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On the Theory of Probabilistic Metric Spaces with Applications*

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§ 0. Introduction

The purpose of this paper is to investigate the theory of probabilistic metric spaces (PM-spaces) and its applications. In § 1 we introduce a kind of Menger PM-spaces. By virtue of their basic properties and the Menger-Hausdorff metric defined for this kind of spaces, in § 2 we shall give some fixed point theorems for multi-valued mappings on PM-spaces. In addition, in § 3 we shall give some fixed point theorems for one-valued mappings on PM-space, which generalize and improve some recent results of [1—4]. As an application, in § 4 we shall use results in § 2 and § 3 to study the fixed point theorems for multi-valued mappings on PM-spaces and the existence and uniqueness of the solution of nonlinear Volterra integral equations on Banach spaces.

Throughout this paper let $R=(-\infty, \infty)$, $R^+=[0, \infty)$, Z^+ be the set of all positive integers, $\mathscr{D}$ the set of all (left-continuous) distribution functions on R, and H a special distribution function defined by

$$H(t)=\begin{cases}1, & t>0,\\ 0, & t\leqslant 0.\end{cases}$$

The definitions, terms and notations in this paper relating to PM-space are adopted from [7, 5].

As Schweizer and Sklar[5] point out that if $(E, \mathscr{F}, \Delta)$ is a Menger PM-space (briefly Menger space) with a continuous t-norm Δ, then $(E, \mathscr{F}, \Delta)$ is a Hausdorff space in the topology $\mathscr{T}$ induced by the family of neighborhoods

$$\{U_p(s, \lambda): p\in E,\ s>0,\ \lambda>0\},$$

where

$$U_p(s, \lambda)=\{x\in E:\ F_{x,p}(s)>1-\lambda\}.$$

§ 1. Menger Space $(E, \mathscr{F}, \Delta)$

Definition 1. Let $(E, \mathscr{F}, \Delta)$ be a Menger space with continuous t-norm Δ,

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and A a nonempty subset of E. If $\sup_{t>0} D_A(t)=1$, where

$$D_A(t)=\sup_{s<t}\inf_{p,q\in A} F_{p,q}(s),$$

then A is called a probabilistically bounded set, and $D_A(t)$ the probabilistic diameter of A.

Proposition 1.1. *Let $(E, \mathscr{F}, \Delta)$ be a Menger space with continuous t-norm Δ.*

(i) *If A is a probabilistically bounded set, then $D_A(t)$ is a distribution function.*

(ii) *If A, $B\subset E$ are any two probabilistically bounded sets, then $A\cup B$ is also a probabilistically bounded set of E.*

Proof. (i) Since A is probabilistically bounded, by definition it is easy to see that $D_A(t)$ is nondecreasing in t, $D_A(0)=0$, $\sup_{t>0} D_A(t)=1$ and $D_A(t)$ is left-continuous in t. This shows that $D_A(t)$ is a distribution function.

(ii) Since A and B are probabilistically bounded, so is $B\backslash A$. From Theorem 10 of [8] we have

$$D_{A\cup B}(t)=D_{A\cup B\backslash A}(t)\geqslant\Delta\Big(D_A\Big(\frac{t}{2}\Big),\ D_{B\backslash A}\Big(\frac{t}{2}\Big)\Big),$$

and therefore by the continuity of Δ

$$\sup_{t>0} D_{A\cup B}(t)\geqslant\sup_{t>0}\Delta\Big(D_A\Big(\frac{t}{2}\Big),\ D_{B\backslash A}\Big(\frac{t}{2}\Big)\Big)$$
$$=\Delta\Big(\sup_{t>0} D_A\Big(\frac{t}{2}\Big),\ \sup_{t>0} D_{B\backslash A}\Big(\frac{t}{2}\Big)\Big)=\Delta(1, 1)=1. \qquad \blacksquare$$

In the rest of this section we always assume that $(E, \mathscr{F}, \Delta)$ is a Menger space with continuous t-norm Δ, and Ω the family of all nonempty $\mathscr{T}$-closed probabilistically bounded sets. We define a mapping $\tilde{\mathscr{F}}$ as follows (we denote $\tilde{\mathscr{F}}(A, B)$ by $\tilde{F}_{A,B}$ and the value of $\tilde{F}_{A,B}$ at $t\in R$ by $\tilde{F}_{A,B}(t)$):

$$\tilde{F}_{A,B}(t)=\sup_{s<t}\Delta(\inf_{a\in A}\sup_{b\in B} F_{a,b}(s),\ \inf_{b\in B}\sup_{a\in A} F_{a,b}(s)),\quad \forall A,\ B\in\Omega. \tag{1.1}$$

$\tilde{\mathscr{F}}$ is called the Menger-Hausdorff metric induced by $\mathscr{F}$.

Proposition 1.2. *$(\Omega, \tilde{\mathscr{F}}, \Delta)$ is a Menger space, i.e. $\tilde{\mathscr{F}}$ is a mapping from $\Omega\times\Omega$ into $\mathscr{D}$ satisfying the following conditions:*

(i) $\tilde{F}_{A,B}(t)=1$, $\forall t>0$ *if and only if* $A=B$;

(ii) $\tilde{F}_{A,B}(0)=0$;

(iii) $\tilde{F}_{A,B}=\tilde{F}_{B,A}$;

(iv) $\tilde{F}_{A,B}(t_1+t_2)\geqslant\Delta(\tilde{F}_{A,C}(t_1),\ \tilde{F}_{C,B}(t_2))$, $\forall A,\ B,\ C\in\Omega$ *and* $t_1,\ t_2\geqslant 0$.

