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Locally Farkas-Minkowski Linear Inequality Systems

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Abstract

This paper introduces the *locally Farkas-Minkowski* (LFM) linear inequality systems in a finite dimensional Euclidean space. These systems are those ones that satisfy that any consequence of the system that is active at some solution point is also a consequence of some finite subsystem. This class includes the Farkas-Minkowski systems and verifies most of the properties that these systems possess. Moreover, it contains the locally polyhedral systems, which are the natural external representation of quasi-polyhedral sets. The LFM systems appear to be the natural external representation of closed convex sets. A characterization based on their properties under the union of systems is provided. In linear semi-infinite programming, the LFM property is the more general constraint qualification such that the Karush-Kuhn-Tucker condition characterizes the optimal points. Furthermore, the pair of Haar dual problems has no duality gap.

Key Words: semi-infinite linear inequality systems, Farkas-Minkowski systems, locally polyhedral systems, semi-infinite linear programming.
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1 Introduction

We will deal with consistent systems $\sigma = \{a_t' x \geq b_t,\ t \in T\}$, of linear inequalities in $\mathbb{R}^n$ where T is any set (possibly infinite) of indexes. They are known as linear semi-infinite inequality systems ($LSIS$). Zhu (1966) and Fan (1968) provided the first general results based on the geometrical properties of certain cones associated to the system. Goberna and López (1988)

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presented many results about boundedness, dimension and boundary of the solution set of $LSIS$ in a unified treatment. In order to deal with the interior, relative interior, boundary, relative boundary, dimension and extreme points of the solution set of a $LSIS$, some closedness conditions on the cones are useful. In this way, various families of systems appear, namely: closed, compact, canonically closed, normal, Farkas-Minkowski systems. We will deal basically with this last kind of systems. The Farkas-Minkowski (FM) systems are those consistent systems for which any inequality that is a consequence of the system is also a consequence of some finite subsystem. They share several good geometrical properties with finite systems (Goberna and López (1988).

Marchi et Al. (1997) have studied some properties about the so called p-systems, which have the Weyl property: the extreme points of the solution set F are those solution points such that the gradient vectors of the active constraints form a complete set. This well known property for finite linear systems is due to Weyl (1950). Recently, Anderson et Al. (1997) have introduced the locally polyhedral (LOP) systems, which include the p-systems. These systems are the natural external representations of the quasi-polyhedral sets, introduced by Klee (1959) in the context of separation of convex sets. This class of systems shares various geometrical properties with the Farkas-Minkowski systems, namely, the interior of the solution set F is the set of the strict solutions of the system (having deleted the trivial inequalities); the affine hull of F is the solution set of certain subsystem; the boundary of F is the union of all the faces of σ, etc. (Anderson et Al. (1997), Goberna and López (1988)).

In this paper we define a new wider class of $LSIS$ which includes the FM and the LOP systems, as well. These systems will be called *locally Farkas-Minkowski systems* (LFM). The main motivation for the study of this kind of systems is the behavior "at infinity" of some closed convex sets obtained as solution sets of FM or LOP systems. These LFM systems possess most of the properties that the FM systems have. The LFM class is closed under finite union whenever the relative interiors of the corresponding solution sets have a common point; this is also a property of the FM class. These results can be applicable in the field of semi-infinite optimization where some relevant papers (see references in Hettich and Kortanek (1993)) have shown many interesting applications related to some families of systems, (Goberna and López (1995)). In fact, we show an application to the theory of linear semi-infinite programing, namely the LFM property guarantees that the sufficient optimality Karush-Kuhn-Tucker condition (KKT) is also necessary. Indeed the LFM condition is the more general constraint qualification for the KKT property. Furthermore, the LFM property yields that the pair of dual problems, in Haar's sense (Charnes et Al. (1962)) has no duality gap whenever the primal problem is solvable.

The outline of the paper is as follows: in section 2, we state terminology and some preliminary results on linear semi-infinite systems. An associated homogeneous system in $\mathbb{R}^{n+1}$ is introduced, which allows us to control the behavior of the points at infinity. In section 3, the LFM systems are defined; some of their properties, the scope of the class and the relations with the FM and LOP systems are argued. Section 4 presents some geometrical properties and a characterization of the LFM systems based on their properties under the union of systems, that is analogous to the corresponding one for the LOP systems. Lastly, an application to linear semi-infinite optimization is discussed in section 5.

2 Preliminaries and Notation

For any non-empty set $F \subset \mathbb{R}^n$, $\operatorname{conv} F$, $\operatorname{cone} F$, $\operatorname{span} F$, $\operatorname{aff} F$, $F^{\perp}$ and F^o stand for the convex hull, the convex cone generated by F, the linear span of F, the affine hull, the orthogonal space and the positive polar cone of F, respectively. Moreover, the interior, closure, boundary, relative interior and relative boundary of F are denoted by $\operatorname{int} F$, $\operatorname{cl} F$, $\operatorname{bd} F$, $\operatorname{ri} F$, and $\operatorname{rbd} F$, respectively. If F is convex, $\dim F$ denotes the dimension of $\operatorname{aff} F$ and $\operatorname{lin} F$ is the linearity space of F. The classical reference for all of these concepts is the book by Rockafellar (1986).

Let σ be a linear inequality system in the Euclidean space $\mathbb{R}^n$, $\sigma = \{a_t' x \geq b_t\,,\ t \in T\}$ where T is any set of indexes. Henceforth, the systems σ we consider are always consistent, i.e., the solution set $F = F(\sigma)$ is not empty. Its characteristic cone, Zhu (1966), is

$$K = K(\sigma) = \operatorname{cone}\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T\,;\ \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}.$$

A vector $\binom{a}{b} \in \mathbb{R}^{n+1}$ where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ will also be denoted by $(a; b)$. An inequality $a'x \geq b$ is a consequence of σ if $a'\bar{x} \geq b$ holds for every $\bar{x} \in F$. A system σ is Farkas-Minkowski (FM) if any consequence of σ is a consequence of a finite subsystem of σ. By the generalized Farkas Lemma,

$a'x \geq b$ is a consequence of σ if, and only if, $(a;b) \in \text{cl } K$; a system σ is FM if, and only if, K is closed.

