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Complete minimal hypersurfaces in hyperbolic n-manifolds

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This paper is concerned with the existence and basic properties of minimal hypersurfaces in hyperbolic n-manifolds. A powerful and general method fo constructing minimal hypersurfaces in complete Riemannian manifolds N^n is given by geometric measure theory. For example, it is known that there exists an area-minimizing hypersurface, with small singular set, in any codimension one homology class of N. More recently, Schoen–Yau [SY] and Sachs–Uhlenbeck [SU] have constructed smooth branched minimal immersions of surfaces $f:\Sigma_g \to N$, area-minimizing in a conjugacy class of homomorphisms $\pi_1(\Sigma_g) \to \pi_1(N)$, provided $f_\#$ is injective on π_1. In case N is a 3-manifold, these surfaces are smooth immersions and in fact embeddings in case f is homotopic to an embedding (see [FHS]).

Restricting ourselves to hyperbolic manifolds (or more generally manifolds of negative curvature), we prove existence theorems for minimal hypersurfaces related to the above results, but distinct in several ways. The method, briefly stated, is as follows. Let N^n be a complete manifold of strictly negative sectional curvature $c_2 \leq K_N \leq c_1 < 0$ and let $\tilde{N}^n$ be its universal cover. Using geometric measure theory, we produce complete area-minimizing hypersurfaces in $\tilde{N}^n$, with prescribed behaviour at infinity; if Γ is a discrete group of isometries of $\tilde{N}^n$ whose action at infinity is sufficiently tame, we prove the existence of Γ-invariant area-minimizing hypersurfaces in $\tilde{N}^n$. Thus when Γ acts freely, one obtains complete immersed minimal hypersurfaces in N^n, provided $\Gamma \subset \pi_1(N^n)$.

In dimensions greater than three, these existence results are new; however, the generality of the result is unclear, since the action at infinity of discrete subgroups of isometries is not well understood in these dimensions.

In dimension three, these results partially overlap with those of [SY] and [SU]; in many respects, their results are much stronger. However, the lifts of least area incompressible surfaces to the universal cover are not in general area-minimizing, so that there is reason to believe the two methods may produce different surfaces in the quotient 3-manifolds. We show in sections §4 and §5 that this is in fact so and is related to the non-uniqueness of minimal surfaces in a given homotopy class. Prèvious examples of such non-uniqueness are due to Thurston and discussed in [SU]; see also the interesting work of Uhlenbeck [U] for related discussion.

From a somewhat different point of view, the results for Γ-invariant minimal surfaces complement the construction of Lawson [L] on complete minimal surfaces in S^3 and Nagano–Smyth [NS] on surfaces in $\mathbb{R}^3$ invariant under discrete groups of isometries. The construction of surfaces in H^3 and H^n is simpler and more complete than in the other space forms, due to the structure of H^n at infinity.

We now present our results and organization of the paper in more detail. The first section is of a preliminary nature, providing the necessary background in geometric measure theory and hyperbolic geometry. In §2, we prove a general existence theorem for complete area-minimizing hypersurfaces in H^n with prescribed behavior at infinity; for example, one may choose the boundary at infinity in H^3 to be an arbitrary Jordan curve (perhaps non-rectifiable). The constructions used in this theorem occur repeatedly throughout the paper. We also remark that a similar result holds for manifolds of negative curvature $c_2 \leq K_N \leq c_1 < 0$, although we do not give a proof here.

In §3, we discuss the action of discrete groups Γ of isometries on H^n ("Kleinian groups") and prove the existence of Γ-invariant area-minimizing hypersurfaces provided the limit set Λ_Γ is sufficiently tame; this class includes in particular the case of quasi-Fuchsian groups in all dimensions. This leads to a new method of constructing closed minimal hypersurfaces in manifolds of negative curvature in dimensions greater than three.

The last two sections are concerned with dimension 3, where a great deal more can be said. We first prove that for any torsion free quasi-Fuchsian group Γ acting on H^3, there is a complete smoothly embedded Γ-invariant minimal disc; when $\Gamma \subset \pi_1(M^3)$ for M^3 a hyperbolic 3-manifold, one obtains in this fashion stable incompressible minimal surfaces in M^3 in the given homotopy class. This duplicates a special case of general results of [SY] and [SU] in the case Γ has no cusps or torsion. (Our method encompasses this case also.) The method of proof relies on the work of Almgren–Simon [AS] on embedded solutions to the Plateau problem; based on this work, one may prove the existence of curves γ on $S^2(\infty)$ in H^3 such that any complete absolutely area-minimizing surface Σ asymptotic to γ has genus greater than a fixed g_0.

