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Ekeland's Variational Principle and the Mountain Pass Lemma

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Ekeland's variational principle is a fundamental theorem in nonconvex analysis. Its general statement is as the following:

Ekeland's Variational Principle[1,2]. *Let V be a complete metric space, and F: $V\to\mathbf{R}\cup\{+\infty\}$ a lower semicontinuous function, not identically $+\infty$ and bounded from below. Let $\varepsilon>0$ be given, and a point $u\in V$ such that*

$$F(u)\leqslant\inf_V F+\varepsilon.$$

Then there exists some point $v\in V$ such that

$$F(v)\leqslant F(u),$$
$$d(u, v)\leqslant 1,$$
$$F(w)>F(v)-\varepsilon d(v, w),\quad \textit{for any } w\neq v.$$

This principle has extremely extensive applications (see [2]), and probably, its potential is not yet brought into full play. In this paper we shall give a new application of this famous principle, namely, a new brief proof of the generalized Mountain Pass Lemma.

The Mountain Pass Lemma is a very useful argument for finding critical points of a function f which is unbounded from above and below. Its initial formulation is the following:

Mountain Pass Lemma[3,4,5]. *Let f be a C^1 real function defined on a Banach space X and satisfying (PS)-condition, i.e.*

(PS) *Any sequence $\{x_n\}\subset X$ such that $\{f(x_n)\}$ is bounded and $\|f'(x_n)\|\to 0$ in X^* (the dual space of X) has a convergent subsequence.*

If there is an open neighbourhood Ω of 0 and a point $x_0\notin\bar{\Omega}$ such that

$$f(0),\ f(x_0)<c_0\leqslant\inf_{\partial\Omega} f,$$

then the following number is a critical value of f:

$$c=\inf_{g\in\Gamma}\max_{t\in[0,1]} f(g(t))\geqslant c_0,$$

where

$$\Gamma=\{g\in C([0,1];X)\mid g(0)=0 \text{ and } g(1)=x_0\},$$

and c is said to be a critical value of f, if there exists $\bar{x}\in X$ *such that* $f(\bar{x})=c$ *and* $f'(\bar{x})=0$.

The Mountain Pass Lemma has many extensions and variations (see [5—7] and others); particularly, Chang Kung-Ching[8] generalizes this lemma to locally Lipschitz functions. Our new proof is also given in this general case. For this purpose, we recall the definitions of a locally Lipschitz function, of its generalized gradient and the corresponding (PS)-condition.

$f: X\to\boldsymbol{R}$ is said to be a locally Lipschitz function, if for any $x\in X$, there exists $\delta_x>0$ and $c_x>0$ such that for any $x_1, x_2\in B(x,\delta_x)=\{y\in X\mid \|y-x\|<\delta_x\}$,

$$|f(x_1)-f(x_2)|\leqslant c_x\|x_1-x_2\|.$$

The generalized gradient $\partial f(x)$ of a locally Lipschitz function f at x is the subset of X^* defined by

$$\partial f(x)=\{x^*\in X^*\mid \langle x^*, v\rangle\leqslant f^0(x;v),\ \forall v\in X\},$$

where

$$f^0(x;v)=\limsup_{\substack{y\to x\\ \lambda\downarrow 0}}\frac{f(y+\lambda v)-f(y)}{\lambda}.$$

For a detailed discussion on the generalized gradient, we refer to Clarke[9,10]. Here, we shall use the following properties:

i) $f^0(x;v)$ is upper semicontinuous as a function of (x, v), and as a function of v alone, is sublinear, i.e. positively homogeneous and subadditive.

ii) ∂f is weak*-closed, i.e. if $x_\alpha\to x$, $x^*_\alpha\in\partial f(x_\alpha)$ and $x^*_\alpha\xrightarrow{w^*}x^*$, then $x^*\in\partial f(x)$.

iii) ∂f is locally bounded, i.e. for any $x\in X$, there exist $\delta_x>0$ and $c_x>0$ such that

$$\|x^*\|\leqslant c_x,\quad \forall x_1\in B(x,\delta_x) \text{ and } \forall x^*\in\partial f(x_1).$$

iv) ∂f satisfies the mean-value theorem, i.e. for any $x_1, x_2\in X$, there exists $\theta\in(0,1)$ and $x^*\in\partial f(x_1+\theta(x_2-x_1))$ such that

$$f(x_2)-f(x_1)=\langle x^*, x_2-x_1\rangle.$$

We can find the proof for these properties in Clarke[10].