The set of active indexes at $\bar{x} \in F$ is $T(\bar{x}) = \{t \in T : a_t' \bar{x} = b_t\}$ and the cone of active constraints of σ at the point $\bar{x}$ is

$$A(\bar{x}) = A(\sigma, \bar{x}) = \text{ cone}\{a_t \ , \ t \in T(\bar{x})\}.$$

An index $t \in T$ is carrier for σ whenever $a_t' x = b_t$ for all $x \in F$. T_C denotes the set of carrier indices. F contains a Slater-point for σ, that is, a point $\bar{x}$ with $a_t' \bar{x} > b_t$, for any $t \in T$, if, and only if, $T_C = \emptyset$, (Goberna and López (1988)).

A feasible direction of a convex set F at a point $\bar{x} \in F$ is any non-null element of the (convex) cone:

$$D(F, \bar{x}) = \{u \in \mathbb{R}^n : \bar{x} + \theta u \in F, \text{ for some } \theta \in \mathbb{R}, \theta > 0\}.$$

It is known that

$$A(\bar{x}) \subset D(F, \bar{x})^o = \left\{a \in \mathbb{R}^n : \left(a; a'\bar{x}\right) \in \text{ cl} K\right\}. \qquad (2.1)$$

A system σ is said to be locally polyhedral (LOP) if $D(F, \bar{x}) = A(\sigma, \bar{x})^o$ for all $\bar{x} \in F$ (Anderson et Al. (1997)). A set $P \subset \mathbb{R}^n$ is said to be quasi-polyhedral if its non-empty intersections with polytopes are polytopes (Klee (1959)).

The recession cone of F is

$$0^+F = \{v \in \mathbb{R}^n : \bar{x} + \lambda v \in F, \text{ for any } \lambda \in \mathbb{R}, \lambda \geq 0, \bar{x} \in F\}.$$

Any non-null $v \in 0^+F$ is called a direction of recession of F. As regards the behavior "at infinity" of unbounded closed convex sets, the directions of recession of F may be interpreted as ideal points of F lying at infinity. If x is any point of F, the half-lines starting at x could be thought of as the segments joining x with those ideal points of F at infinity. To put these ideas in a formal way and following Rockafellar (1986), Section 8, we will consider the natural correspondence between points of $\mathbb{R}^n$ and points of the hyperplane $M = \{(x; -1) : x \in \mathbb{R}^n\}$ in $\mathbb{R}^{n+1}$. The ray $\{\lambda(x; -1) : \lambda \geq 0\}$ represents the point $x \in \mathbb{R}^n$. Finally, the directions of $\mathbb{R}^n$ are represented by the rays $\{\lambda(x; 0) : \lambda \geq 0\}$, $x \neq 0$.

For a given n-dimensional system σ, let us consider the following associated homogeneous $(n+1)$-dimensional system:

$$\sigma^* = \left\{a_t' x + b_t x_{n+1} \geq 0,\ t \in T\right\} \cup \left\{-x_{n+1} \geq 0\right\}, \tag{2.2}$$

with $x \in \mathbb{R}^n$ and $x_{n+1} \in \mathbb{R}$. Its solution set $F(\sigma^*)$ will also be denoted by F^* and its active cones $A(\sigma^*, (x; x_{n+1}))$, by: $A^*(x;\, x_{n+1})$.

Lemma 2.1. *The solution set of σ^* is the following closed convex cone:*

$$\begin{aligned} F^* &= \mathit{cone}(F \times \{-1\}) \cup 0^+F \times \{0\} \\ &= \mathit{cl}\ \mathit{cone}(F \times \{-1\}). \end{aligned}$$

Moreover,

$$A^*(0;0) = K \quad, \textit{ and} \tag{2.3}$$

$$F^* = D(F^*, (0;0)) = A^*(0;0)^o = K^o. \tag{2.4}$$

Proof. It easy to see the first equality through a little manipulation of the definition of σ^* and the fact that $v \in 0^+F$ if, and only if, $a_t' v \geq 0$, for all $t \in T$. The second equality is a known result (Rockafellar (1986), Theorem 8.2). Finally, all the equalities in (2.3) and (2.4) follow because σ^* is a homogeneous system. □

It is easy to prove that the feasible direction cone at any point y in F^* is: $D(F^*, y) = F^* + \text{cone}\{-y\}$. However, the following description shows the relationship between the feasible direction cones at points in F and F^*.

Lemma 2.2. *For any $\bar{x} \in F$, $\mu > 0$, the cones of feasible directions satisfy that:*

$$D(F^*, \mu(\bar{x}; -1)) = D(F, \bar{x}) \times \{0\} + \mathit{span}\{(\bar{x}; -1)\}.$$

Proof. Let $(u;z) \in D(F^*, \mu(\bar{x}; -1))$ and let $\theta > 0$ be small enough such that $\mu(\bar{x}; -1) + \theta(u;z) \in \text{cone}(F \times \{-1\}) \subset F^*$. That is, there exist $x \in F$, $\lambda > 0$ such that

$$\mu\begin{pmatrix} \bar{x} \\ -1 \end{pmatrix} + \theta\begin{pmatrix} u \\ z \end{pmatrix} = \lambda\begin{pmatrix} x \\ -1 \end{pmatrix}.$$

Then,

$$\begin{pmatrix} u \\ z \end{pmatrix} = \frac{\lambda}{\theta}\begin{pmatrix} x - \bar{x} \\ 0 \end{pmatrix} + \frac{(\lambda - \mu)}{\theta}\begin{pmatrix} \bar{x} \\ -1 \end{pmatrix},$$

and $(u; z) \in D(F, \bar{x}) \times \{0\} + \text{span}\,\{(\bar{x}; -1)\}$.

