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Link genus and the Conway moves

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Let L_+, L_- and L_0 be three links in S^3 related by the standard Conway moves:

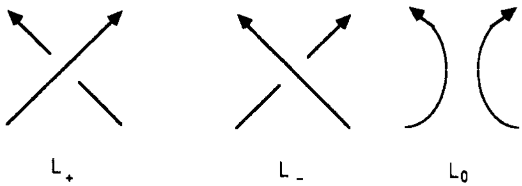

Figure 1

The Conway potential functions $\nabla_+(z)$, $\nabla_-(z)$ and $\nabla_0(z)$ of the three links are related as follows [Co]:

$$\nabla_+(z) - \nabla_-(z) = z\nabla_0(z)$$

Hence in particular, at least two of ∇_+, ∇_-, and $z\nabla_0$ have the same degree, which is no smaller than the degree of the third.

A Seifert surface for an oriented link L in a 3-manifold is a compact oriented surface none of whose components are closed and whose boundary is the link. Define $\chi(L)$ to be the maximal Euler characteristic of all Seifert surfaces for L. If L is a non-split alternating link in S^3 then $\deg(\nabla_L) = 1 - \chi(L)$ [Cr]. Hence if L_+, L_- and L_0 are all non-split alternating links, then two of $\chi(L_+)$, $\chi(L_-)$ and $\chi(L_0) - 1$ are equal and are no larger than the third. We will show that this relation remains true for arbitrary links. Two consequences are:

a) the height of the Conway skein diagram for a link L is bounded below by $-\chi(L)$. In particular, this gives an unexpected lower bound for the complexity of calculating the new oriented knot polynomials.

b) doubled knots are precisely those knots whose genus and unknotting number are both 1.

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1. The main theorem

1.1. DEFINITIONS. Following Thurston [Th], define the *complexity* $\chi^-(S)$ of an oriented surface S to be $-\chi(C)$, where C is the union of all non-simply connected components of S and $\chi(C)$ is its Euler characteristic. For M a compact oriented 3-manifold and N a (possibly empty) surface in ∂M, assign to any homology class α in $H_2(M, N; Z)$ the minimum complexity $x(\alpha)$ of all oriented imbedded surfaces whose fundamental class represents α. The function $x: H_2(M, N; Z) \to Z_+$ is called the *Thurston norm*. An oriented surface $(S, \partial S) \subset (M, \partial M)$ is *taut* if it is incompressible and $\chi^-(S) = x([S, \partial S])$ in $H_2(M, \eta(\partial S))$, where $\eta(\partial S)$ is a bicollar neighborhood of ∂S in ∂M.

1.2 LEMMA. *A Seifert surface S for a link L is taut if and only if $\chi(S) = \chi(L)$.*

Proof. Let L_d be the maximal sublink of L which bounds an imbedded collection of disks D_d with interiors disjoint from L. By an innermost disk argument we can take these disks to have interiors disjoint from any given incompressible Seifert surface S for L. Any component of L_d must then bound a disk in S, since S is incompressible, and any disk component of S must have boundary in L_d by maximality of L_d. Hence $\chi^-(S) = d - \chi(S)$. Then an incompressible Seifert surface minimizing χ^- must maximize χ and vice versa. ||

1.3 DEFINITION. An arbitrary link L is isotopic to the distant union of its non-splittable sublinks. The number of such non-splittable sublinks is called the *splitting number* of L.

1.4 THEOREM. *Suppose L_+, L_-, and L_0 are three links related by the Conway moves at a crossing. Then two of $\chi(L_+)$, $\chi(L_-)$ and $\chi(L_0) - 1$ are equal and are no larger than the third. The splitting numbers of the same pair of links are equal and are no larger than that of the third.*

Proof. The proof is a modest variation of ideas in [Ga$_3$] and [ST]. Let D be a crossing disk for the crossing, i.e. a disk which intersects L_+ in precisely two points, of opposite orientation (see [ST, 1.1] or figure 2). Note that the knot in S^3 obtained by doing -1 surgery on $K = \partial D$ is precisely L_-.