In §5, these results are used to prove certain non-uniqueness and non-finiteness results. First, we note that there are naturally occurring quasi-circles γ (limit sets of quasi-Fuchsian groups Γ) for which any Γ-invariant area minimizing surface asymptotic to γ at infinity has infinite genus. As corollaries of this, it is shown that such curves must bound an infinite number of complete smoothly embedded minimal surfaces at infinity. Second, such groups Γ have at least two distinct Γ-invariant minimal discs; thus one finds non-uniqueness of incompressible minimal surfaces in a given homotopy class, for a large class of quasi-Fuchsian manifolds of a given genus. Further, such manifolds provide examples where the least area incompressible surfaces of [SY] are not homologically area-minimizing. Finally, we establish a general finiteness result for compact area-minimizing surfaces in hyperbolic 3-manifolds, based on the method of Tomi [To]. These last results answer some questions of Uhlenbeck in [U].

This paper may be viewed as a sequel to [An], which we refer to occasionally. A portion of the results in this paper are based on part of the author's Ph.D. Thesis at U.C. Berkeley. I wish to thank my advisor, H. Blaine Lawson, for his unending guidance and encouragement. Also, I wish to thank Bill Dunbar for helpful conversations on 3-manifolds and orbifolds.

§1. Preliminaries

We discuss briefly in this section basic concepts from geometric measure theory and hyperbolic geometry used throughout the paper.

A natural class of objects in which the Plateau problem admits a solution with desired smoothness properties is the class of integral p-currents; these may be thought of as suitable generalizations of smooth oriented p-manifolds. Recall that given an oriented smooth Riemannian manifold N^n, the space of p-currents on N is defined to be the space of continuous linear functionals $(\Omega^p)^*$ on the space of p-forms of N, endowed with the weak topology. Clearly, there is a natural embedding of the set of smooth oriented p-manifolds S^p of finite volume in $(\Omega^p)^*$, given by

$$[S](\alpha)=\int_S \alpha \qquad \alpha\in\Omega^p(N).$$

More generally, a rectifiable p-current is a convergent sum of such currents $\mathcal{S}=\sum_{j=1}^{\infty} j[S_j]$, where $\{S_j\}_1^{\infty}$ is a collection of mutually disjoint oriented p-rectifiable sets and

$$\mathbf{M}(\mathcal{S})\equiv\sum_{j=1}^{\infty} j\mathcal{H}^p(S_j)<\infty;$$

here $\mathcal{H}^p$ is Hausdorff p-measure for the metric on N. There is a natural mass norm on the space $\mathcal{R}_p(N)$ of rectifiable p-currents, given as

$$\mathbf{M}(\mathcal{S})=\sup\{\mathcal{S}(w):M(w)\leq 1\},$$

where $M(w)=\sup_{x\in N}|w_x|$, $|w_x|=\sup\{w_x(\xi):\xi$ a unit simple p-vector$\}$. The support of $\mathcal{S}=\Sigma j[S_j]$ is defined as $\operatorname{supp}\mathcal{S}=\overline{\bigcup_{j=1}^{\infty}S_j}$; finally, the boundary operator on $(\Omega^p)^*$ is given by

$$(\partial\mathcal{S})(w)=\mathcal{S}(dw).$$

One now defines the space of *integral p-currents* $\mathcal{I}_p(N)$ on N to be the set of currents $\mathcal{S}$ such that $\mathcal{S}$ and $\partial\mathcal{S}$ are rectifiable. One of the deep theorems of geometric measure theory is the

COMPACTNESS THEOREM ([FF]). *Let* $K\subset N^n$ *be a compact set and* $C\in\mathbb{R}^+$. *Then the set*

$$\{\mathcal{S}\in\mathcal{I}_p(N):\operatorname{supp}\mathcal{S}\subset K,\ \mathbf{M}(\mathcal{S})+\mathbf{M}(\partial\mathcal{S})\le C\}$$

is compact in the weak topology.