For the reverse inclusion, if $(w, 0) \in D(F, \bar{x}) \times \{0\}$, then $\bar{x} + \theta w \in F$ for some $\theta > 0$, so

$$\begin{pmatrix} \bar{x} \\ -1 \end{pmatrix} + \theta\begin{pmatrix} w \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{x} + \theta w \\ -1 \end{pmatrix} \in F^*,$$

and $(w; 0) \in D(F^*, (\bar{x}; -1))$. Moreover, $\pm(\bar{x}; -1) \in D(F^*, (\bar{x}; -1))$. Thus, the union of both cones is included in the convex cone $D(F^*, (\bar{x}; -1))$. Therefore, their sum is included as well. □

3 Locally Farkas-Minkowski Systems

Recall that the *LOP* systems were defined by Anderson et Al. (1997), through the property $D(F, \bar{x}) = A(\bar{x})^o$, for all $\bar{x} \in F$. However, the key relation in several proofs that appear there is the following "polar" property: $A(\bar{x}) = D(F, \bar{x})^o$. Therefore, it is natural to introduce a new class of systems that satisfy the latter property.

Definition 3.1. A system σ is said to be locally Farkas-Minkowski (*LFM*) at $\bar{x} \in F$ if

$$A(\bar{x}) = D(F, \bar{x})^o. \tag{3.1}$$

σ is *LFM* if it is *LFM* at every $\bar{x} \in F$.

It is immediate that $A(\bar{x})$ is closed for any *LFM* system. The inclusion $A(\bar{x}) \subset D(F, \bar{x})^o$ is valid for any system, so, it is only necessary to ask that $A(\bar{x}) \supset D(F, \bar{x})^o$ to get a *LFM* system.

The following result says that the *LFM* systems are precisely those systems for which any inequality that is a consequence of it and is active at some solution point must be a consequence of some finite subsystem of active constraints at this point. This interpretation justifies the name locally Farkas-Minkowski. (Another justification appears in section 4, Theorem 4.4).

Theorem 3.1. *The system σ is LFM at $\bar{x} \in F$ if, and only if, whenever $a'x \geq b$ is a consequence of σ that satisfies $a'\bar{x} = b$, then $a \in A(\bar{x})$.*

Proof. In view of the characterization of the polar of the feasible direction cone given in (2.1), the following equality:

$$A(\bar{x}) = \left\{a \in R^n : \left(a; a'\bar{x}\right) \in \text{ cl } K\right\} \tag{3.2}$$

is equivalent to (3.1). Thus, the assertion follows immediately. □

The following lemma for homogeneous systems (as σ^* in (2.2)) shows that the LFM condition at the origin is crucial.

Lemma 3.1. *A homogeneous system σ is LFM at the point $\bar{x} = 0$ if, and only if, it is LFM at any $\bar{x} \in F$, $\bar{x} \neq 0$.*

Proof. Recall that for homogeneous systems $A(0) = \text{cone}\,\{a_t\,,\, t \in T\}$ and so, $F = A(0)^o$. Moreover $F = D(F,0)$ and thus, $F^o = \text{cl}\, A(0) = D(F,0)^o$.

Suppose $D(F,\bar{x})^o = A(\bar{x})$ for every $\bar{x} \in F$, $\bar{x} \neq 0$. To show that σ is LFM at 0, it is enough to prove that $D(F,0)^o \subset A(0)$. Let $a \in D(F,0)^o = \text{cl}\, A(0)$. If $a \in \text{bd}\, A(0)$, then, there is a direction $u \neq 0$ such that $a + \theta u \notin \text{cl}\, A(0) = F^o$, for any $\theta > 0$. Let $\{x_\theta\}$ be a sequence in F, with $\theta \searrow 0$, such that

$$(a + \theta u)' x_\theta < 0\,. \tag{3.3}$$

Since F is a cone, we may assume by the standard argument of normalizing vectors and by passing to a subsequence if necessary that

$$\lim_{\theta \to 0} \|x_\theta\|^{-1} x_\theta = \tilde{x}\,, \tag{3.4}$$

for some $\tilde{x} \in F$, $\tilde{x} \neq 0$. We will show that a belongs to $A(\tilde{x}) \subset A(0)$. The key fact here is that $a \in F^o$, which gives that $a'\bar{x} \geq 0$ for all $\bar{x} \in F$. Then, it is easy to see, by virtue of (3.3) and (3.4), that $a'\tilde{x} = 0$. Theorem 3.1 yields that $a \in A(\tilde{x})$.

For the converse, assume that $D(F,0)^o = A(0)$. We need to show the inclusion $D(F,\bar{x})^o \subset A(\bar{x})$, for any $\bar{x} \in F$, $\bar{x} \neq 0$. Let $0 \neq a \in D(F,\bar{x})^o \subset D(F,0)^o = A(0) = F^o$. From $a \in D(F,\bar{x})^o$ it follows that $a'(x - \bar{x}) \geq 0$, for every $x \in F$. In particular, $x = 0$ yields that $a'\bar{x} \leq 0$. On the other hand,

$a \in F^o$ gives that $a'x \geq 0$, for every $x \in F$; in particular, $a'\bar{x} \geq 0$. Therefore $a'\bar{x} = 0$. Moreover, $a \in A(0)$, hence, $a = \sum \lambda_t\, a_t$ for some $\lambda_t > 0$, with t in a non-empty finite set $T' \subset T(0)$. Hence,

$$0 = a'\bar{x} = \sum_{t \in T'} \lambda_t\, a_t'\,\bar{x} \geq 0,$$

which implies $T' \subset T(\bar{x})$ and so, $a \in A(\bar{x})$. Therefore, $D(F, \bar{x})^o \subset A(\bar{x})$. □

The scope of the LFM class is discussed in the sequel of this section. Theorem 3.1 suggests that any FM system is LFM; in fact, this is true.

