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An Analytic Doubly-expansive Self-homeomorphism of the Open *n*-cube

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§ 0. Introduction

Originally, the definition of expansive homeomorphism was proposed in [9] as a concept in topological dynamics. Since then, it has been involved in some important topics in differentiable dynamics. For instance, the well-known Anosov diffeomorphism and the restriction of Smale's horseshoe map on its Ω-set are typical examples of expansive homeomorphisms. This concept has also been applied, for example, in ergodic theory[10] and the theory of structural stability[4]. In the mathematical description of the strange attractors[8], expansiveness is one of the main characteristics.

The existence of expansive homeomorphisms on some manifolds is a problem evoking many interesting discussions[1,2,3,5,6,7,11]. The one-dimensional case of this problem is easy to solve[1]. However, in the case of two-dimensional compact manifolds, a definite answer has been obtained only for orientable surfaces with positive genus[6]. Recently, the existence of expansive homeomorphism on an n-dimensional open ball ($n \geqslant 2$) was thoroughly solved[2,5]. It is shown in [5] that the self-homeomorphisms of an open disk given by Reddy[7] and Williams[11] is actually not expansive, and the positively expansive self-homeomorphism of $n(\geqslant 2)$-dimensional open ball given in [5] can be made C^r-smooth through a simple modification. In [3], the same existence result follows from a more general fact; but the smoothing of the positive expansive homeomorphism in that paper seems difficult.

In this paper, an expansive homeomorphism f is constructed on an open n-cube I^n ($n \geqslant 2$) (or equivalently, on an open n-ball B^n[1]). Compared with [3, 5] it has the following outstanding features:

1) f is doubly-expansive, analytic and f^{-1} is also analytic;
2) f can be imbedded in an analytic flow F on I^n;
3) f has a simpler analytical expression and the proof is shorter.

§ 1. Definitions and the Key Lemma

Throughout this paper, we denote by I the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, R^n the n-dimensional Euclidean space, I^n the n-dimensional open cube in R^n, and J (J_+) the set of all integers (the set of all nonnegative integers) respectively.

Definition. A self-homeomorphism f of a metric space (S, d) is said to be expansive if there is a positive number e (called an expansive constant of f) such that for any $x, y \in S$, $x \neq y$, one has $d(f^m x, f^m y) > e$ for some $m \in J$.

If, moreover, J is replaced by J_+, then f is said to be positively expansive. f is called negatively expansive if f^{-1} is positively expansive, and f is doubly-expansive if f is both positively and negatively expansive.

The main result of this paper depends on the following key lemma.

Lemma 1.1. *Let*

$$h(r, s) = s[1+\pi r^2+\pi(1+r^{126})^{\frac{1}{64}}]+r^2(1+r^2)^{\frac{1}{4}}[\sin 2\pi(1+r^2)^{\frac{1}{4}}-\sin 2\pi(1+r^2)^{\frac{1}{8}}]$$

and let (λ_1, μ_1) *and* (λ_2, μ_2) *be two defferent points in* R^2. *Then for any* $N \in J_+$, *there is an integer* $m \geqslant N$ *such that at least one of the following inequalities hold:*

$$h(\lambda_1+m, \mu_1) < -1 < 1 < h(\lambda_2+m, \mu_2), \tag{1.1}$$

$$h(\lambda_2+m, \mu_2) < -1 < 1 < h(\lambda_1+m, \mu_1). \tag{1.2}$$

Proof. (1) Firstly consider the case $\lambda_1+\mu_1 \neq \lambda_2+\mu_2$. Let

$$\Delta\lambda = \lambda_2-\lambda_1, \quad \Delta\mu = \mu_2-\mu_1, \quad \delta = \Delta\lambda+\Delta\mu.$$

Then $\delta \neq 0$. Take a number $c \in \left[\frac{1}{3}, \frac{2}{3}\right]$ so that $\lambda_1+\mu_1+c\delta$ is a rational number. Let $\lambda_0 = \lambda_1+c\Delta\lambda$, $\mu_0 = \mu_1+c\Delta\mu$, and take a positive integer n such that $n_0+\lambda_0+\mu_0 > 0$. We set

$$a = \sqrt{\frac{n_0+\lambda_0+\mu_0}{12}},$$

$$K = \{k \mid k \in J_+,\ k > 100(|\lambda_1|+|\lambda_2|+|\mu_1|+|\mu_2|+n_0) \text{ and } a^2 k \text{ is the square of an even integer}\}.$$