Proof. By the definition of $\tilde{\mathscr{F}}$ it is easy to see that $\tilde{F}_{A,B}(t)$ is nondecreasing and left-continuous in t, and $\tilde{F}_{A,B}(0)=0$. Now we prove that

$$\sup_{t>0}\tilde{F}_{A,B}(t)=1.$$

In fact, since A, $B\in\Omega$, we have $A\cup B\in\Omega$. By the continuity of Δ we have

$$\sup_{t>0}\tilde{F}_{A,B}(t)=\sup_{t>0}\sup_{s<t}\Delta(\inf_{a\in A}\sup_{b\in B} F_{a,b}(s),\ \inf_{b\in B}\sup_{a\in A} F_{a,b}(s))$$
$$\geqslant\Delta(\sup_{t>0}\sup_{s<t}\inf_{a\in A}\inf_{b\in B} F_{a,b}(s)\ \sup_{t>0}\sup_{s<t}\inf_{b\in B}\inf_{a\in A} F_{a,b}(s))$$
$$\geqslant\Delta(\sup_{t>0} D_{A\cup B}(t),\ \sup_{t>0} D_{A\cup B}(t))=\Delta(1, 1)=1.$$

This shows that $\mathscr{F}$ is a mapping from $\Omega\times\Omega$ into $\mathscr{D}$. It is obvious that $\mathscr{F}$ satisfies conditions (ii) and (iii). From Theorem 17 of [8] it follows that $\mathscr{F}$ satisfies condition (iv). To show $\widetilde{\mathscr{F}}$ satisfies condition (i), we first suppose $\forall t>0$, $\widetilde{F}_{A,B}(t)=1$. By the continuity of Δ, for any $\varepsilon>0$ we have

$$1=\widetilde{F}_{A,B}(\varepsilon)=\sup_{s<\varepsilon}\Delta(\inf_{a\in A}\sup_{b\in B}F_{a,b}(s),\ \inf_{b\in B}\sup_{a\in A}F_{a,b}(s))$$
$$=\Delta(\sup_{s<\varepsilon}\inf_{a\in A}\sup_{b\in B}F_{a,b}(s),\ \sup_{s<\varepsilon}\inf_{b\in B}\sup_{a\in A}F_{a,b}(s)).$$

This implies that

$$\sup_{s<\varepsilon}\inf_{a\in A}\sup_{b\in B}F_{a,b}(s)=1; \tag{1.2}$$

$$\sup_{s<\varepsilon}\inf_{b\in B}\sup_{a\in A}F_{a,b}(s)=1. \tag{1.3}$$

From (1.2) we have $\sup_{b\in B}F_{a,b}(\varepsilon)=1$, $\forall a\in A$. Therefore for any $a\in A$ and any $\lambda>0$ there exists $b_*\in B$ such that

$$F_{a,b_*}(\varepsilon)>1-\lambda.$$

This shows that a is a $\mathscr{T}$-accumulation point of B, hence $a\in B$, i.e. $A\subseteq B$.

Similarly we can prove that $B\subseteq A$. Therefore we have $A=B$.

Conversely, if $A=B$, then for any $t>0$ and any $s\in(0,t)$ we have

$$\widetilde{F}_{A,B}(t)\geqslant\Delta(\inf_{a\in A}\sup_{b\in B}F_{a,b}(s),\ \inf_{b\in B}\sup_{a\in A}F_{a,b}(s))=\Delta(1,1)=1. \qquad \blacksquare$$

Definition 2. Let $A\in\Omega$ and $x\in E$. The probabilistic distance between point x and set A is the function $F_{x,A}$ defined by

$$F_{x,A}(t)=\sup_{s<t}\sup_{y\in A}F_{x,y}(s),\quad \forall t\geqslant0.$$

Proposition 1.3. *Let $A\in\Omega$, and x, y be arbitrary points of E. Then*

(i) $F_{x,A}(t)=1$, $\forall t>0$ *if and only if* $x\in A$;

(ii) $F_{x,A}(t_1+t_2)\geqslant\Delta(F_{x,y}(t_1),\ F_{y,A}(t_2))$, $\forall t_1,\ t_2\geqslant0$;

(iii) *For any A, $B\in\Omega$ and $x\in A$,*

$$F_{x,B}(t)\geqslant\widetilde{F}_{A,B}(t),\quad\forall t\geqslant0.$$

Proof. (i) If $x\in A$, then for any $t>0$ and any $s\in(0,t)$ we have

$$F_{x,A}(t)\geqslant\sup_{y\in A}F_{x,y}(s)\geqslant F_{x,x}(s)=1.$$

This shows that

$$F_{x,A}(t)=1,\quad\forall t>0.$$

Conversely, if $F_{x,A}(t)=1$, $\forall t>0$, then for any $\varepsilon>0$ we have

$$1=F_{x,A}(\varepsilon)=\sup_{s<\varepsilon}\sup_{y\in A}F_{x,y}(s)=\sup_{y\in A}F_{x,y}(\varepsilon).$$

This implies that for any $\lambda>0$ there exists $y_*\in A$ such that

$$F_{x,y_*}(\varepsilon)>1-\lambda.$$

It follows that x is a $\mathscr{T}$-accumulation point of A and $x\in A$.

(ii) By the Menger triangle inequality and the continuity of Δ we have

$$F_{x,A}(t_1+t_2)=\sup_{s_1+s_2<t_1+t_2}\sup_{z\in A}F_{x,z}(s_1+s_2)\geqslant\sup_{s_1+s_2<t_1+t_2}\Delta(F_{x,y}(s_1),\ \sup_{z\in A}F_{y,z}(s_2))$$
$$\geqslant\Delta(\sup_{s_1<t_1}F_{x,y}(s_1),\ \sup_{s_2<t_2}\sup_{z\in A}F_{y,z}(s_2))=\Delta(F_{x,y}(t_1),\ F_{y,A}(t_2)).$$

(iii) If $x\in A$, then we have

$$F_{x,B}(t)=\sup_{s<t}\sup_{b\in B}F_{x,b}(s)\geqslant\sup_{s<t}\inf_{a\in A}\sup_{b\in B}F_{a,b}(s)$$
$$=\sup_{s<t}\Delta(\inf_{a\in A}\sup_{b\in B}F_{a,b}(s),1)$$
$$\geqslant\sup_{s<t}\Delta(\inf_{a\in A}\sup_{b\in B}F_{a,b}(s),\inf_{b\in B}\sup_{a\in A}F_{a,b}(s))=\tilde{F}_{A,B}(t),\quad\forall t\geqslant 0.\quad\blacksquare$$

§ 2. Fixed Point Theorems for Multi-valued Mappings

Throughout this section let $(E,\mathscr{F},\Delta)$ be a $\mathscr{T}$-complete Menger space with continuous t-norm Δ satisfying $\Delta(t,t)\geqslant t$, $\forall t\in[0,1]$. We denote by Ω the family of all nonempty $\mathscr{T}$-closed probabilistically bounded sets and by $\tilde{F}$ the Menger-Hausdorff metric defined by (1.1).