Theorem 3.2. *If σ is a Farkas-Minkowski system then σ is LFM.*

Proof. Let $\bar{x} \in F$ and consider a non trivial active consequence of σ, $0 \neq (a; a'\bar{x}) \in \operatorname{cl} K = K$. Then, there is a finite set $T' \subset T$ such that

$$a = \sum_{t \in T'} \lambda_t\, a_t \;, \quad a'\,\bar{x} \leq \sum_{t \in T'} \lambda_t\, b_t \;,$$

for some positive real numbers λ_t. But

$$a'\bar{x} = \sum_{t \in T'} \lambda_t\, a_t'\,\bar{x} \geq \sum_{t \in T'} \lambda_t\, b_t \geq a'\bar{x} \;,$$

which implies that $T' \subset T(\bar{x})$ and so, $a \in A(\bar{x})$. By Theorem 3.1, σ is LFM at $\bar{x}$, for any $\bar{x} \in F$ and so σ is LFM. □

Anderson et Al. (1997) have proved that the active cones are closed for LOP systems and that their solution sets are quasi-polyhedral. These two properties will be applied in the following theorem.

Theorem 3.3. *σ is a LOP system if, and only if, σ is LFM and $F(\sigma)$ is quasi-polyhedral.*

Proof. Assume σ is a LOP system, then $D(F, \bar{x}) = A(\bar{x})^o$, for any $\bar{x} \in F$. Moreover, $F(\sigma)$ is quasi-polyhedral and the active cone $A(\bar{x})$ is closed (Anderson et Al. (1997)). By taking the polar of these sets, σ is LFM.

Conversely, if σ is a LFM system then $A(\bar{x}) = D(F, \bar{x})^o$. By considering the polar of these cones, the proposition follows because the cone $D(F, \bar{x})$ is closed when F is quasi-polyhedral. □

The following example shows that the LFM class is bigger than both classes, the FM and the LOP.

Example 3.1. Consider the system $\sigma = \{(1-t)\,t^{-1}x + t\,(1-t)^{-1}\,y \geq 2,\ \ 0 < t < 1\}$. The solution set F is the convex hull of the set $\{(x,y) \in \mathbb{R}^2 : xy = 1,\ x > 0,\ y > 0\}$. The system σ is neither a FM system nor a LOP system. However, it is LFM.

The following property shows the relationship between the systems σ and σ^* and the condition of being FM and LFM, respectively.

Theorem 3.4. *σ^* is a LFM system if, and only if, σ is FM.*

Proof. Assume σ^* is LFM; by (2.3), $K = A^*(0;0) = D(F^*,(0;0))^o$ is closed, which yields the assertion about σ.

Conversely, suppose σ is a FM system, i.e., cl $K = K$. By (2.4), $(F^*)^o = \text{cl } K = K$. By recalling that $K = A^*(0;0)$, it follows that $A^*(0;0) = D(F^*,(0;0))^o$, that is, the homogeneous system σ^* is LFM at $(0;0)$. By Lemma 3.1, σ^* is a LFM system. □

Theorem 3.5. *σ^* is a LOP system if, and only if, σ is FM and F is polyhedral.*

Proof. If σ^* is LOP then F^* is quasi-polyhedral (Anderson et Al. (1997)). Therefore $D(F^*,y)$ is a polyhedral set for any $y \in F^*$. In particular, the following sets are polyhedral: $D(F^*,(0;0)) = F^*$ by using (2.4), $F^* \cap (\mathbb{R}^n \times \{-1\}) = F \times \{-1\}$ and finally F. Moreover, if σ^* is LOP then σ^* is LFM (Theorem 3.3), so the active cone $A^*(0;0)$ is closed. From (2.3), K is closed too, which yields that σ is FM.

For the reverse implication, assume that σ is FM and F is polyhedral. By Theorem 3.4, σ^* is LFM. Because F is polyhedral, $\text{cl}\,\text{cone}\,(F \times \{-1\}) = F^*$ is a polyhedral cone (Rockafellar (1986), Theorem 19.7). An application of Theorem 3.3 to the system σ^* yields that σ^* is LOP. □

Theorem 3.6. *If σ is a LFM system with a bounded solution set, then σ is FM.*

Proof. There are no directions of recession of the bounded solution set F. By Lemma 2.1, $F^* = \text{cone}\,(F \times \{-1\})$ and by Lemma 2.2,

$$D(F^*, (\bar{x}; -1)) = D(F, \bar{x}) \times \{0\} + \text{span}\{(\bar{x}; -1)\}.$$

The polar of the sum of two non-empty convex cones is the intersection of their polars. Hence

$$\begin{aligned} D(F^*, (\bar{x}; -1))^o &= (D(F, \bar{x}) \times \{0\} + \text{span}\{(\bar{x}; -1)\})^o \\ &= (D(F, \bar{x}) \times \{0\})^o \cap (\text{span}\{(\bar{x}; -1)\})^o \\ &= A(\bar{x}) \times I\!R \cap \{(\bar{x}; -1)\}^{\perp}. \end{aligned}$$

Therefore, if $(a; b) \neq 0$ is in this polar then $a = \sum \lambda_t a_t$ for finitely many $a_t \in A(\sigma, \bar{x})$ and some $\lambda_t \geq 0$. Moreover, $(a; b)$ is orthogonal to $(\bar{x}; -1)$. Thus,

$$0 = a'\bar{x} - b = \sum \lambda_t \left(a_t' \bar{x}\right) - b = \sum \lambda_t b_t - b,$$

which implies that $b = \sum \lambda_t b_t$, and hence $(a; b)$ is in the active cone at $(\bar{x}; -1)$. Then, $D(F^*, (\bar{x}; -1))^o \subset A^*(\bar{x}; -1)$. Lemma 3.1 and Theorem 3.4 complete the proof. □

For the sake of completeness, we state some results about LOP systems, whose proofs are direct, and others about homogeneous ones.

