *Scientia Sinica*, Series A, **28**(1985), 1018—1022,

[3] Schmidt, W.M., Diophantine Approximations, Lecture Notes in Math., 785, Springer—Verlag, Berlin, Heidelberg, New York, 1980.

[4] Bundschuh, P., *J.Number Theory*, **18**(1984), 91—98.

[5] Wylegala, F.J., Approximationsmaße und spezielle Systeme algebraisch unabhängiger p-adischer Zahlen Inaugural-Dissertation zur Erlangung des Doktorgrades der Mathematisch-Naturwissenschaftlichen Fakultät der Universität zu Köln, 1980.

[6] van der Poorten, A.J., Linear forms in logarithms in the p-adic case, in "Transcendence Theory: Advances and Applications" (A. Baker and D.W. Masser Eds), p 29—57, *Academic Press*, London, New York, San Francisco, 1977.

[7] Mo Deze, To appear in *Kexue Tongbao, Sinica*.

[8] Waldschmidt, M., *Acta Arith.*, **37**(1980), 257—283.

[9] Bachman, G., Introduction to p-adic Numbers and Valuation Theory, *Academic Press*, New York, London, 1964.

[10] Artin,E. and Tate,J., Class Field Theory.