Theorem 2.1. *Let $\{T_i\}$: $E\to\Omega$ be a sequence of multi-valued mappings. Suppose that there exists a constant $k>1$ such that for any $i,j\in Z^+$, $i\neq j$ and any $x,y\in E$*

$$\tilde{F}_{T_ix,T_jy}(t)\geqslant\min\{F_{x,y}(kt),F_{x,T_ix}(kt),F_{y,T_jy}(kt)\},\quad\forall t\geqslant 0.\tag{2.1}$$

Suppose further that for any $x\in E$, any $n=1,2,\cdots$ and any $a\in T_nx$ there exists $b\in T_{n+1}a$ such that

$$F_{a,b}(t)\geqslant\tilde{F}_{T_nx,T_{n+1}a}(t),\quad\forall t\geqslant 0.\tag{2.2}$$

Then there exists $x_*\in E$ such that $x_*\in\bigcap_{i=1}^{\infty}T_ix_*$.

Proof. For any $x_0\in E$ take $x_1\in T_1x_0\in\Omega$. By the supposition there exists $x_2\in T_2x_1\in\Omega$ such that

$$F_{x_1,x_2}(t)\geqslant\tilde{F}_{T_1x_0,T_2x_1}(t),\quad\forall t\geqslant 0.$$

Similarly there exists $x_3\in T_3x_2$ such that

$$F_{x_2,x_3}(t)\geqslant\tilde{F}_{T_2x_1,T_3x_2}(t),\quad\forall t\geqslant 0.$$

Continuing this procedure we can obtain a sequence $\{x_n\}$ satisfying the following conditions:

(i) $x_n\in T_nx_{n-1}$, $n=1,2,\cdots$;

(ii) $F_{x_n,x_{n+1}}(t)\geqslant\tilde{F}_{T_nx_{n-1},T_{n+1}x_n}(t)$, $\forall t\geqslant 0$.

It is easy to prove that $\{x_n\}\subset E$ is a Cauchy sequence. By the $\mathscr{T}$-completeness of $(E,\mathscr{F},\Delta)$ we can suppose $x_n\xrightarrow{\mathscr{T}}x_*\in E$.

Now we prove that x_* is a common fixed point of $\{T_i\}_{i=1}^{\infty}$.

In fact, it follows from Proposition 1.3 (ii) and (iii) that

$$\begin{aligned}F_{x_*,T_ix_*}(t)&\geqslant\Delta\Big(F_{x_*,x_{n+1}}\Big(\Big(1-\frac{1}{\beta}\Big)t\Big),F_{x_{n+1},T_ix_*}\Big(\frac{t}{\beta}\Big)\Big)\\&\geqslant\Delta\Big(F_{x_*,x_{n+1}}\Big(\Big(1-\frac{1}{\beta}\Big)t\Big),\tilde{F}_{T_{n+1}x_n,T_ix_*}\Big(\frac{t}{\beta}\Big)\Big)\\&\geqslant\Delta\Big(F_{x_*,x_{n+1}}\Big(\Big(1-\frac{1}{\beta}\Big)t\Big),\min\Big\{F_{x_n,x_*}\Big(\frac{kt}{\beta}\Big),\\&\qquad F_{x_n,T_{n+1}x_n}\Big(\frac{kt}{\beta}\Big),\ F_{x_*,T_ix_*}\Big(\frac{kt}{\beta}\Big)\Big\}\Big).\end{aligned}\tag{2.3}$$

In addition, by Proposition 1.3 (i) and (ii) we have

$$F_{x_n,T_{n+1}x_n}\left(\frac{kt}{\beta}\right)\geqslant\Delta\left(F_{x_n,x_{n+1}}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right),\ F_{x_{n+1},T_{n+1}x_n}\left(\frac{t}{\beta^2}\right)\right)$$
$$=F_{x_n,x_{n+1}}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right).$$

Substituting the above inequality into (2.3) and letting $n\to\infty$ we have by the continuity of Δ

$$F_{x_*,T_ix_*}(t)\geqslant F_{x_*,T_ix_*}\left(\frac{k}{\beta}t\right)\geqslant\cdots\geqslant F_{x_*,T_ix_*}\left(\left(\frac{k}{\beta}\right)^m t\right),\quad m=1,2,\cdots.$$

Letting $m\to\infty$ on the right we have

$$F_{x_*,T_ix_*}(t)=1,\quad \forall t>0,\ i=1,2,\cdots,$$

and therefore $x_*\in T_ix_*$, $i=1, 2, \cdots$ by Proposition 1.3 (i), i.e.

$$x_*\in\bigcap_{i=1}^{\infty}T_ix_*.$$ ∎

Moreover, Theorem 2.1 is equivalent to the following result.

Theorem 2.2. *Let $\{T_i\}_{i=1}^{\infty}:E\to\Omega$ be a sequence of multi-valued mappings. Suppose that there exists a constant $k>1$ such that for any $i, j\in Z^+$, $i\neq j$ and any $x, y\in E$*

$$\tilde{F}_{T_ix,T_jy}(t)\geqslant\min\{F_{x,y}(kt),\ F_{x,T_ix}(kt),\ F_{y,T_jy}(kt),\ F_{x,T_jy}(2kt),\ F_{y,T_ix}(2kt)\},$$
$$\forall t\geqslant 0. \tag{2.4}$$

Suppose further for any $x\in E$, any $n=1, 2, \cdots$ and any $a\in T_nx$ there exists a point $b\in T_{n+1}a$ such that

$$F_{a,b}(t)\geqslant\tilde{F}_{T_nx,T_{n+1}a}(t),\quad\forall t\geqslant 0.$$

Then there exists $x_*\in E$ such that

$$x_*\in\bigcap_{i=1}^{\infty}T_ix_*.$$

Proof. It can be seen that condition (2.4) and condition (2.1) are equivalent, and therefore Theorem 2.2 is equivalent to Theorem 2.1. ∎