For a locally Lipschitz function f, (PS)-condition is as follows:

(PS) Any sequence $\{x_n\}\subset X$ such that $\{f(x_n)\}$ is bounded and $\min_{x^*\in\partial f(x_n)}\|x^*\|\to 0$ has a convergent subsequence.

Theorem 1. *Let f be a locally Lipschitz function defined on a Banach space X and satisfying* (PS)*-condition. If there is an open neighbourhood* Ω *of* 0 *and a point* $x_0\notin\bar{\Omega}$ *such that*

$$f(0),\ f(x_0)<c_0\leqslant\inf_{\partial\Omega} f, \tag{1}$$

then the following number is a critical value of f:

$$c=\inf_{g\in\Gamma}\max_{t\in[0,1]} f(g(t))\geqslant c_0, \tag{2}$$

where

$$\Gamma=\{g\in C([0,1];X)\mid g(0)=0 \text{ and } g(1)=x_0\}, \tag{3}$$

and c is said to be a critical value of f, if there exists $\bar{x}\in X$ such that $f(\bar{x})=c$ and $0\in\partial f(\bar{x})$.

The idea of our proof is very simple: Considering

$$F(g)=\max_{t\in[0,1]} f(g(t))$$

as a function defined on the closed linear manifold Γ of $C([0,1];X)$, it is easy to check that F is a locally Lipschitz function on Γ. Then, by Ekeland's variational principle, F has almost minimizers satisfying some particular conditions. Using a sequence of these points on Γ, we shall associate this sequence of almost minimizers a sequence on X, which satisfies the requirement in (PS)-condition for f. Finally, the limit of a subsequence in this sequence on X is just a critical point of f. The difficult point in this process is to establish a relationship between the sequence of almost minimizers of F and a sequence on X, which satisfies the requirements in (PS)-condition for f.

We decompose the proof of this theorem into several propositions. These propositions will be proved for a general case, in which $[0,1]$ is replaced by a compact metric space K, and then they can be also used for some more general extensions of Mountain Pass Lemma.

Proposition 1. *Let f: $X\to\mathbf{R}$ be a locally Lipschitz function on a Banach space X, and K a compact metric space. Then F: $C(K;X)\to\mathbf{R}$, defined by*

$$F(g)=\max_{t\in K} f(g(t)),\quad \forall g\in C(K;X)$$

is a locally Lipschitz function on $C(K;X)$.

Proof. For any $g\in C(K;X)$, as a continuous image of the compact space K, $g(K)$ is a compact subset. Since f is a locally Lipschitz function, for any $t\in K$, there are $\delta_t>0$ and $c_t>0$ such that

$$\forall x_1, x_2\in B(g(t),\delta_t),\quad |f(x_1)-f(x_2)|\leqslant c_t\|x_1-x_2\|. \tag{4}$$

Then $\{B(g(t),\delta_t)\}_{t\in K}$ constitutes an open covering of $g(K)$, and there are $t_1,\cdots,t_k\in K$ such that

$$g(K)\subset\bigcup_{i=1}^{k} B(g(t_i),\delta_{t_i}). \tag{5}$$

On the other side, by Lebesgue lemma, there exists a Lebesgue number $\delta>0$ depending on $g(K)$ such that for any $x\in g(K)$, there exists some i, satisfying

$$B(x,\delta)\subset B(g(t_i),\delta_{t_i}). \tag{6}$$

Set $c_g=\max_{1\leqslant i\leqslant k} c_{t_i}$. By (4)—(6), we have that

$$\forall t\in K,\ \forall x_1, x_2\in B(g(t),\delta),\quad |f(x_1)-f(x_2)|\leqslant c_g\|x_1-x_2\|.$$

Thus, when $h_1, h_2\in C(K;X)$ satisfy

$$\|h_i-g\|_{C(K;X)}=\max_{t\in K}\|h_i(t)-g(t)\|<\delta,\quad i=1,2,$$

we have

$$|F(h_1)-F(h_2)|=|\max_{t\in K} f(h_1(t))-\max_{t\in K} f(h_2(t))|\leqslant\max_{t\in K}|f(h_1(t))-f(h_2(t))|$$
$$\leqslant c_g\max_{t\in K}\|h_1(t)-h_2(t)\|=c_g\|h_1-h_2\|. \quad ∎$$