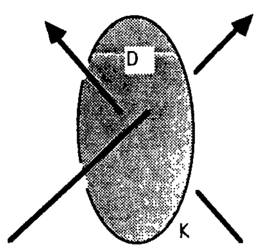

Figure 2

An innermost circle argument shows that any essential sphere in $S^3-(L_+\cup K)$ can be isotoped off of D in S^3-L_+. Any sphere in $S^3-(L_+\cup D)$ which separates a sublink of L_+ from K persists in L_- and L_0. Hence, with no loss of generality, we restrict further to the case in which $S^3-(L_+\cup D)$ is irreducible.

Let $M=S^3-\eta(K\cup L_+)$ and let M_+, M_- and M_0 be the manifolds obtained from M by filling in a torus along $\partial\eta(K)$ with framings ∞, -1, and 0 respectively. Then $M_+=S^3-\eta(L_+)$ and $M_-=S^3-\eta(L_-)$. It is not quite true that $M_0=S^3-\eta(L_0)$, but there is a close connection (see claim 2 below). Let S be a Seifert surface for L_+ in M which has maximal χ among all Seifert surfaces for L_+ *in* M.

CLAIM 1. At least two of M_+, M_- and M_0 are irreducible; in those two manifolds, S still maximizes χ.

Proof of claim 1.

CASE 1. L_+ lies in a knotted solid torus τ in $S^3-\eta(K)$ whose linking number with K is non-trivial and $\partial\tau$ is incompressible in $\tau-L_+$ (i.e. τ is a companion of L_+).

Since L_+ pierces D twice, with opposite orientation, in fact τ pierces D precisely once (in a subdisk of D). Then $D-\tau$ is an annulus whose boundary circle on $\eta(K)$ has slope 0. Since τ is knotted no other slope on $\partial\eta(K)$ can be that of a boundary circle of an essential spanning annulus in $M-\tau$. Hence $T=\partial\tau$ is incompressible in M_+ and M_-.

Subclaim (a) $M_\pm$ is irreducible.

Proof. $M_\pm-\tau$ is irreducible since $M_\pm-\tau$ is a knot complement. $\tau-L_+\subset M$ is irreducible since M is irreducible and T is incompressible. Since $M_\pm$ is obtained from gluing $M_\pm-\tau$ to $\tau-L_+$ along the incompressible T, $M_\pm$ is irreducible.

Subclaim (b) S maximizes χ in $M_\pm$.

Proof. The argument is essentially that found in [Sh]: Suppose Σ is a Seifert surface for L_+ in $M_\pm$. Without decreasing $\chi(\Sigma)$ do 2-surgeries to Σ so that each component of $\Sigma\cap T$ is essential in T. Let $\Sigma_Y=\Sigma\cap\tau$ and $\Sigma_X=\Sigma-\tau$. Since K and L_+ have trivial linking number, $\Sigma\cap T$ is homologically trivial in T, hence it is possible to cap off the components of $\partial\Sigma_Y$ lying in T with annuli near T to get a Seifert surface Σ' which is disjoint from K. On the other hand, no component of Σ_X is a disk, since T is incompressible in $M_\pm$, so each component of Σ_X has non-positive Euler characteristic. Hence $\chi(\Sigma)\le\chi(\Sigma')\le\chi(S)$, by definition of S.

This verifies claim 1 in this case.

CASE 2. No such torus exists.

Then according to [Ga_2, Cor. 2.4] there is at most one way of filling in $\partial\eta(K)$ to get a manifold which is either reducible or in which S is not taut. This and 1.2 verify claim 1.

Next consider the connection between M_0 and $S^3 - \eta(L_0)$:

Isotope S so that it intersects D in an arc α joining the boundary components of $\eta(L_+) \cap D$. Define S_0 to be the surface obtained from S by deleting a neighborhood of α in S. Then $L_0 = \partial S_0$, i.e. S_0 is a Seifert surface for L_0. Equivalently, $S^3 - \eta(S_0)$ is obtained from $S^3 - \eta(S)$ by attaching a 2-handle to $\partial\eta(S)$ along the circle $\beta = \partial\eta(S) \cap D = \partial\eta(\alpha) \cap D$ (cf. Figure 3).