§ 3. Fixed Point Theorems for One Valued Mappings

Throughout this section we always assume that $(E, \mathscr{F}, \Delta)$ is a $\mathscr{F}$-complete Menger space with continuous t-norm Δ, and the function Φ satisfies the following condition(Φ):

$$(\Phi)\quad\begin{cases}\Phi: R^+\to R^+ \text{ is strictly increasing, } \Phi(0)=0 \text{ and } \Phi^n(t)\to\infty,\ \forall t>0,\\ \text{where } \Phi^n \text{ denotes the } n\text{-th iteration of } \Phi.\end{cases}$$

In addition, in Theorem 3.1 and Corollary 3.2 we assume that

$$\Delta(a, b)\geqslant\max\{a+b-1, 0\},\quad\forall a, b\in[0, 1].$$

We have the following results.

Theorem 3.1. *Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of self-mappings on $(E, \mathscr{F}, \Delta)$, and $\{m_i\}_{i=1}^{\infty}$: $E\to Z^+$ a sequence of mappings such that for each $i\in Z^+$, $m_i(x)\mid m_i(T_ix)$, $\forall x\in E$. Suppose that for any $i, j\in Z^+$, $i\neq j$ and any $x, y\in E$*

$$F_{T_i^{m_i(x)}x,T_j^{m_j(y)}y}(t)\geqslant\min_{p,q\in\{x,y,T_i^{m_i(x)}x,T_j^{m_j(y)}y\}}F_{p,q}(\Phi(t)),\quad\forall t\geqslant 0. \tag{3.1}$$

Suppose further that there exist $x_0 \in E$ and $G \in \mathscr{D}$, $G(0)=0$ such that

$$\inf_{p,q \in \{x_n\}_{n=0}^{\infty}} F_{p,q}(t) \geqslant G(t), \quad \forall t \geqslant 0, \tag{3.2}$$

where

$$x_n = T_n^{m_n(x_{n-1})} x_{n-1}, \quad n=1,2,\cdots. \tag{3.3}$$

Then there exists a unique common fixed point x_* of $\{T_i\}_{i=1}^{\infty}$ in E and $x_n \xrightarrow{\mathscr{T}} x_*$.

Take $\Phi(t) = \frac{t}{\alpha}$, $\alpha \in (0, 1)$, $t \geqslant 0$, It is easy to see that Φ satisfies the condition (Φ). Therefore from Theorem 3.1 we have the following result.

Corollary 3.2. *Let $\{T_i\}_{i=1}^{\infty}$, $(E, \mathscr{F}, \Delta)$ and $\{m_i\}_{i=1}^{\infty}$ be the same as in Theorem* 3.1. *Suppose that for any $i, j \in Z^+$, $i \neq j$ and any $x, y \in E$*

$$F_{T_i^{m_i(x)}x, T_j^{m_j(y)}y}(t) \geqslant \min_{p,q \in \{x, y, T_i^{m_i(x)}x, T_j^{m_j(y)}y\}} F_{p,q}\left(\frac{t}{\alpha}\right), \quad \forall t \geqslant 0.$$

Suppose further there exist $x_0 \in E$ and $G \in \mathscr{D}$ with $G(0)=0$ such that the sequence $\{x_n\}$ defined by (3.3) *satisfies the condition* (3.2). *Then the conclusion of Theorem* 3.1 *still holds.*

Remark 1. The special case of Corollary 3.2 with $T_i = T$ and $m_i = m$ for $i \in Z^+$ appears in Istrătescu[1]. Moreover, the main result of [3] is also a special case of Corollary 3.2. ∎

Theorem 3.3. *Let T_1, T_2 be two self-mappings on $(E, \mathscr{F}, \Delta)$ and m_1, m_2: $E \to Z^+$ be two mappings such that $m_i(x) \mid m_i(T_i x)$, $\forall x \in E$, $i=1, 2$, and that for any $x, y \in E$ and any $t \geqslant 0$*

$$F_{T_1^{m_1(x)}x, T_2^{m_2(y)}y}(t) \geqslant \min\{F_{x,y}(\Phi(t)), F_{x, T_1^{m_1(x)}x}(\Phi(t)), F_{y, T_2^{m_2(y)}y}(\Phi(t))\}. \tag{3.4}$$

Suppose that there exist $x_0 \in E$ and $G \in \mathscr{D}$ with $G(0)=0$ such that the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{2n+1} = T_1^{m_1(x_{2n})} x_{2n}, \; x_{2n+2} = T_2^{m_2(x_{2n+1})} x_{2n+1}, \quad n=0, 1, 2, \cdots \tag{3.5}$$

satisfies

$$\inf_{p,q \in \{x_n\}_{n=0}^{\infty}} F_{p,q}(t) \geqslant G(t), \quad \forall t \geqslant 0. \tag{3.6}$$

Then there exists a unique common fixed point $x_* \in E$ of T_1 and T_2 with $x_n \xrightarrow{\mathscr{T}} x_*$.

Proof. Let $\{x_n\}$ be the sequence in (3.5). Then for any $i, j \in Z^+$ with $i+j=$ odd (without loss of generality we can suppose i is odd and j is even) we have

$$F_{x_i, x_j}(t) = F_{T_1^{m_1(x_{i-1})}x_{i-1}, T_2^{m_2(x_{j-1})}x_{j-1}}(t)$$

$$\geqslant \min\{F_{x_{i-1}, x_{j-1}}(\Phi(t)), F_{x_{i-1}, x_i}(\Phi(t)), F_{x_{j-1}, x_j}(\Phi(t))\}, \quad \forall t \geqslant 0.$$

It follows from (3.6) that for any $m, n \in Z^+$ with $m < n$

$$\inf_{\substack{m \leqslant i, j \leqslant n \\ i+j=\text{odd}}} F_{x_i, x_j}(t) \geqslant \inf_{\substack{m-1 \leqslant i, j \leqslant n \\ i+j=\text{odd}}} F_{x_i, x_j}(\Phi(t)), \quad \forall t \geqslant 0. \tag{3.7}$$