Proposition 2. *Let f, F, X, K be as in Proposition 1 and assume that for $h \in C(K; X)$,*

$$F^0(g; h) = \limsup_{\substack{u \to g \\ \lambda \downarrow 0}} \frac{F(u+\lambda h) - F(u)}{\lambda},$$

and

$$M(g) = \{s \in K \mid f(g(s)) = F(g) = \max_{t \in K} f(g(t))\}.$$

Then

$$F^0(g; h) \leqslant \max_{s \in M(g)} f^0(g(s); h(s)).$$

Proof. We choose two suitable sequences $\{u_i\} \subset C(K; X)$ and $\{\lambda_i\} \subset \boldsymbol{R}_+$ such that $\|u_i - g\| = \max_{t \in K} |u_i(t) - g(t)| \to 0$, $\lambda_i \downarrow 0$ as $i \to \infty$ and

$$F^0(g; h) = \lim_{i \to \infty} \frac{F(u_i + \lambda_i h) - F(u_i)}{\lambda_i}. \tag{7}$$

Pick any $s_i \in M(u_i + \lambda_i h)$, $i = 1, 2, \cdots$, then it follows that

$$\frac{F(u_i + \lambda_i h) - F(u_i)}{\lambda_i} \leqslant \frac{f(u_i(s_i) + \lambda_i h(s_i)) - f(u_i(s_i))}{\lambda_i}. \tag{8}$$

By the mean-value theorem, there exist $\theta_i \in (0, 1)$ and $x_i^* \in \partial f(u_i(s_i) + \theta_i \lambda_i h(s_i))$ such that

$$\frac{f(u_i(s_i) + \lambda_i h(s_i)) - f(u_i(s_i))}{\lambda_i} = \langle x_i^*, h(s_i) \rangle, \quad i = 1, 2, \cdots. \tag{9}$$

Since K is a compact metric space, $\{s_i\}$ has a convergent subsequence, denoted again by $\{s_i\}$, such that $s_i \to s \in K$. Then, it is obvious that

$$u_i(s_i) + \theta_i \lambda_i h(s_i) \to g(s).$$

By the local boundness and the weak*-closeness of the generalized gradient, $\{x_i^*\}$ has a weak*-cluster point $x^* \in \partial f(g(s))$. We may suppose $\langle x_i^*, h(s) \rangle \to \langle x^*, h(s) \rangle$, and then, by (7)—(9), we have that

$$F^0(g; h) \leqslant \lim_{i \to \infty} \langle x_i^*, h(s_i) \rangle = \lim_{i \to \infty} \langle x_i^*, h(s_i) - h(s) \rangle + \lim_{i \to \infty} \langle x_i^*, h(s) \rangle.$$

Finally, we merely have to check that $s \in M(g)$. Indeed, from $s_i \in M(u_i + \lambda_i h)$ we have

$$f(u_i(s_i) + \lambda_i h(s_i)) \geqslant f(u_i(t) + \lambda_i h(t)), \quad \forall t \in K.$$

By taking limits, we conclude

$$f(g(s)) \geqslant f(g(t)), \quad \forall t \in K.$$ ∎

Proposition 3. *Let f, F, X, K and others be as above and $K_0 \subset K$ a closed subset. If for $g \in C(K; X)$,*

$$M(g) \subset K \setminus K_0, \tag{10}$$

and there exists $\varepsilon > 0$ such that

$$\forall h \in C_0(K; X) = \{h \in C(K; X) \mid h(t) = 0, \quad \forall t \in K_0\},$$

$$F^0(g; h) \geqslant -\varepsilon \|h\|, \tag{11}$$

then there exists $s \in M(g)$ such that

$$f^0(g(s);\ v)\geqslant -\varepsilon\|v\|,\quad \forall v\in X. \tag{12}$$

Proof. If there does not exist such s, then for any $t\in M(g)$, there exists $v_t\in X$ with $\|v_t\|=1$ such that

$$f^0(g(t);\ v_t)<-\varepsilon.$$

Since g is continuous and f^0 is upper semicontinuous, we have that for any $t\in M(g)$, there exists $v_t\in X$ with $\|v_t\|=1$ and $\delta_t>0$ such that

$$\forall s\in B(t,\ \delta_t)=\{s\in K\,|\,d(s,\ t)<\delta_t\},\quad f^0(g(s);\ v_t)<-\varepsilon, \tag{13}$$