Clearly, K is an infinite subset of J_+. For every $k \in K$, let

$$m = m(k) = k^4+8ak^{\frac{5}{2}}+4ak^{\frac{3}{2}}+16a^2k+2ak^{\frac{1}{2}}+n_0.$$

then m is also an integer and $\lim_{k\to\infty} m(k) = \infty$. For $-1 \leqslant s < 1$, let

$$\lambda = \lambda(s) = \lambda_0+s\Delta\lambda,$$
$$\mu = \mu(s) = \mu_0+s\Delta\mu,$$
$$r = r(k, s) = m+\lambda = m(k)+\lambda(s).$$

We have

$$(1+r^2)^{\frac{1}{4}}=\sqrt{r}\,(1+r^{-2})^{\frac{1}{4}}=\sqrt{r}\,(1+O(r^{-2}))=\sqrt{m+\lambda}+O(r^{-\frac{3}{2}})$$
$$=k^2+4ak^{\frac{1}{2}}+2ak^{-\frac{1}{2}}+ak^{-\frac{3}{2}}-\left(8a^2-\frac{\lambda+n_0}{2}\right)k^{-2}+O(k^{-3}). \qquad ∎$$

Remark. Suppose that g_1 is a function of i ($\geqslant 1$) variables defined on some domain D, and g_2 is a function of x. If there exists a positive number M such that $|g_1(x, y_2, \cdots, y_i)|\leqslant M\cdot g_2(x)$ for any $(x, y_2, \cdots, y_i)\in D$, then $g(x, y_2, \cdots, y_i)$ will be briefly denoted by $O(g_2(x))$ when we are not interesting in the expression of g_1 and the concrete value of M. In other words, $O(g_2(x))$ will denote a function whose absolute value does not exceed $g_2(x)$ multiplied by a positive number. ∎

Thus we can write

$$(1+r^2)^{\frac{1}{8}}=\sqrt[4]{m+\lambda}+O(r^{-\frac{7}{4}})=\sqrt{\sqrt{m+\lambda}}+O(r^{-\frac{7}{4}})$$
$$=k+2ak^{-\frac{1}{2}}+ak^{-\frac{3}{2}}-2a^2k^{-2}+O(k^{-\frac{5}{2}}),$$
$$(1+r^{126})^{\frac{1}{64}}=r^{\frac{63}{32}}[1+O(r^{-126})]=k^{\frac{63}{8}}+O(k^{-504}).$$

From

$$\sin x=x-\frac{x^3}{6}+O(x^5)$$

we have

$$\sin 2\pi(1+r^2)^{\frac{1}{4}}=2\pi\left[2ak^{-\frac{1}{2}}+ak^{-\frac{3}{2}}-\left(8a^2-\frac{\lambda+n_0}{2}\right)k^{-2}+O(k^{-3})\right]$$
$$-\frac{4}{3}\pi^3\,[8a^3k^{-\frac{3}{2}}+O(k^{-\frac{5}{2}})]+O(k^{-\frac{5}{2}}),$$
$$\sin 2\pi(1+r^2)^{\frac{1}{8}}=2\pi\,[2ak^{-\frac{1}{2}}+ak^{-\frac{3}{2}}-2a^2k^{-2}+O(k^{-\frac{5}{2}})]$$
$$-\frac{4}{3}\pi^3[8a^3k^{-\frac{3}{2}}+O(k^{-\frac{5}{2}})]+O(k^{-\frac{5}{2}}),$$

and it follows that

$$h(m+\lambda,\mu)=\pi\mu[k^8+O(k^{7\frac{7}{8}})]$$
$$+[k^8+O(k^7)]\,[k^2+O(k)]\cdot 2\pi\left[\left(\frac{\lambda+n_0}{2}-6a^2\right)k^{-2}+O(k^{-\frac{5}{2}})\right]$$
$$=\pi(\mu+\lambda+n_0-12a^2)k^8+O(k^{7\frac{7}{8}})=\pi\varepsilon\delta k^8+O(k^{7\frac{7}{8}}). \tag{1.3}$$

Let $\varepsilon_1=-c$, $\varepsilon_2=1-c$. Then $\lambda_i=\lambda(\varepsilon_i)$, $\mu_i=\mu(\varepsilon_i)$, $i=1, 2$. By (1.3), it is easily shown that, for $k\in K$ large enough, if $\delta>0$, then (1.1) holds; and if $\delta<0$, then (1.2) holds. Hence, in the case $\lambda_1+\mu_1\neq\lambda_2+\mu_2$, the lemma is proved.