In view of the arbitrariness of m and n we obtain

$$\inf_{\substack{i, j \geqslant m \\ i+j=\text{odd}}} F_{x_i, x_j}(t) \geqslant \inf_{\substack{i, j \geqslant m-1 \\ i+j=\text{odd}}} F_{x_i, x_j}(\Phi(t)), \quad \forall t \geqslant 0.$$

Inductively we can prove that

$$\inf_{\substack{i,j\geq m\\ i+j=\text{odd}}} F_{x_i,x_j}(t)\geq \inf_{\substack{i,j>0\\ i+j=\text{odd}}} F_{x_i,x_j}(\Phi^m(t))\geq \sup_{s<\Phi^m(t)} \inf_{\substack{i,j>0\\ i+j=\text{odd}}} F_{x_i,x_j}(s),\quad \forall t\geq 0. \tag{3.8}$$

Letting $m\to\infty$ and noting condition (Φ) we have

$$\lim_{m\to\infty} \inf_{\substack{i,j\geq m\\ i+j=\text{odd}}} F_{x_i,x_j}(t)\geq \sup_{s>0} \inf_{\substack{i,j>0\\ i+j=\text{odd}}} F_{x_i,x_j}(s)\geq \sup_{s>0} G(s)=1,\quad \forall t>0.$$

Therefore for any $\varepsilon>0$ and $\lambda>0$ there exists $N=N(\varepsilon,\lambda)\in Z^+$ such that

$$F_{x_i,x_j}(\varepsilon)>1-\lambda, \tag{3.9}$$

whenever $i, j\geq m\geq N$ with $i+j=\text{odd}$.

Moreover, by the continuity of t-norm Δ, for any $\lambda>0$ there exists λ_1, $0<\lambda_1<\min\{\lambda, 1\}$, such that

$$\Delta(1-\lambda_1, 1-\lambda_1)>1-\lambda. \tag{3.10}$$

Then it follows from (3.9) that for any $\varepsilon>0$ and for given λ_1 there exists $N_1=N_1(\varepsilon,\lambda_1)$ such that

$$F_{x_i,x_j}(\varepsilon)>1-\lambda_1,\quad \forall i, j\geq N_1,\ i+j=\text{odd}. \tag{3.11}$$

Now we consider the case that $i, j\in Z^+$ with $i+j=\text{even}$. It is sufficient to discuss the case that i, j are both odd since the opposite case is similar. By using Menger triangle inequality we have

$$F_{x_i,x_j}(2\varepsilon)\geq\Delta(F_{x_i,x_n}(\varepsilon), F_{x_n,x_j}(\varepsilon)),\quad i+j=\text{even},$$

where n is an arbitrary even. Therefore it follows from (3.11) and (3.10) that for any $i, j, n\geq N_1$

$$F_{x_i,x_j}(2\varepsilon)\geq\Delta(1-\lambda_1, 1-\lambda_1)>1-\lambda,\quad i+j=\text{even}. \tag{3.12}$$

Combining (3.9) and (3.12) we see that for any $\varepsilon>0$ and $\lambda>0$ there exists a positive integer $N_2=\max\{N, N_1\}$ such that

$$F_{x_i,x_j}(2\varepsilon)>1-\lambda,\quad \forall i, j\geq N_2. \tag{3.13}$$

This means that $\{x_n\}$ is a $\mathscr{T}$-Cauchy sequence of E. By the $\mathscr{T}$-completeness of E, we can suppose that $x_n \xrightarrow{\mathscr{T}} x_*\in E$, hence it can be proved that x_* is the unique common fixed point of T_1 and T_2. ∎

The following theorem gives a necessary and sufficient condition for a pair of mappings to have a common fixed point.

Theorem 3.4. *Let T_1, T_2 be two $\mathscr{T}$-continuous self-mappings on $(E, \mathscr{F}, \Delta)$. Then T_1, T_2 have a unique common fixed point in E if and only if there exists a $\mathscr{T}$-continuous mapping A, which commutes with T_1 and T_2 and satisfies the following conditions:*

(i) $A(E)\subset T_1(E)\subset T_2(E)$;

(ii) *For any $x, y\in E$ and any $t\geq 0$*

$$F_{Ax,Ay}(t)\geq\min\{F_{T_1x,T_2y}(\Phi(t)), F_{T_1x,Ax}(\Phi(t)), F_{T_2y,Ay}(\Phi(t))\}.$$

Proof. Necessity. Suppose that x_* is the unique common fixed point of T_1 and T_2. We define the mapping A: $E\to E$ as follows:

$$Ax = x_*, \quad \forall x \in E.$$

It is obvious that

$$AT_1x = x_* = T_1Ax, \quad \forall x \in E; \quad AT_2x = x_* = T_2Ax, \quad \forall x \in E.$$

This implies that condition (i) is satisfied. The $\mathscr{T}$-continuity of A is trivial. Moreover, since

$$F_{Ax,Ay}(t) = F_{x_*,x_*}(t) = H(t), \quad \forall x, y \in E,$$

and H is greater than any other distribution functions, it is obvious that (ii) is true.

Sufficiency. For any $x_0 \in E$, by condition (i) we can choose $x_1, x_2, \cdots$ inductively such that $Ax_0 = T_1x_1$, $Ax_1 = T_2x_2$, $\cdots$ $Ax_{2n} = T_1x_{2n+1}$, $Ax_{2n+1} = T_2x_{2n+2}$, $\cdots$. Let $y_n = Ax_n$, $n = 1, 2, \cdots$. By the same way as stated in Theorem 3.3 we can prove that $\{y_n\}$ is a $\mathscr{T}$-Cauchy sequence in E. By the $\mathscr{T}$-completeness of E let $y_n \xrightarrow{\mathscr{T}} y$. By virtue of the $\mathscr{T}$-continuity of A, T_1 and T_2 we have $T_1y_{2n+1} \xrightarrow{\mathscr{T}} T_1y_*$, $Ay_{2n} \xrightarrow{\mathscr{T}} Ay_*$. Moreover, Menger triangle inequality implies

$$F_{T_1y_*,Ay_*}(t) \geqslant \Delta\left(F_{T_1y_*,T_1y_{2n+1}}\left(\frac{t}{2}\right), F_{T_1y_{2n+1},Ay_*}\left(\frac{t}{2}\right)\right)$$
$$= \Delta\left(F_{T_1y_*,T_1y_{2n+1}}\left(\frac{t}{2}\right), F_{Ay_{2n},Ay_*}\left(\frac{t}{2}\right)\right), \quad \forall t \geqslant 0,$$

and letting $n \to \infty$ we have

$$F_{T_1y_*,Ay_*}(t) \geqslant H\left(\frac{t}{2}\right) = H(t), \quad \forall t \geqslant 0.$$