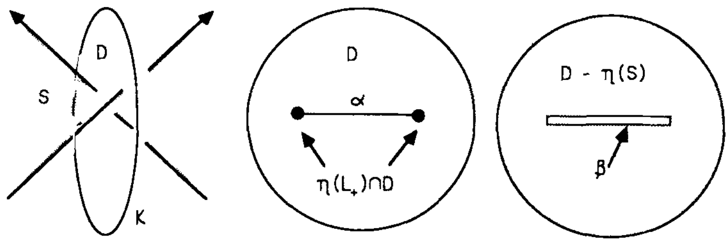

Figure 3

CLAIM 2. If M_0 is irreducible and S is taut in M_0 then $S^3 - \eta(L_0)$ is irreducible and S_0 is a taut Seifert surface for L_0.

Proof of claim 2. $D - \eta(\alpha)$ is an annulus with boundary components β and K, and the end of the annulus at K has framing 0. Hence β bounds in M_0 a disk D', the union of this annulus and a meridional disk of the solid torus filled in to produce M_0 from M. Attaching to $S^3 - \eta(S)$ a 2-handle along β is equivalent to deleting from $M_0 - \eta(S)$ a neighborhood of the disk D'. Now if M_0 is irreducible and S is taut in M_0 then the induced sutured manifold structure on $M_0 - \eta(S)$ is taut (cf. [Ga_1], [Sc]). D' is a disk in $M_0 - \eta(S)$ whose boundary crosses precisely two sutures so it is a product disk. Deleting product disks preserves tautness [Ga_1, 3.12], [Sc, 4.2]. Hence $(M_0 - \eta(S)) - \eta(D') = S^3 - \eta(S_0)$ is a taut sutured manifold. But this implies that $S^3 - \eta(L_0)$ is taut (i.e. irreducible) and that S_0 is taut [Ga_1, 3.6], [Sc, 3.3].

The theorem follows from Claims 1 and 2, together with the observation that a Seifert surface for L_- in M_- corresponds precisely to a Seifert surface for L_- in S^3. ||

2. Application to skein trees

Any link L can be reduced to unlinks by a series of "skein moves", that is, replacing L_+ (resp. L_-) with the pair of links L_- (resp. L_+) and L_0. To any such process (called a skein decomposition) we can associate a binary tree [Gi, §8], called a skein tree, with a node for each link and edges between a link and the pair of links obtained by a skein move.

2.1 DEFINITIONS. Let T be a skein tree for a link L. Then there is one end (the root) λ of T representing L; the other ends (called leaves) $\{\varepsilon_i\}$ represent unlinks. Define the *width* $\omega(\varepsilon_i)$ of a leaf to be the number of components in the unlink it represents, and its *height* $h(\varepsilon_i)$ to be the number of edges in a path in T from λ to ε_i. Define $h(T)$ to be $\max\{h(\varepsilon_i)\}$ and the height $h(L)$ to be $\min\{h(T) \mid T \text{ a skein tree for } L\}$.

Similarly the *weight* (h*eight* − w*idth*) $\mu(\varepsilon_i)$ of ε_i is $h(\varepsilon_i)-\omega(\varepsilon_i)$, $\mu(T)=\max\{\mu(\varepsilon_i \mid \varepsilon_i \text{ in } T\}$ and $\mu(L)=\min\{\mu(T) \mid T \text{ a skein tree for } L\}$.

2.2 *Remarks.* Note that always $\mu(\varepsilon_i)<h(\varepsilon_i)$, so $\mu(L)<h(L)$. Since any edge in a path in T from λ to ε_i represents an increase by at most one in the number of components of the link, $\mu(L)\geq -|L|$, where $|L|$ denotes the number of components of L.

2.3 PROPOSITION. $\mu(L)\geq -\chi(L)$.

Proof. The proof is by induction on the pair $(|L|+\mu(T),\ h(T))$, in lexicographic order, taken over all skein trees T for L. Note that both entries are non-negative, and if both are zero then L is an unlink. For an unlink $\mu(L)=-|L|=-\chi(L)$.