(2) The case $\lambda_1+\mu_1=\lambda_2+\mu_2$. We can suppose $\lambda_2>\lambda_1$. Let

$$\lambda_0=\frac{1}{2}(\lambda_1+\lambda_2),\quad \mu_0=\frac{1}{2}(\mu_1+\mu_2),\quad \delta=\frac{1}{2}(\lambda_2-\lambda_1)$$

and take an integer n_0 so that

$$2\leqslant n_0+\lambda_0+\mu_0<3.$$

Let

$$a_0=\sqrt{\frac{n_0+\lambda_0+\mu_0}{3}}.$$

Then $a_0\in\left[\frac{\sqrt{6}}{3},\ 1\right]$.

Now we set

$$Q=\{q\mid q\in J_+,\ q\geqslant 100,\ \sqrt[4]{q}>|\mu_0|+\delta\}.$$

Clearly Q is an infinite subset of J_+. For any $q\in Q$. Let $p=p(q)$ be an integer such that $\frac{p}{q}\leqslant a_0<\frac{p+1}{q}$. Then $\frac{4}{5}q<p\leqslant q-1$. Clearly, we have

$$\begin{aligned}
&3\left(\frac{p+\sqrt{q}-2}{q}\right)^2-3\left(\frac{p+1}{q}\right)^2\\
&=\frac{3}{q^2}(\sqrt{q}-3)(2p+\sqrt{q}-1)>\frac{3\sqrt{q}}{q}>\frac{|\mu_0|+\sqrt[4]{q}}{q};\\
&3\left(\frac{p}{q}\right)^2-3\left(\frac{p-\sqrt{q}+2}{q}\right)^2\\
&=\frac{2}{q^3}(\sqrt{q}-2)(2p-\sqrt{q}+2)>\frac{3\sqrt{q}}{q}>\frac{|\mu_0|+\sqrt[4]{q}}{q}.
\end{aligned}\tag{1.4}$$

Furthermore, for any integer $i\in[-\sqrt{q},\ \sqrt{q}-1]$, we have

$$3\left(\frac{p+i+1}{q}\right)^2-3\left(\frac{p+i}{q}\right)^2=\frac{3}{q^2}(2p+2i+1)<\frac{3}{q^2}\cdot\frac{22}{10}q<\frac{7}{q}.\tag{1.5}$$

From (1.4) it follows that

$$3\left(\frac{p-\sqrt{q}+2}{q}\right)^2<3a_0^2+\frac{\mu_0\sqrt[4]{q}}{q}<3\left(\frac{p+\sqrt{q}-2}{q}\right)^2.\tag{1.6}$$

From (1.5), (1.6) it can be easily seen that there exists an integer

$$i_0=i_0(q)\in[-\sqrt{q},\ \sqrt{q}-1]$$

such that

$$-\frac{7}{q}<3a_0^2+\frac{\mu_0\sqrt[4]{q}}{q}-3\left(\frac{p+i_0}{q}\right)^2\leqslant 0.\tag{1.7}$$

Let $a_q=\frac{p+i_0}{q}$, $\Delta_q=3a_q^2-3a_0^2$ and $k=k(q)=q^3$. For $0\leqslant a\leqslant 2$ and $\mu_0-|\mu_0|-\delta-1\leqslant\mu\leqslant\mu_0+|\mu_0|+\delta+1$, we denote

$$\begin{aligned}
\eta(k,a,\mu)=&k^8+4k^7+6k^6+(4a+4)k^5+(8a+1)k^4+6ak^3\\
&+(2a+4a^2)k^2+ak+3a^2-\mu.
\end{aligned}$$