This implies that $T_1y_* = Ay_*$. Similarly, we can prove $T_2y_* = Ay_*$. Now put $Ay_* = \tilde{y}$. Then

$$\begin{aligned} F_{\tilde{y},A\tilde{y}}(t) &\geqslant \min\{F_{T_1y_*,T_2\tilde{y}}(\Phi(t)), F_{T_1y_*,Ay_*}(\Phi(t)), F_{T_2\tilde{y},A\tilde{y}}(\Phi(t))\} \\ &= \min\{F_{\tilde{y},A\tilde{y}}(\Phi(t)), H(\Phi(t)), F_{A\tilde{y},A\tilde{y}}(\Phi(t))\} \\ &= F_{\tilde{y},A\tilde{y}}(\Phi(t)), \quad \forall t \geqslant 0. \end{aligned}$$

Hence

$$F_{\tilde{y},A\tilde{y}}(t) \geqslant F_{\tilde{y},A\tilde{y}}(\Phi(t)) \geqslant \cdots \geqslant F_{\tilde{y},A\tilde{y}}(\Phi^n(t)), \quad \forall t \geqslant 0,$$

and therefore letting $n \to \infty$ we have

$$F_{\tilde{y},A\tilde{y}}(t) = H(t), \quad \forall t \geqslant 0,$$

i.e. $\tilde{y} = A\tilde{y}$. On the other hand,

$$\tilde{y} = A\tilde{y} = A^2y_* = AT_1y_* = T_1Ay_* = T_1\tilde{y},$$
$$\tilde{y} = A\tilde{y} = AT_2y_* = T_2Ay_* = T_2\tilde{y},$$

so $\tilde{y}$ is a common fixed point of A, T_1 and T_2. Finally it is easy to prove that $\tilde{y}$ is the unique common fixed point of A, T_1 and T_2. ∎

The following are the immediate consequences of Theorems 3.3 and 3.4 respectively.

Corollary 3.5. *Let $\mathfrak{M}$ be a family of self-mappings on $(E, \mathscr{F}, \Delta)$, and m a mapping of $\mathfrak{M}$ into Z^+, such that for all $x, y \in E$ and all $t \geqslant 0$, and for any two mappings $S, T \in \mathfrak{M}$, $S \neq T$*

$$F_{S^{m(S)}x,T^{m(T)}y}(t)\geqslant\min\{F_{x,y}(\Phi(t)),\ F_{x,S^{m(S)}x}(\Phi(t)),\ F_{y,T^{m(T)}y}(\Phi(t))\}.$$

Then there exists $x_*\in E$ such that $x_*=Tx_*$, $\forall T\in\mathfrak{M}$, and that for any $x_0\in E$ the sequence $\{x_n\}$ defined by

$$x_{2n+1}=S^{m(S)}x_{2n},\quad x_{2n+2}=T^{m(T)}x_{2n+1},\quad n=0,\ 1,\ 2,\ \cdots,$$

$\mathscr{T}$-converges to x_*.

Corollary 3.6. *Let $\mathfrak{M}$ be a family of $\mathscr{T}$-continuous self-mappings on $(E,\mathscr{F},\Delta)$. Then $\mathfrak{M}$ has a unique common fixed point in E if and only if there exists a $\mathscr{T}$-continuous mapping A which commutes with each T of $\mathfrak{M}$ and satisfies the following conditions:*

(i) $A(E)\subset\bigcap_{T\in\mathfrak{M}}T(E)$;

(ii) *For all x, $y\in E$ and all $t\geqslant0$, and for any two mappings S and $T\in\mathfrak{M}$, $S\neq T$*

$$F_{Ax,Ay}(t)\geqslant\min\{F_{Sx,Ty}(\Phi(t)),\ F_{Sx,Ax}(\Phi(t)),\ F_{Ty,Ay}(\Phi(t))\}.$$

§ 4. Applications

First, we shall use the results in § 2 to study the fixed point theorems for multi-valued mappings in metric spaces. We give the following results.

Theorem 4.1. *Let $(E,\ d)$ be a complete metric space, $C(E)$ the family of all nonempty compact sets of E, and $\{T_i\}_{i=1}^{\infty}:E\to C(E)$ a sequence of multi-valued mappings. Suppose that there exists $\alpha\in(0,\ 1)$ such that for any i, $j\in Z^+$, $i\neq j$ and any x, $y\in E$*

$$\rho(T_ix,\ T_jy)\leqslant\alpha\max\{d(x,\ y),\ d(x,\ T_ix),\ d(y,\ T_jy)\},\qquad(4.1)$$

where ρ denotes the Hausdorff metric on $C(E)$. Then there exists $x_*\in E$ such that

$$x_*\in\bigcap_{i=1}^{\infty}T_ix_*.$$

Proof. First we define a mapping $\mathscr{F}$: $E\times E\to\mathscr{D}$ as follows (we denote $\mathscr{F}(x,\ y)$ by $F_{x,y}$):

$$F_{x,y}(t)=H(t-d(x,\ y)),\qquad\forall x,\ y\in E,\ t\in R.\qquad(4.2)$$

It follows from Theorem 3 of [3] that the space $(E,\ \mathscr{F},\ \min)$ with t-norm $\Delta=\min$ is a $\mathscr{T}$-complete Menger space, called Menger space induced by the metric space $(E,\ d)$.