For the inductive step, let T be a tree for which $\mu(T)=\mu(L)$, and which, among all such trees, has minimum height. With no loss of generality assume $L=L_+$. The subtrees T_- and T_0 of T which are skein trees for L_- and L_0, each have height strictly less than T; also $\mu(T_-)+|L_-|<\mu(T)+|L|$ and $\mu(T_0)+|L_0|\leq\mu(T)+|L|$. By induction 2.3 applies to L_- and L_0 so $\mu(L_+)=\max\{\mu(L_0),\ \mu(L_-)\}+1\geq\max\{1-\chi(L_0),\ 1-\chi(L_-)\}$. Now consider the possibilities given by 1.4: Either

a) $-\chi(L_+)=-\chi(L_-)\geq 1-\chi(L_0)$ in which case $\mu(L_+)\geq 1-\chi(L_-)>-\chi(L_+)$
b) $-\chi(L_+)=1-\chi(L_0)>-\chi(L_-)$ in which case $\mu(L_+)\geq 1-\chi(L_0)=-\chi(L_+)$
c) $-\chi(L_-)=1-\chi(L_0)>-\chi(L_+)$ in which case $\mu(L_+)\geq 1-\chi(L_0)>-\chi(L_+)$. ||

2.4 *Remark.* For $d(L)$ the degree of the Conway polynomial, it is classical [To] that $d(L)\leq -\chi(L)+1$. An argument analogous to that of 2.3 applied to the recursion formula for the Conway polynomial shows $d(L)\leq h(L)$. Hence 2.2 and

2.3 complete the picture:

$$d(L) \leq -\chi(L) + 1 \leq \mu(L) + 1 \leq h(L).$$

3. Characterizing doubled knots

Consider the alternate picture of the Conway moves obtained by giving a half-twist to all the diagrams of Figure 1:

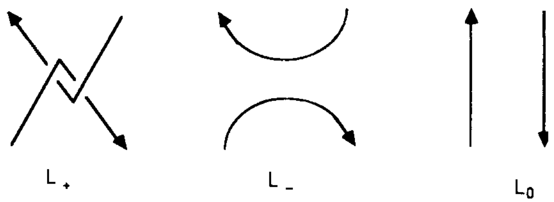

Figure 4

There is the following addendum to 1.4:

3.1 PROPOSITION. *When $\chi(L_+) = \chi(L_0) - 1 < \chi(L_-)$ there are taut Seifert surfaces S' for L_+ and S for L_0 which appear as in Figure 5 near the crossing, i.e. S' is obtained from S by plumbing on a Hopf band: (An analogous conclusion holds when $\chi(L_-) = \chi(L_0) - 1 < \chi(L_+)$.)*

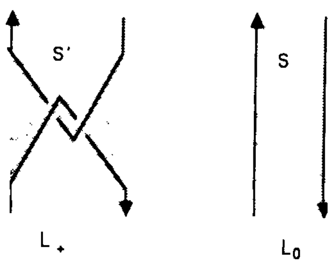

Figure 5

Proof. Consider the crossing circle K' for L_0 shown in Figure 6 below (note this is *not* a crossing circle for the crossing above). For the crossing change determined by K' note that L_- is obtained from L_0 by smoothing, so the roles of L_0 and L_- in the ensuing argument are the reverse of those in 1.4.

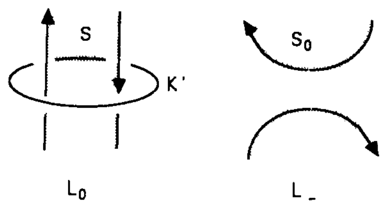

Figure 6

Let S be a Seifert surface for L_0 which is taut in $S^3 - K'$. Then it appears as shown in Figure 6.

CLAIM. S is a taut Seifert surface for L_0 in S^3.

Proof of claim. Claim 1 of 1.4 shows that S remains taut either in S^3 or in the manifold obtained by doing 0-surgery to K'. In the latter case, it follows from 1.4 Claim 2 that the surface S_0 for L_- obtained by altering S locally as in Figure 6 is a taut Seifert surface for L_- in S^3. Note $\chi(S_0) = \chi(S) + 1$.

Thus if S is not a taut Seifert surface for L_0 then $\chi(L_0) > \chi(S) = \chi(S_0) - 1 = \chi(L_-) - 1$. But our hypothesis includes $\chi(L_0) < \chi(L_-) + 1$. Thus $\chi(L_0) = \chi(L_-)$. But this is impossible, because $\chi(L)$ has the parity of $|L|$, and $|L_0|$ and $|L_-|$ have different parity. This verifies the claim.