Theorem 3.7. *For any LOP system σ, it holds that:*

(i) If F is bounded then σ is FM and F is polyhedral.

(ii) If F is polyhedral then σ is FM.

Theorem 3.8. *If σ is a homogeneous system, then:*

(i) σ is LFM if, and only if, σ is FM.

(ii) σ is LOP if, and only if, σ is FM and F is polyhedral.

Proof. (*i*) By virtue of Theorem 3.4 it only remains to prove that any homogeneous LFM system σ is FM. $A(0)$ is closed for a LFM system because $A(0) = D(F, 0)^o$. On the other hand, for a homogeneous system, $A(0) = \text{cone}\{a_t, t \in T\}$ which implies that $K = A(0) \times (-\infty, 0]$. Hence, K is closed and σ is FM.

(*ii*) Any *LOP* system σ is *LFM*, so part (*i*) gives that σ is *FM*. Moreover, its solution set F is quasi-polyhedral. Since σ is homogeneous, F is a quasi-polyhedral convex cone. Corollary 19.7.1 in Rockafellar (1986) gives that F is polyhedral because it is the convex cone generated by a polytope, namely $F = \text{cone}\ (F \cap P)$, where P is a closed n-dimensional rectangle centered at the origin with diameter one. □

Remark 3.1. A natural external representation σ of any closed convex set F is the one consisting of the inequalities that define all the supporting half-spaces of F. This kind of representation is *LFM*, by Theorem 3.1. If F is quasi-polyhedral, this representation is *LOP* by Theorem 3.3. If F is either bounded, polyhedral or conical, then σ is *FM* by Theorems 3.6, 3.7 or 3.8, respectively.

4 Properties of *LFM* systems

Now, we proceed to establish the main geometrical properties of the solution set of a *LFM* system. In doing this, the following lemma shows useful.

Lemma 4.1. *If σ is a LFM system and $\bar{x} \in F$, then:*

(*i*) $\quad (F - \bar{x})^{\perp} = \mathit{lin}\, A(\bar{x})$.

(*ii*) $\quad \mathit{aff}\, F = \bar{x} + \mathit{span}\{F - \bar{x}\} = \bar{x} + \{\mathit{lin}\, A(\bar{x})\}^{\perp}$.

(*iii*) $\quad \mathit{lin}\, A(\bar{x}) = \mathit{span}\{a_t,\, t \in T_C\}$.

Proof. (*i*) Given $\bar{x} \in F$, we have $A(\bar{x}) = D(F, \bar{x})^o = (F - \bar{x})^o$, so that

$$\text{lin}\, A(\bar{x}) = \text{lin}\,((F - \bar{x})^o) = (F - \bar{x})^{\perp}.$$

(*ii*) is an immediate consequence of (*i*) above.

(*iii*) If $a \in \text{lin}\ A(\bar{x})$ then, $\pm a \in A(\bar{x})$. Assume $a \neq 0$ and let $\lambda_t > 0$, for t in some non-empty finite set $T' \subset T(\bar{x})$, such that

$$a = \sum_{t \in T'} \lambda_t\, a_t\ .$$

For any $x \in F$, $(x - \bar{x}) \in D(F, \bar{x}) \subset A(\bar{x})^o \subset \{\pm a\}^o$, hence

$$0 = a'(x - \bar{x}) = \sum_{t \in T'} \lambda_t\, a_t'\ (x - \bar{x}).$$

Since $a_t'(x-\bar{x}) \geq 0$ for all $t \in T'$, it follows that $a_t'\,x = a_t'\,\bar{x} = b_t$, for all $x \in F$, $t \in T'$. Therefore, $T' \subset T_C$ and so, $a \in \operatorname{span}\{a_t,\, t \in T_C\}$.

Conversely, if $t \in T_C$ then $\pm(a_t\,; a_t\,\bar{x}) \in \operatorname{cl} K$ and by (3.2), $\pm a_t \in A(\bar{x})$, so that, $a_t \in \operatorname{lin} A(\bar{x})$. □

As it was stated in the introduction, the LFM systems possess the main geometrical properties that the Farkas-Minkowski systems and the LOP ones share. Theorem 3.2 in Anderson et Al. (1997) and Theorem 3.2, Corollary 4.2.1, Theorems 4.1 and 4.2 in Goberna and López (1988), give almost identical properties for LOP and FM systems, respectively. Indeed these properties are valid for the more general class of LFM systems, as the following propositions show (The proofs of the above lemma and the next theorem are partially taken from Anderson et Al. (1997) and reproduced here for the sake of completeness):

Theorem 4.1. *Let σ be a LFM system, then:*

(i) $A(\bar{x})$ *is closed for any* $\bar{x} \in F$,

(ii) $\dim F = n - \dim \mathit{span}\{a_t,\; t \in T_C\}$, *and so*

$$\mathit{aff}\, F = \{x \in \mathbb{R}^n :\; a_t' x = b_t,\; t \in T_C\},$$

(iii) $\mathit{ri}\, F = \{x \in \mathbb{R}^n :\; a_t' x > b_t,\; t \in T \backslash T_C,\quad a_t' x = b_t,\; t \in T_C\}$.

Proof. (*i*) is immediate from the definition of LFM systems.

(*ii*) is a consequence of Lemma 4.1, (*ii*) and (*iii*).