Then we have

$$\begin{aligned}
\sqrt{\eta(k,a,\mu)}=&k^4+2k^3+k^2+2ak+ak^{-1}-ak^{-2}+\frac{3}{2}ak^{-3}\\
&-\left(\frac{a^2}{2}+2a+\frac{\mu}{2}\right)k^{-4}+O(k^{-5}),
\end{aligned}$$

$$\sqrt[4]{\eta(k, a, \mu)} = k^2 + k + ak^{-1} - ak^{-2} + \frac{3}{2} ak^{-3} - \left(\frac{a^2}{2} + 2a\right)k^{-4} + O(k^{-5}).$$

Let $\lambda = \lambda(\mu) = \lambda_0 + \mu_0 - \mu$. Since $a_q k = \frac{p+i_0}{q} \cdot q^3$ is an integer, we have

$$\eta(k, a_q, \mu + \Delta q) \equiv 3a_q^2 - \mu - \Delta q = n_0 + \lambda_0 + \mu_0 + \Delta q - \mu - \Delta q \equiv \lambda \pmod 1,$$

$$\sqrt{\eta(k, a_q, \mu + \Delta q)} \equiv a_q k^{-1} - a_q k^{-2} + \frac{3}{2} a_q k^{-3} - \left(\frac{a_q^2}{2} + 2a_q + \frac{\mu + \Delta q}{2}\right)k^{-4} + O(k^{-5}) \pmod 1,$$

$$\sqrt[4]{\eta(k, a_q, \mu + \Delta q)} \equiv a_q k^{-1} - a_q k^{-2} + \frac{3}{2} a_q k^{-3} - \left(\frac{a_q^2}{2} + 2a_q\right)k^{-4} + O(k^{-5}) \pmod 1.$$

Let $m = m(k, a_q, \mu + \Delta q) = \eta(k, a_q, \mu + \Delta q) - \lambda$. Then m is an integer. Since

$$m + \lambda = \eta(k, a_q, \mu + \Delta q) = k^8 + O(k^7),$$

it follows that

$$\begin{aligned}
h(m+\lambda, \mu) &= \pi\mu[k^{16} + k^{15\frac{3}{4}} + O(k^{15})] + [k^{16} + O(k^{15})][k^4 + O(k^3)] \\
&\quad \times \{\sin 2\pi\sqrt{m+\lambda}[1 + O(m^{-2})] - \sin 2\pi \sqrt[4]{m+\lambda}[1 + O(m^{-2})]\} \\
&= \pi\mu[k^{16} + k^{15\frac{3}{4}} + O(k^{15})] + [k^{20} + O(k^{19})] \\
&\quad \times \left\{2\pi\left[a_q k^{-1} - a_q k^{-2} + \frac{3}{2} a_q k^{-3} - \left(\frac{a_q^2}{2} + 2a_q + \frac{\mu + \Delta q}{2}\right)k^{-4}\right]\right. \\
&\quad \left. - 2\pi\left[a_q k^{-1} - a_q k^{-2} + \frac{3}{2} a_q k^{-3} - \left(\frac{a_q^2}{2} + 2a_q\right)k^{-4}\right] + O(k^{-5})\right\} \\
&= \pi\mu[k^{16} + k^{15\frac{3}{4}} + O(k^{15})] + [k^{20} + O(k^{19})] \\
&\quad \times 2\pi\left[-\frac{\mu + \Delta q}{2} k^{-4} + O(k^{-5})\right] \\
&= \pi[\mu k^{15\frac{3}{4}} - \Delta q k^{16} + O(k^{15})] = \pi q^{48}\left[\frac{\mu\sqrt[4]{q}}{q} - \Delta q + O(q^{-3})\right].
\end{aligned}$$

Thus it follows from (1.7) that when q is large enough, we have

$$\begin{aligned}
h(m+\lambda_1, \mu_1) &= \pi q^{48}\left[\frac{(\mu_0 + \delta)\sqrt[4]{q}}{q} - \Delta q + O(q^{-3})\right] \\
&> \pi q^{48}\left[\frac{\delta\sqrt[4]{q}}{q} - \frac{7}{q} + O(q^{-3})\right] > 1
\end{aligned}$$

and

$$\begin{aligned}
h(m+\lambda_2, \mu_2) &= \pi q^{48}\left[\frac{(\mu_0 - \delta)\sqrt[4]{q}}{q} - \Delta q + O(q^{-3})\right] \\
&\leqslant \pi q^{48}\left[-\frac{\delta\sqrt[4]{q}}{q} + O(q^{-3})\right] < -1.
\end{aligned}$$

Hence, the lemma is also valid for $\lambda_1 + \mu_1 = \lambda_2 + \mu_2$ and the proof is completed.