Let $x\in E$ and $A\in C(E)$. We define a probabilistic distance $F_{x,A}$ as follows:

$$F_{x,A}(t)=H(t-d(x,\ A)).$$

It is easy to prove that the Menger-Hausdorff metric $\tilde{\mathscr{F}}$ induced by $\mathscr{F}$ [defined by (4.2) in terms of (1.1)] has the following form:

$$\tilde{F}_{A,B}(t)=H(t-\rho(A,\ B)),\quad A,\ B\in C(E).\qquad(4.3)$$

Therefore for any x, $y\in E$, any i, $j\in Z^+$, $i\neq j$, and any $t\geqslant0$, we have by (4.1)

$$\begin{aligned}\tilde{F}_{T_ix,T_jy}(t)&=H(t-\rho(T_ix,\ T_jy))\\&\geqslant H(t-\alpha\max\{d(x,\ y),\ d(x,\ T_ix),\ d(y,\ T_jy)\})\\&=H\Big(\frac{t}{\alpha}-\max\{d(x,\ y),\ d(x,\ T_ix),\ d(y,\ T_jy)\}\Big)\\&=\min\Big\{F_{x,y}\Big(\frac{t}{\alpha}\Big),\ F_{x,T_ix}\Big(\frac{t}{\alpha}\Big),\ F_{y,T_jy}\Big(\frac{t}{\alpha}\Big)\Big\},\quad\forall t\geqslant0.\end{aligned}$$

Moreover, it follows from [6] that for any $x\in E$, any $n=1, 2, \cdots$ and any $a\in T_n x$, there exists $b\in T_{n+1}a$ such that

$$d(a, b)\leqslant\rho(T_n x, T_{n+1}a),$$

hence

$$F_{a,b}(t)=H(t-d(a, b))\geqslant H(t-\rho(T_n x, T_{n+1}a))=\widetilde{F}_{T_n x, T_{n+1}a}(t),\quad \forall t\geqslant 0.$$

Therefore all conditions of Theorem 2.1 are satisfied, and Theorem 4.1 follows from Theorem 2.1 immediately. ▮

In what follows we shall use results in § 3 to study the existence and uniqueness of the solution of nonlinear Volterra integral equation on Banach space. For the sake of convenience we first introduce some notations and basic definitions as follows.

We assume that $[0, a]$ is a given real interval $(0<a<\infty)$, $(E, \|\cdot\|_E)$ a real Banach space, $C([0, a]; E)$ the Banach space of all E-valued continuous functions defined on $[0, a]$ with the norm

$$\|x\|_C=\sup_{0\leqslant t\leqslant a}\|x(t)\|_E,$$

and $C([0, a]\times[0, a]\times C)[0, a]; E); E)$ the linear space of all E-valued continuous functions defined on $[0, a]\times[0, a]\times C([0, a]; E)$. Besides the norm $\|\cdot\|_C$, the space $C([0, a]; E)$ can be endowed with another norm

$$\|x\|_*=\max_{0\leqslant t\leqslant a}(e^{-Lt}\|x(t)\|_E),\quad x\in C([0, a]; E), \tag{4.4}$$

where L is an arbitrary positive number. It is obvious that the norm $\|\cdot\|_*$ is equivalent to the norm $\|\cdot\|_C$.

Now we define a mapping $\mathscr{F}$: $C([0, a]; E)\times C([0, a]; E)\to\mathscr{D}$ as follows:

$$F_{x,y}(t)=H(t-\|x-y\|_*),\quad t\in R;\ x, y\in C([0, a]; E).$$

By Theorem 3 of [3] we know that $(C([0, a]; E), \mathscr{F}, \min)$ is the $\mathscr{T}$-complete Menger space induced by $C([0, a]; E)$. In addition, we can prove that in the space $(C([0, a]; E), \mathscr{F}, \min)$ the convergences in topology $\mathscr{T}$, in norm $\|\cdot\|_*$ and in norm $\|\cdot\|_C$ are equivalent each to other.

Now we consider the existence and uniqueness of the solution of the nonlinear Volterra integral equation of the type

$$x(t)=\tilde{x}(t)+\int_0^t K(t, s, x(s))ds,\quad 0\leqslant t\leqslant a<\infty, \tag{4.5}$$

where the kernel K is assumed to satisfy the following conditions:

(i) $K\in C([0, a]\times[0, a]\times C([0, a]; E); E)$ and

$$\|K\|_C=\sup_{\substack{t, s\in[0,a]\\ x\in C([0,a];E)}}\|K(t, s, x(s))\|_E<\infty;$$

(ii) There exist $m\in Z^+$ and $L>0$ such that for all $x, y\in C([0, a]; E)$ and all $t, s\in[0, a]$

$$\|K(t, s, T^{m-1}x(s))-K(t, s, T^{m-1}y(s))\|_E\leqslant L\cdot\max_{p,q\in\{x,y,T^m x,T^m y\}}\{\|p-q\|_E\},$$

where the mappings T, T^n, $n=1, 2,\cdots$, are self-mappings on $C([0, a]; E)$ defined by

$$Tx(t)=\tilde{x}(t)+\int_0^t K(t,\ s,\ x(s))ds,\qquad \tilde{x}\in C([0,\ a];\ E).$$
$$T^n x(t)=\tilde{x}(t)+\int_0^t K(t,\ s,\ T^{n-1}x(s))ds,$$

Theorem 4.2. *Let* $(C([0,\ a];\ E),\ \mathscr{F},\ \min)$ *be the Menger space induced by* $C([0,\ a];\ E)$. *Suppose there exists some* $x_0\in C([0,\ a];\ E)$ *such that the sequence* $\{x_n\}$ *defined by*

$$x_n(t)=T^m x_{n-1}(t),\quad n=1,\ 2,\ \cdots \tag{4.6}$$

is bounded, where m is the positive integer appeared in the above condition (ii). *Then there exists a unique solution of equation* (4.5) *in* $C([0,\ a];\ E)$ *and the sequence* $\{x_n\}$ *is* $\mathscr{F}$*-convergent (hence convergent in norms* $\|\cdot\|_*$ *and* $\|\cdot\|_C$*) to this unique solution.*