(*iii*) The inclusion "$\subset$" is well known (Goberna and López (1988), Theorem 3.1). For the other inclusion, it is clear that if $\bar{x}$ is in the right hand set then $\bar{x} \in F$ and $T(\bar{x}) = T_C$. If $\bar{x} \in \operatorname{rbd} F$ there exists a proper supporting hyperplane of F at $\bar{x}$, i.e., there exists $0 \neq a \in \mathbb{R}^n$ such that $a'x \geq a'\bar{x}$ for all $x \in F$ and $a'y > a'\bar{x}$ for some $y \in F$. Theorem 3.1 yields that $a \in A(\bar{x}) = \operatorname{cone}\{a_t,\, t \in T_C\}$. Hence, $a = \sum \lambda_t\, a_t$ for finitely many a_t and some $\lambda_t > 0,\; t \in T_C$. A standard argument gives the following contradiction:

$$a'\bar{x} < a'y = \sum \lambda_t\, a_t' y = \sum \lambda_t\, a_t' \bar{x} = a'\bar{x}.$$

Therefore, $\bar{x} \in \operatorname{ri} F$. □

By recalling that any system σ has a Slater point if, and only if, $T_C = \emptyset$, (Goberna and López (1988)), the following corollary is a consequence of the above theorem.

Corollary 4.1. *If σ is a LFM system, then:*

(i) $\text{int}\, F$ is the set of Slater-points of σ (by deleting the trivial inequalities).

(ii) F is full-dimensional if and only if σ has a Slater point.

In order to describe the boundary and the relative boundary of the solution set of a given LFM system σ, recall that the set $F_t = \{x \in F : a_t' x = b_t,\ t \in T\}$ is a face of σ, whenever $a_t' x = b_t$, is not the trivial equality. Obviously, $F_t = F$ for any carrier index t.

Corollary 4.2. *If σ is a LFM system, then*

(i) $bd\, F$ is the union of all the faces of σ, and

(ii) $rbd\, F$ is the union of all the faces F_t of σ, $t \in T \backslash T_C$.

Now, we proceed to analyze the properties of the LFM systems under the operation of union of systems. For a system σ^i we will denote by F^i its solution set and by K^i its characteristic cone.

Lemma 4.2. *Let σ^1 and σ^2 be any two systems. Then $ri\, F^1 \cap ri\, F^2 \neq \emptyset$ if, and only if, $cl K^1 \cap -cl K^2 = -cl K^1 \cap cl K^2$ (i.e., it is a linear space).*

Proof. The condition $\text{ri}\, F^1 \cap \text{ri}\, F^2 \neq \emptyset$ is known to be equivalent to the non-existence of a hyperplane separating F^1 and F^2 properly. So it is enough to show that $\text{cl}\, K^1 \cap -\text{cl}\, K^2 = -\text{cl}\, K^1 \cap \text{cl}\, K^2$ if and only if there is no such hyperplane. Suppose H is a hyperplane with normal vector $(a; b)$ that separates F^1 and F^2 properly, say for instance,

$$
\begin{aligned}
a'x &\geq b, \text{ for } x \in F^1, && (4.1)\\
a'x &\leq b, \text{ for } x \in F^2 \text{ and} && (4.2)\\
a'\bar{x} &> b, \text{ for some } \bar{x} \in F^1, \text{ or, } a'\tilde{x} < b, \text{ for some } \tilde{x} \in F^2. && (4.3)
\end{aligned}
$$

Then, by taking into account that an inequality $a'x \geq b$ is a consequence of any system if, and only if, $(a; b)$ is an element of the closure of its characteristic cone, conditions 4.1-4.3 are equivalent to $(a; b) \in \text{cl}\, K^1 \cap -\text{cl}\, K^2$ and $(a; b) \notin -\text{cl}\, K^1 \cap \text{cl}\, K^2$. □

Theorem 4.2. *The union of any two FM systems σ^1 and σ^2 that satisfy that $ri\,F^1 \cap ri\,F^2 \neq \emptyset$, is another FM system.*

Proof. It is enough to show that $\mathrm{cl}\left(K^1 + K^2\right) = \mathrm{cl}\,K^1 + \mathrm{cl}\,K^2$ because $K^1 + K^2$ is the characteristic cone of $\sigma^1 \cup \sigma^2$ and each of the cones K^1 and K^2 is closed. Recall the property that says that $\mathrm{cl}\left(K^1 + K^2\right) = \mathrm{cl}\,K^1 + \mathrm{cl}\,K^2$, whenever K^1 and K^2 are non-empty convex cones satisfying the following condition: if $a_i \in \mathrm{cl}\,K^i$, for $i = 1, 2$, and $a_1 + a_2 = 0$, then $a_i \in \mathrm{lin}\,\mathrm{cl}\,K^i$, for $i = 1, 2$ (Rockafellar (1986), Corollary 9.1.3). In our case, from the above Lemma, for any $a_1 \in \mathrm{cl}\,K^1$, $a_2 \in \mathrm{cl}\,K^2$ with $a_1 + a_2 = 0$, it holds that

$$a_1 = -a_2 \in cl\,K^1 \cap -cl\,K^2 = -cl\,K^1 \cap cl\,K^2,$$

which provides the required condition. □

Lemma 4.3. *Let σ^1 and σ^2 be any two systems with $F^1 \cap F^2 \neq \emptyset$. Then $ri\,F^1 \cap ri\,F^2 \neq \emptyset$ if, and only if, $ri\,D\left(F^1, \bar{x}\right) \cap ri\,D\left(F^2, \bar{x}\right) \neq \emptyset$, for any $\bar{x} \in F^1 \cap F^2$.*

Proof. Suppose $\mathrm{ri}\,F^1 \cap \mathrm{ri}\,F^2 \neq \emptyset$ and let $\bar{x} \in F^1 \cap F^2$. Then:

$$\begin{aligned}
\emptyset &\neq \left(\mathrm{ri}\,F^1 \cap \mathrm{ri}\,F^2\right) - \bar{x} \\
&= \left(\mathrm{ri}\,F^1 - \bar{x}\right) \cap \left(\mathrm{ri}\,F^2 - \bar{x}\right) \\
&= \mathrm{ri}\left(F^1 - \bar{x}\right) \cap \mathrm{ri}\left(F^2 - \bar{x}\right) \\
&\subset \mathrm{ri}\left(\mathrm{cone}\left(F^1 - \bar{x}\right)\right) \cap \mathrm{ri}\left(\mathrm{cone}\left(F^2 - \bar{x}\right)\right) \\
&= \mathrm{ri}\,D\left(F^1, \bar{x}\right) \cap \mathrm{ri}\,D\left(F^2, \bar{x}\right).
\end{aligned}$$