§ 2. An Analytic Doubly-expansive Homeomorphism of I^n ($n\geq 2$)

Now let us construct a mapping ρ: $R^n\to R^n$ such that

$$\rho(r, s_2, \cdots, s_n)=(r, h(r, s_2), \cdots, h(r, s_n)),$$

for any $(r, s_2, \cdots, s_n)\in R^n$. Since h is an analytic function of R^2 into R, the mapping ρ is analytic on the whole R^n. Let

$$g(r, s')=\frac{s'-r^2(1+r^2)^{\frac{1}{4}}[\sin 2\pi(1+r^2)^{\frac{1}{4}}-\sin 2\pi(1+r^2)^{\frac{1}{8}}]}{1+\pi r^2+\pi(1+r^{126})^{\frac{1}{64}}}.$$

It is easy to see that $g(r, h(r, s))=s$, $h(r, g(r, s'))=s'$ and

$$\rho^{-1}(r, s_2', \cdots, s_n')=(r, g(r, s_2'), \cdots, g(r, s_n')).$$

Hence, ρ is a homeomorphism and ρ^{-1} is also analytic. Thus the Jacobian of ρ and ρ^{-1} cannot be zero at any point of R^n.

Now we construct mappings P: $R^n\times R\to R^n\times R$, T: $R^n\times R\to R^n$ and T_1: $R^n\to R^n$ respectively such that

$$P(v, t)=(\rho(v), t),$$
$$T(v, t)=(r+t, s_2, \cdots, s_n),$$
$$T_1(v)=T(v, 1),$$

for any $v=(r, s_2, \cdots, s_n)\in R^n$ and $t\in R$. T is, obviously, a continuous uniform "translational motion"; and T_1 translates every point in R^n by a unit distance. If T_1 is regarded as a self-homeomorphism and T as a flow, then T_1 can be imbedded in T.

We further construct a flow Φ in R^n and a self-homeomorphism φ such that

$$\Phi=PTP^{-1},\quad \varphi=\rho T_1\rho^{-1}.$$

In other words, for any $v=\rho(r, s_2, \cdots, s_n)=(r, h(r, s_2), \cdots, h(r, s_n))\in R^n$ and $t\in R$, we let

$$\Phi(v, t)=\rho(r+t, s_2, \cdots, s_n),$$
$$\varphi(v)=\rho(r+1, s_2, \cdots, s_n).$$

It is clear that Φ and φ are analytic and φ can be imbedded in Φ.

Lemma 2.1. *The homeomorphism φ is expansive. Moreover for any*

$$v_i=\rho(\lambda_i, \mu_{2i}, \cdots, \mu_{ni})\in R^n,\quad i=1, 2$$

and $N\in J_+$, $j\in\{2, \cdots, n\}$, if $\lambda_1\neq\lambda_2$ or $\mu_{j1}\neq\mu_{j2}$, then there exists an integer $m\geq N$ such that at least one of the following inequalities holds

$$c_j\varphi^m(v_1)<-1<1<c_j\varphi^m(v_2), \tag{2.1}$$
$$c_j\varphi^m(v_2)<-1<1<c_j\varphi^m(v_1). \tag{2.2}$$

Here, c_j is a mapping of R^n to R defined by

$$c_j(s_1, s_2, \cdots, s_n)=s_j,\quad \forall(s_1, s_2,\cdots, s_n)\in R^n.$$

Proof. Since $c_j\varphi^m(v_i)=h(\lambda_i+m, \mu_{ji})$, by Lemma 1.1 we know that there is an integer m satisfying (2.1) or (2.2).