Proof. Let us consider the norm $\|\cdot\|_*$ defined by (4.4) with the positive number L given in condition (ii). Then

$$\|T^m x-T^m y\|_*\leqslant\max_{0\leqslant t\leqslant a}\int_0^t e^{L(s-t)}e^{-Ls}\|K(s,\ t,\ T^{m-1}x(s))-K(t,\ s,\ T^{m-1}y(s))\|_E\,ds$$
$$\leqslant L\cdot\max_{p,\ q\in\{x,y,T^m x,T^m y\}}\{\|p-q\|_*\}\cdot\max_{0\leqslant t\leqslant a}\int_0^t e^{L(s-t)}\,ds$$
$$\leqslant(1-e^{-La})\cdot\max_{p,q\in\{x,y,T^m x,T^m y\}}\{\|p-q\|_*\},\quad\forall x,\ y\in C([0,\ a];\ E).$$

Putting $\beta=1-e^{-La}$, we have for any $r\in R^+$

$$F_{T^m x,T^m y}(r)\geqslant H(r-\beta\cdot\max_{p,q\in\{x,y,T^m x,T^m y\}}\|p-q\|_*)$$
$$=\min_{p,q\in\{x,y,T^m x,T^m y\}}H\left(\frac{r}{\beta}-\|p-q\|_*\right)$$
$$=\min_{p,q\in\{x,y,T^m x,T^m y\}}F_{p,q}\left(\frac{r}{\beta}\right),\quad\forall x,\ y\in C([0,\ a];\ E)$$

Furthermore, since $\{x_n\}$ is a bounded sequence, the function

$$\inf_{p,q\in\{x_n\}_{n=0}^\infty}F_{p,q}(t)=\inf_{p,q\in\{x_n\}_{n=0}^\infty}H(t-\|p-q\|_*)$$

is a distribution function with value 0 for $t=0$. Therefore Theorem 4.2 follows immediately from Corollary 3.2. ∎

Finally we consider the existence and uniqueness of the common solution of the system of two nonlinear Volterra integral equations of the type

$$\left.\begin{aligned}x(t)&=\tilde{x}(t)+\int_0^t K_1(t,\ s,\ x(s))ds\\ y(t)&=\tilde{x}(t)+\int_0^t K_2(t,\ s,\ y(s))ds\end{aligned}\right\}\quad 0\leqslant t\leqslant a<\infty, \tag{4.7}$$

where the kernel K_i, $i=1,\ 2$, are assumed to satisfy the following conditions:

(iii) $K_i\in C([0,\ a]\times[0,\ a]\times C([0,\ a];\ E);\ E)$, $i=1,\ 2$, and

$$\|K_i\|_C=\sup_{\substack{s,t\in[0,a]\\ x\in C([0,a];\ E)}}\|K_i(t,\ s,\ x(s))\|_E<\infty;$$

(iv) There exist $\tilde{m},\ \tilde{n}\in Z^+$ and $L>0$ such that for all $x,\ y\in C([0,\ a];\ E)$ and all $t,\ s\in[0,\ a]$

$$\|K_1(t,\ s,\ T_1^{\tilde{m}-1}x(s))-K_2(t,\ s,\ T_2^{\tilde{n}-1}y(s))\|_E$$

$$\leqslant L\cdot\max\{\|x-y\|_E,\ \|x-T_1^{\bar m}x\|_E,\ \|y-T_2^{\bar n}y\|_E\},$$

where the mappings T_i, T_i^n, $i=1, 2$, $n=1, 2, \cdots$ are defined as above.

We have the following result.

Theorem 4.3. *Let $(C([0, a]; E), \mathscr{F}, \min)$ be the $\mathscr{T}$-complete Menger space induced by $C([0, a]; E)$. Suppose that there exists $x_0\in C([0, a]; E)$ such that the sequence $\{x_n\}$ defined by*

$$x_{2n+1}(t)=T_1^{\bar m}x_{2n}(t),\ x_{2n+2}(t)=T_2^{\bar n}x_{2n+1}(t),\quad n=0, 1, 2, \cdots$$

is bounded. Then the system of equation (4.7) has a unique common solution $x_*\in C([0, a]; E)$ with $x_n\xrightarrow{\mathscr{T}}x_*$ and $x_*\xrightarrow{\|\cdot\|_*}x_*$.

Proof. We consider the norm $\|\cdot\|_*$ defined by (4.4) with the positive number L given in condition (iv), and therefore we have

$$\|T_1^{\bar m}x-T_2^{\bar n}y\|_*\leqslant L\cdot\max\{\|x-y\|_*,\ \|x-T_1^{\bar m}x\|_*,\ \|y-T_2^{\bar n}y\|_*\}\cdot\max_{0<t<a}\int_0^t e^{L(s-t)}\,ds$$

$$\leqslant(1-e^{-La})\cdot\max\{\|x-y\|_*,\ \|x-T_1^{\bar m}x\|_*,\ \|y-T_2^{\bar n}y\|_*\},\quad \forall x, y\in C([0, a]; E).$$

Let $\beta=1-e^{-La}$. It follows from the above inequality that for all $r\in R^+$ and for any $x, y\in C([0, a]; E)$

$$F_{T_1^{\bar m}x,T_2^{\bar n}y}(r)\geqslant H\Big(\frac{r}{\beta}-\max\{\|x-y\|_*,\ \|x-T_1^{\bar m}x\|_*,\ \|y-T_2^{\bar n}y\|_*\}\Big)$$

$$=\min\Big\{F_{x,y}\Big(\frac{r}{\beta}\Big),\ F_{x,T_1^{\bar m}x}\Big(\frac{r}{\beta}\Big),\ F_{y,T_2^{\bar n}y}\Big(\frac{r}{\beta}\Big)\Big\}.$$

Since $\{x_n\}$ is bounded, we see that $\inf\limits_{p,q\in\{x_n\}_{n=0}^\infty}F_{p,q}(t)$ is a distribution function and $\inf\limits_{p,q\in\{x_n\}_{n=0}^\infty}F_{p,q}(0)=0$. Therefore Theorem 4.3 follows from Theorem 3.3 immediately. ∎

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