Conversely, let $\bar{x} \in F^1 \cap F^2$ be such that $\mathrm{ri}\,D\left(F^1, \bar{x}\right) \cap \mathrm{ri}\,D\left(F^2, \bar{x}\right) \neq \emptyset$. Suppose that $\mathrm{ri}\,F^1 \cap \mathrm{ri}\,F^2 = \emptyset$, then, there is a hyperplane $H = \{z : a'z = b\}$ separating F^1 and F^2 properly, i.e., $a'x \geq b \geq a'y$ for all $x \in F^1$, $y \in F^2$, and $a'\hat{x} > a'\hat{y}$ for some $\hat{x} \in F^1$, $\hat{y} \in F^2$. Since $\bar{x} \in F^1 \cap F^2$, it follows that $a'\bar{x} = b$.

Let $z \in \mathrm{ri}\,D\left(F^1, \bar{x}\right) \cap \mathrm{ri}\,D\left(F^2, \bar{x}\right)$, then there exist some positive numbers θ_1 and θ_2 such that $\bar{x} + \theta_1 z \in F^1$ and $\bar{x} + \theta_2 z \in F^2$. Hence,

$$a'\left(\bar{x} + \theta_1 z\right) \geq b \geq a'\left(\bar{x} + \theta_2 z\right),$$

which implies that $a'z = 0$. Now, suppose that $a'\hat{x} > b$. Since $\hat{x} - \bar{x}$ is a feasible direction of F^1 at $\bar{x}$ and $z \in \operatorname{ri} D\left(F^1, \bar{x}\right)$, there exists $\mu > 1$ such that

$$v = (1-\mu)(\hat{x} - \bar{x}) + \mu z \in D\left(F^1, \bar{x}\right).$$

But then, there is a positive β for which $\bar{x} + \beta v \in F^1$. Hence,

$$\begin{aligned} b &\leq a'(\bar{x} + \beta v) \\ &= a'\bar{x} + \beta(1-\mu)a'(\hat{x} - \bar{x}) + \beta\mu a'z \\ &= b + \beta(1-\mu)\left(a'\hat{x} - b\right) \\ &< b \quad , \end{aligned}$$

which is a contradiction. The case $a'\hat{y} < b$ is handled in a similar manner. Therefore, $\operatorname{ri} F^1 \cap \operatorname{ri} F^2$ is non empty. □

Theorem 4.3. *If σ^1 and σ^2 are two LFM systems with $\operatorname{ri} F^1 \cap \operatorname{ri} F^2 \neq \emptyset$ then $\sigma^1 \cup \sigma^2$ is a LFM system as well.*

Proof. The facts that $D\left(F^1 \cap F^2, \overline{x}\right) = D\left(F^1, \overline{x}\right) \cap D\left(F^2, \overline{x}\right)$ and that the polar of the intersection of cones with a common relative interior point (by Lemma 4.3) is the sum of their polar cones (Rockafellar (1986), Corollary 16.4.2), imply that:

$$\begin{aligned} D\left(F^1 \cap F^2, \overline{x}\right)^o &= D\left(F^1, \overline{x}\right)^o + D\left(F^2, \overline{x}\right)^o \\ &= A\left(\sigma^1, \overline{x}\right) + A\left(\sigma^2, \overline{x}\right) \\ &= A\left(\sigma^1 \cup \sigma^2, \overline{x}\right), \end{aligned}$$

since σ^1 and σ^2 are both LFM systems. Therefore $\sigma^1 \cup \sigma^2$ is LFM. □

Note. The LFM systems are closed by aggregation of consequent inequalities. Moreover, from Theorem 4.3, they are also closed by finite aggregation of properly cutting inequalities. The same properties hold for the FM and LOP systems.

The LOP systems take their name from the fact that they are those systems whose solution sets satisfy that the non-empty intersections with polytopes are polytopes too (Anderson et Al. (1997)). Since the polytopes are just the bounded polyhedral sets, the following characterization of the LFM systems is in some sense analog to that one of the LOP systems. Furthermore, this characterization justifies the name " locally" Farkas-Minkowski for LFM systems.

Theorem 4.4. *σ is a LFM system if, and only if, $\sigma \cup \sigma^1$ is a FM system for any FM system σ^1 with $\operatorname{ri} F \cap \operatorname{ri} F^1 \neq \emptyset$ and F^1 bounded.*

Proof. The direct implication follows from Theorems 4.3 and 3.6. For the converse, let any $\bar{x} \in F$ and consider a FM system σ^1 whose solution set F^1 is an n-dimensional bounded set such that $\bar{x} \in \operatorname{int} F^1$. Then, $\operatorname{ri} F \cap \operatorname{ri} F^1 \neq \emptyset$ and thus $\sigma \cup \sigma^1$ is FM. Moreover, the fact that $\bar{x}$ is an interior point of F^1 gives that $A(\sigma^1, \bar{x}) = \{0\}$ and $D(F^1, \bar{x}) = \mathbb{R}^n$; hence,

$$A(\sigma, \bar{x}) = A(\sigma \cup \sigma^1, \bar{x}) = D(F \cap F^1, \bar{x})^o = D(F, \bar{x})^o .$$

Therefore, σ is a LFM system. □

The following two examples show that the conditions in Theorems 4.2, 4.3 and 4.4 are not superfluous:

Example 4.1. The following system (recall Example 3.1) is LFM:

$$\sigma^1 = \{(1-t)\, t^{-1} x + t\,(1-t)^{-1}\, y \geq 2\ , \quad 0 < t < 1\}.$$

Consider the FM system $\sigma^2 = \{x + y \geq 0\}$, whose solution set F^2 is unbounded. In this case, $F^1 \cap F^2 = F^1$, $\operatorname{ri} F^1 \cap \operatorname{ri} F^2 \neq \emptyset$ and $\sigma^1 \cup \sigma^2$ is not a FM system.