$\{B(t,\ \delta_t)\}_{t\in M(g)}$ is an open covering of $M(g)$ and from the compactness of $M(g)$ and the relation (10), we may suppose

$$K_0\cap B(t,\ \delta_t)=\phi,\quad \forall t\in M(g), \tag{14}$$

and there are finite $t_1,\ \cdots,\ t_k\in M(g)$ such that

$$M(g)\subset\bigcup_{i=1}^{k}B(t_i,\ \delta_{t_i}). \tag{15}$$

For any $t\in K$, we define

$$\rho_0(t)=\min_{s\in M(g)}d(t,\ s), \tag{16}$$

$$\rho_i(t)=\min_{s\in K\setminus B(t_i,\delta_{t_i})}d(t,\ s),\quad i=1,\ 2,\ \cdots,\ k. \tag{17}$$

From (15), we have

$$K=\left\{\bigcup_{i=1}^{k}B(t_i,\ \delta_{t_i})\right\}\cup\{K\setminus M(g)\},$$

and it follows that

$$\sum_{i=0}^{k}\rho_i(t)>0,\quad \forall t\in K. \tag{18}$$

Set

$$h(t)=\frac{\sum\limits_{i=1}^{k}v_{t_i}\rho_i(t)}{\sum\limits_{i=0}^{k}\rho_i(t)}.$$

Then, by (13)—(18), we have

$$h\in C_0(K;\ X)\quad\text{and}\quad \|h\|\leqslant 1.$$

Since $f^0(x;\ v)$ is sublinear in v,

$$f^0(g(t);h(t))\leqslant\frac{\sum\limits_{i=1}^{k}\rho_i(t)f^0(g(t);v_{t_i})}{\sum\limits_{i=0}^{K}\rho_i(t)}.$$

By (13), (16) and (17), for any $t\in M(g)$, we have

$$\rho_0(t)=0\ \text{and}\ [\rho_i(t)>0]\Rightarrow[f^0(g(t);\ v_{t_i})<-\varepsilon].$$

Then

$$\forall t\in M(g).\quad f^0(g(t);\ h(t))<-\varepsilon\leqslant-\varepsilon\|h\|,$$

From Proposition 2, it follows

$$F^0(g;\ h)\leqslant\max_{s\in M(g)}f^0(g(s),\ h(s))<-\varepsilon\|h\|,$$

and we have a contradiction to (11). Hence, (12) is proved. ∎

Proof of Theorem 1.

Set $K=[0, 1]$ and $K_0=\{0, 1\}$. Since $\partial\Omega$ separates 0 and x_0, for any $g\in\Gamma$, we have

$$g(K)\cap\partial\Omega=g([0, 1])\cap\partial\Omega\neq\phi \tag{19}$$

and by (1) and (3),

$$\max_{t\in[0,1]} f(g(t))\geqslant\inf_{\partial\Omega} f\geqslant c_0>f(g(0)),\ f(g(1)). \tag{20}$$

Therefore,

$$\forall g\in\Gamma,\ M(g)=\{s\in[0, 1]\,|\,f(g(s))=\max_{t\in[0,1]} f(g(t))\}\subset(0, 1)=K\setminus K_0.$$

Γ is a closed linear manifold of $C([0, 1];\ X)$ and it is a complete metric space for the distance determined by its norm. We define F: $\Gamma\to\boldsymbol{R}$ by

$$F(g)=\max_{t\in[0,1]} f(g(t)),\quad \forall g\in\Gamma.$$

Then, by Proposition 1, F is a locally Lipschitz function on Γ and from (20) and (2), it is bounded from below. According to Ekeland's variational principle, for any positive sequence $\{\varepsilon_n\}$, $\varepsilon_n\downarrow 0$, there exists a sequence $\{g_n\}\subset\Gamma$ such that

$$c\leqslant F(g_n)\leqslant c+\varepsilon_n,$$

and

$$F(u)>F(g_n)-\varepsilon_n\|u-g_n\|,\quad \forall u\neq g_n,\ n=1, 2, \cdots.$$

Thus, for any $h\in C_0([0, 1];\ X)=\{h\in C([0, 1];\ X)\,|\,h(0)=h(1)=0\}$, we have

$$F^0(g_n;\ h)\geqslant\lim_{\lambda\downarrow 0}\sup\frac{F(g_n+\lambda h)-F(g_n)}{\lambda}\geqslant-\varepsilon_n\|h\|,\quad n=1, 2, \cdots.$$