Now let θ: $I\to R$ be a mapping such that

$$\theta(x) = \operatorname{tg} x, \quad \forall x \in I.$$

Then $\theta^{-1}(r) = \operatorname{arctg} r$ ($\forall r \in R$). Furthermore, we let ψ: $I^n \to R^n$ and Ψ: $I^n \times R \to R^n \times R$ be

$$\psi(v) = (\theta(x), \theta(y_2), \cdots, \theta(y_n)),$$

$$\Psi(v, t) = (\psi(v), t).$$

Obviously, ψ and Ψ are homeomorphisms and ψ, Ψ, ψ^{-1}, Ψ^{-1} are analytic. Let F be a flow on I^n and f be its self-homeomorphism such that

$$F = \psi^{-1}\Phi\Psi = \psi^{-1}\rho T P^{-1}\Psi^{-1},$$

$$f = \psi^{-1}\varphi\psi = \psi^{-1}\rho T_1\rho^{-1}\psi.$$

That is, for any $w = \psi^{-1}\rho(r, s_2, \cdots, s_n) \in I^n$ and any $t \in R$ we let

$$F(w, t) = \psi^{-1}\rho(r+t, s_2, \cdots, s_n),$$

$$f(w) = \psi^{-1}\rho(r+1, s_2, \cdots, s_n).$$

Since ψ, Ψ, φ, Φ, ρ, P, T_1, T and their inverses are all analytic, F and f are analytic as well, f can clearly be imbedded in F. This completes the proof. ∎

Furthermore, we have

Lemma 2.2. *The above self-homeomorphism f of I^n is doubly-expansive. Moreover, for any $w = \psi^{-1}\rho(r_i, s_{2i}, \cdots, s_{ni}) \in I^n$ ($i = 1, 2$) and $N \in J_+$, $j \in \{2, \cdots, n\}$, when $r_1 \neq r_2$ or $s_{j1} \neq s_{j2}$, there exists $m > N$ such that*

$$d(f^m(w_1), f^m(w_2)) > \frac{\pi}{2} \tag{2.3}$$

and at least one of the inequalities

$$c_j f^m(w_1) < -\frac{\pi}{4} < \frac{\pi}{4} < c_j f^m(w_2), \tag{2.4}$$

$$c_j f^m(w_1) < -\frac{\pi}{4} < \frac{\pi}{4} < c_j f^m(w_1) \tag{2.5}$$

is true.

Proof. Let $v_i = f(r_i, s_{2i}, \cdots, s_{ni})$ and m be as in Lemma 2.1. Then $f^m(w_i) = \psi^{-1}\varphi^m(v_i)$. Since $\operatorname{arctg} x < -\frac{\pi}{4}$ for $x < -1$ and $\operatorname{arctg} x > \frac{\pi}{4}$ for $x > 1$, the inequality (2.1) implies (2.4) and (2.2) implies (2.5). Since either (2.1) or (2.2) must be true, so does (2.4) or (2.5). Hence, f is positively expansive.

In order to show that f is negatively expansive, let $w_i' = \psi^{-1}\rho(-r_i, s_{2i}, \cdots, s_{ni})$. By the above result there exists an integer $m' > N$ satisfying at least one of the following inequalities:

$$c_j f^{m'}(w_1') < -\frac{\pi}{4} < \frac{\pi}{4} < c_j f^{m'}(w_2'), \tag{2.6}$$

$$c_j f^{m'}(w_2') < -\frac{\pi}{4} < \frac{\pi}{4} < c_j f^{m'}(w_1'). \tag{2.7}$$

Because the mapping ψ, ρ are symmetric about the first $(n-1)$-dimensional coordinate plane (i.e., the plane given by $r = 0$) in R^n, we have

$$c_j f^{-m'}(w_i) = c_j\psi^{-1}\rho(r_i - m', s_{2i}, \cdots, s_{ni}) = c_j\psi^{-1}\rho(-r_i + m', s_{2i}, \cdots, s_{ni}) = c_j f^{m'}(w_i'). \tag{2.8}$$

(2.8), (2.6) and (2.7) imply that f is negatively expansive, completing the proof. ∎

By all of above results, we finally obtain

Theorem. *An n-dimensional open cube I^n ($n\geqslant 2$) admits an analytic doubly-expansive self-homeomorphism which can be imbedded in an analytic flow.*

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