Example 4.2. $\sigma^1 = \{-(\cos t)x - (\sin t)y \geq -1,\ t \in [0, 2\pi)\}$ and $\sigma^2 = \{y \geq 1\}$ are both FM systems. The solution set F^1 is the unitary disk in $\mathbb{R}^2$ and $F^1 \cap F^2$ is the set consisting of the single point $(0,1)$. The inequality $x \geq 0$ is a consequence of the system $\sigma^1 \cup \sigma^2$ which is not a consequence of only finitely many inequalities in $\sigma^1 \cup \sigma^2$. Hence, $\sigma^1 \cup \sigma^2$ is not a FM system. Notice that $\operatorname{ri} F^1 \cap \operatorname{ri} F^2 = \emptyset$. Furthermore, it is clear that we may replace the system σ^2 by any one with a bounded solution set and still having the same features, namely $\sigma^3 = \{2 \geq y \geq 1,\ 1 \geq x \geq -1\}$.

Note. To close this section, let us remark that besides the obvious relation through the polar correspondence in the definitions of the LFM and the LOP systems, we have shown other analogies:

(i) Theorems 3.4 and 3.5 show the relations with the $(n+1)$-dimensional system σ^*.

(ii) The characterization of the LFM systems provided by Theorem 4.4 and the one given in Corollary 4.1 in Anderson et Al. (1997): "$F \neq \emptyset$ is the solution set of some *LOP* if, and only if, the non-empty intersections of F with polytopes is polytope too".

(*iii*) Both classes, the *LFM* and the *LOP*, are closed under finite union of systems (with the additional condition in Theorem 4.3 for *LFM* systems).

Furthermore, by taking into account Remark 3.1, concerning the natural external representation of closed convex sets, one can realize certain analogy between quasi-polyhedral sets - *LOP* systems - on one side, and closed convex sets - *LFM* systems - on the other. There is also a certain duality regarding the polytopes as an absorbent class into the quasi-polyhedral sets (under the intersection), and the *FM* systems with bounded solution sets as an absorbent class into the *LFM* systems (under the union).

5 An application to optimization theory

In this section we apply the *LFM* property to the optimization theory in linear semi-infinite programming, which can be considered as the natural extension of the classical Karush-Kuhn-Tucker (*KKT*) theory for ordinary non-linear programming.

Consider the following Haar-dual pair, (P) - (D), of problems:

$$(P) \quad \inf c'x \quad (c \neq 0) \qquad\qquad (D) \quad \sup \psi(\lambda) = \sum_{t \in T} \lambda_t \, b_t$$

$$\text{s.t.} \quad a_t' \, x \geq b_t \,, \; t \in T. \qquad\qquad \text{s.t.} \quad \sum_{t \in T} \lambda_t \, a_t = c \,, \; \lambda \in R_+^{(T)},$$

where $R_+^{(T)}$ denotes the space of non-negative generalized finite sequences on T. $v\,(P)$ and $v\,(D)$ indicate the value of (P) and (D), respectively. The defect is $\delta(P;D) = v\,(P) - v\,(D)$; we say that there is no duality gap when $\delta(P;D) = 0$.

Let $\overline{x} \in F$ be a feasible point of (P). The (KKT) condition can be stated as follows: $c \in A(F, \overline{x})$; in other words, c is asked to be an element of the active cone at $\overline{x}$. This condition is sufficient for the optimality of $\overline{x}$ for any (P). However, for the necessity, some constraint qualifications are required. (P) is said to be a *LFM* problem whenever the corresponding constraint system is a *LFM* system. The following theorem shows that (KKT) condition is sufficient and necessary for optimality for *LFM* problems.

Theorem 5.1. *Let (P) be a LFM problem and $\overline{x} \in F$. Then $\overline{x}$ is a solution of (P) if, and only if, $c \in A(F, \overline{x})$ (KKT condition).*

Proof. For any LSI problem (P) , if $c \in A(F, \overline{x}) \subset D(F, \bar{x})^o = (F - \bar{x})^o$, then $c'(y - \overline{x}) \geq 0$ for any $y \in F$, which gives the optimality of $\overline{x}$. Now, for LFM systems, the equality $A(\sigma, \bar{x}) = D(F, \bar{x})^o$ holds true; suppose that $\overline{x}$ is a solution of (P) and that $c \notin A(F, \overline{x})$. Then $c \notin D(F, \bar{x})^o$ and there exists some $y \in F$ such that $c'(y - \overline{x}) < 0$, which contradicts the optimality of $\overline{x}$. □

It is clear now that the LFM property is the more general constraint qualification for the KKT condition.

Theorem 5.2. *Let (P) be a solvable LFM problem. Then the dual problem (D) is solvable and there is no duality gap.*

Proof. Let $\overline{x}$ be a solution of (P). Because of the previous theorem, $c \in A(F, \overline{x})$ and hence $c = \sum_{t \in T'} \overline{\lambda}_t a_t$ for some positive numbers $\overline{\lambda}_t$ and some finite set $T' \subset T(\overline{x})$, which implies that the dual problem (D) is feasible. Let $\overline{\lambda}$ indicate the so obtained generalized finite sequence, then

$$\psi(\overline{\lambda}) = \sum_{t \in T'} \overline{\lambda}_t b_t = \sum_{t \in T'} \overline{\lambda}_t a_t' \overline{x} = c' \overline{x} = v(P).$$

Therefore, the weak duality property gives $v(P) = v(D)$. □

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