By Proposition 3, there exists $s_n\in M(g_n)$ such that

$$f(g_n(s_n))=F(g_n),$$

$$f^0(g_n(s_n);\ v)\geqslant-\varepsilon_n\|v\|,\quad \forall v\in X,\ n=1, 2, \cdots.$$

Then, setting $x_n=g_n(s_n)$, $n=1, 2, \cdots$, we have

$$f(x_n)\to c$$

and

$$0\in\partial f(x_n)+\varepsilon_n\overline{B}^*,\quad n=1, 2, \cdots,$$

where $\overline{B}^*\subset X^*$ is the closed unit ball of X^*. By (PS)-condition, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$. Suppose that $\bar{x}$ is the limit of $\{x_{n_i}\}$, and then $\bar{x}$ satisfies

$$f(\bar{x})=\lim_{n_i\to\infty} f(x_{n_i})=c$$

and

$$\forall v\in X,\ f^0(\bar{x};\ v)\geqslant\lim_{n_i\to\infty}\sup f^0(g_{n_i}(s_{n_i});\ v)\geqslant-\lim_{n_i\to\infty}\varepsilon_{n_i}\|v\|=0,$$

i.e. $0\in\partial f(\bar{x})$.

The technique used in proving Theorem 1 is also adapted to more general extensions (see [4, 5, 6]). For example, we have

Theorem 2. *Let f be a locally Lipschitz function defined on a Banach space X and satisfying* (PS)*-condition. Assume that X has a direct sum decomposition*

$$X=X_1\oplus X_2,\quad \text{where } \dim X_1=k<+\infty.$$

Let S_1 (*resp.* B_1) *be the unit sphere* (*resp. ball*) *in* X_1, *and* S_2 *a sphere* $\{x\in X_2 \mid \|x\|=r\}$ *with* $r>0$. *If for* $c_0, c_1\in \boldsymbol{R}$ and $\varphi\in C(S_1; X)$,

i) $f(x)\geqslant c_0>c_1, \quad \forall x\in S_2$;

ii) $f(x)\leqslant c_1, \quad \forall x\in \varphi(S_1)$;

iii) *the linking number* $l(S_2, \varphi(S_1))\neq 0$;

then the following number is a critical value of f:

$$c=\inf_{g\in \Gamma}\max_{t\in \bar{B}_1} f(g(t))\geqslant c_0,$$

where

$$\Gamma=\{g\in C(\bar{B}_1; X) \mid g(x)=\varphi(x), \ \forall x\in S_1\}.$$

For the proof of this theorem, we merely need to take $\bar{B}_1=K$ and $S_1=K_0$. The condition iii) ensures

$$g(\bar{B}_1)\cap S_2\neq\phi, \quad \forall g\in\Gamma,$$

which replace (19), and then

$$\max_{t\in \bar{B}_1} f(g(t))\geqslant \inf_{S_2} f\geqslant c_0>c_1\geqslant \sup_{\varphi(S_1)} f.$$

Hence

$$M(g)\subset B_1=\bar{B}_1\backslash S_1, \quad \forall g\in\Gamma.$$

The rest is the same as in the proof of Theorem 1.

The most general version, including Theorems 1 and 2 as special cases, is in the following:

Theorem 3. *Let* f *be a locally Lipschitz function defined on a Banach space* X *and satisfying* (PS)-*condition. Assume that* K *is a compact metric space,* K_0 *a closed subset of* K, $\varphi\in C(K_0; X)$ and $S\subset X$. *If*

i) $f(x)\geqslant c_0>c_1$, $\forall x\in S$;

ii) $f(x)\leqslant c_1$, $\forall x\in\varphi(K_0)$;

iii) *for any* $g\in\Gamma=\{g\in C(K; X) \mid g(x)=\varphi(x), \ \forall x\in K_0\}$,

$$g(K)\cap S\neq\phi;$$

then the following number is a critical value of f:

$$c=\inf_{g\in\Gamma}\max_{t\in K} f(g(t))\geqslant c_0.$$

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Remark. After completion of this paper, the author has learned that in the new book of J. P. Aubin and I. Ekeland, Applied Nonlinear Analysis, Wiley-Interscience, New York, 1984, p. 272, there is a different proof, also using Ekeland's variational principle, for a strengthened version of the Mountain Pass Lemma, but their proof is not suitable for a locally Lipschitz function. In addition, by private communication, the author has still learned that Prof. J. Mawhin at University of Bruxelles has too a similar proof for a C^1 function. Nevertheless, our proof is new, even in the C^1 case.

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