Since S is a taut Seifert surface for L_0, $\chi(L_+) = \chi(L_0) - 1 = \chi(S) - 1$. Then the Seifert surface S' for L_+ obtained from that of S by plumbing on a Hopf band as shown in Figure 5 has $\chi(S') = \chi(S) - 1 = \chi(L_+)$ and so is taut. $\|$

3.2 COROLLARY. *A knot is a doubled knot if and only if its genus and unknotting number are both* 1.

Proof. It is obvious that a doubled knot has genus and unknotting number both 1.

So suppose K has genus and unknotting number both 1. Then with no loss of generality there is a crossing change for which $K = K_+$ and K_- is the unknot. Since $-1 = \chi(K_+) < \chi(K_-)$ it follows from 1.4 that $\chi(K_0) = \chi(K_+) + 1 = 0$. That is, an annulus is a Seifert surface for K_0 of maximal Euler characteristic. Then by 3.1 there is an annulus Seifert surface for K_0 whose core, when doubled, gives $K = K_+$. $\|$

(Remark: This has since been proven independently by Kobayashi [Ko], using similar methods.)

3.3 DEFINITION. A knot k is *totally knotted*, if, for any minimal genus Seifert surface of K with regular neighborhood $\eta(S)$ in S^3, $\partial\eta(S)$ is incompressible in $S^3 - \eta(S)$.

For an example, see [ST, Fig. 1.1].

3.4 COROLLARY. *No crossing change can lower the genus of a totally knotted knot.*

Proof. Suppose changing a crossing on the knot K reduced it's genus. With no loss take $K = K_+$ so $\chi(K_+) < \chi(K_-)$. Then for the taut Seifert surface S' for K in Figure 5, $\partial\eta(S')$ is clearly compressible in $S^3 - \eta(S')$, so K is not totally knotted. ||

P.P.A: We have shown that links arising from the Conway moves have related Euler characteristics. This relation is easily demonstrated for non-split alternating links by the simple iteration formula of the Alexander polynomial. Here we have demonstrated it for all links using the deep machinery of Gabai.

For any non-split prime alternating link L the Jones polynomial can be used to show that the minimal crossing number $c(L)$ is realized by an alternating projection without nugatory crossings [Mu]. It follows that if L_+, L_- and L_0 are all non-split prime alternating links and an alternating projection of L_+ is chosen for which L_0 is irreducible, then $c(L_+) = c(L_0) + 1 \geq c(L_-)$.

Is there a geometric invariant of arbitrary links, specializing to crossing number for alternating links, which satisfies a similar inequality?

REFERENCES

[Co] J. H. CONWAY, *An enumeration of knots and links*, in *Computational problems in abstract algebra*, Pergamon Press, Oxford, 1969, pp. 329–358.

[Cr] R. CROWELL, *Genus of alternating link types*, Ann. of Math. (2) *69* (1959), 258–275.

[Ga_1] D. GABAI, *Foliations and the topology of 3-manifolds*, J. Diff. Geom. *18* (1983) 445–503.

[Ga_2] D. GABAI, *Foliations and the topology of 3-manifolds, II*, J. Diff. Geom. *26* (1987) 461–478.

[Ga_3] D. GABAI, *Genus is superadditive under band-connected sum*, Topology *26* (1987) 209–210.

[Gi] C. GILLER, *A family of links and the Conway calculus*, T.A.M.S. *270* (1982) 75–109.

[Ko] T. KOBAYASHI, *Minimal genus Seifert surfaces for unknotting number 1 knots*, preprint.

[Mu] K. MURASUGI, *Jones polynomials and classical conjectures in knot theory*, Topology *26* (1987) 187–194.

[Sc] M. SCHARLEMANN, *Sutured manifolds and generalized Thurston norms*, to appear in J. Diff. Geom.

[ST] M. SCHARLEMANN and A. THOMPSON, *Unknotting number, genus and companion tori*, Math. Ann. *280* (1988) 191–205.

[Sh] H. SCHUBERT, *Knotten in vollringe*, Acta Math., *90* (1953) 131–286.

[Th] W. THURSTON, *A norm for the homology of 3-manifolds*, Mem. Amer. Math. Soc. #339, *59* (1986), 99–130.
[To] G. TORRES, *On the Alexander polynomial*, Annals of Math. (2), *57* (1953), 57–89.

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