It follows easily from the definition that the mass norm is lower semicontinuous in the weak topology; this, together with the compactness theorem, allows one to solve the Plateau problem in the category of integral currents. Thus, if B^{p-1} is a $(p-1)$ manifold (or integral $(p-1)$-current) such that $B^{p-1}=\partial\mathcal{S}$, for some $\mathcal{S}\in\mathcal{I}_p(N)$, then there is an $\mathcal{S}_0\in\mathcal{I}_p(N)$ satisfying $\partial\mathcal{S}_0=B$ and

$$\mathbf{M}(\mathcal{S}_0)\le\mathbf{M}(\mathcal{S}),\quad\forall\mathcal{S}\text{ s.t. }\ \partial\mathcal{S}=B.$$

One says that $\mathcal{S}_0$ is absolutely area minimizing for the boundary B. We will often work with currents of non-compact support. One defines the group $\mathcal{I}_p^{\mathrm{loc}}$ of locally integral p-currents as the currents $\mathcal{S}$ such that for all $x\in N$, there is a $\tau\in\mathcal{I}_p(N)$ of compact support such that $x\notin\operatorname{supp}(\mathcal{S}-\tau)$. We then say $\mathcal{S}\in\mathcal{I}_p^{\mathrm{loc}}(N)$ is absolutely area-minimizing if, for all compact sets $K\subset N$, one has

$$\mathbf{M}(\mathcal{S}\llcorner K)\le\mathbf{M}(\tau),$$

for any $\tau\in\mathcal{I}_p(N)$ with $\partial(\mathcal{S}\llcorner K)=\partial\tau$.

Next, we briefly discuss the regularity properties of area-minimizing currents. A point $a\in\operatorname{supp}(\mathcal{S})\setminus\operatorname{supp}(\partial\mathcal{S})$ is regular if there is a neighborhood W of p such that $W\cap\operatorname{supp}(S)$ is a connected p-dimensional C^2-submanifold of N^n. If a is regular, then the manifold $B=W\cap\operatorname{supp}\mathcal{S}$ is oriented by $\vec{\mathcal{S}}|_B$ and $\mathcal{S}$ is given by integration over B, up to multiplicity. A fundamental theorem in the subject is the

REGULARITY THEOREM (c.f. [F]). *Let $\mathcal{S}$ be an absolutely area-minimizing integral $(n-1)$-current in $U \subset N^n$. Then the interior singular set Z of $\mathcal{S}$ has codimension ≥ 8, i.e. $\mathcal{H}^q(Z)=0$, for all $q>n-8$.*

In particular, if $n \leq 7$, then any area-minimizing $(n-1)$-current $\mathcal{S}$ is the standard orientation current over a smoothly embedded hypersurface.

For further information and details regarding geometric measure theory, we refer to the basic references [Al], [F].

Throughout much of this paper, the ambient space N^n will be hyperbolic space H^n of constant curvature -1, or a quotient of H^n by a discrete group of isometries. Usually we identify H^n with the unit ball $B^n(1)$ of Euclidean space via the Poincaré model. In this model, the unit sphere represents the *sphere at infinity* $S^{n-1}(\infty)$ of H^n and provides a natural conformal compactification of H^n; every point $p \in S^{n-1}(\infty)$ represents an asymptote class of geodesics in H^n. Analogously, we define the *asymptotic boundary* $\mathcal{A}$ of a locally integral p-current Σ in H^n by

$$\mathcal{A} = \overline{\operatorname{supp} \Sigma} \cap S^{n-1}(\infty),$$

where $\overline{}$ denotes closure in the Euclidean topology.

Recall that in the Poincaré model, geodesics are arcs of circles intersecting the sphere at infinity orthogonally; similarly, totally geodesic k-planes are domains on Euclidean k-spheres having orthogonal intersection with $S^{n-1}(\infty)$. One defines the *convex hull* $\mathcal{C}(S)$ of a set S in $\overline{H^n}$ as the intersection of all half-spaces containing S; a half-space is a component of $\overline{H^n} - P$, where P is a totally geodesic hyperplane.

Finally, we use standard notation and results from Riemannian geometry; geodesic balls of radius r are denoted by B_r^p or $B^p(r)$, where p is dimension.

§2. The boundary-value problem at infinity

In this section, we will prove the existence of complete area-minimizing hypersurfaces in H^n asymptotic to a rather general class of boundaries in $S^{n-1}(\infty)$; such boundaries arise naturally as limit sets of discrete groups acting on H^n.

Given compact sets A, B in a metric space (X, d), recall that the Hausdorff distance between A and B is given by

$$\rho(A, B) = \max(\rho_A(B), \rho_B(A)),$$

where $\rho_A(B) = \sup\{d(x, B) : x \in A\}$.

We now state the main existence theorem of this section; both the theorem and its proof will be used often in the sequel.

THEOREM 2.1. *Let $S \subset S^{n-1}(\infty)$ be a closed set such that $S^{n-1}(\infty)\setminus S$ has exactly 2 connected components. Suppose there are $(n-2)$-dimensional smooth, closed, connected manifolds $M_j \subset S^{n-1}(\infty)$ such that*

$$\lim_{j\to\infty} \rho(M_j, S) = 0.$$

Then there exists an absolutely area-minimizing integral $(n-1)$-current Σ asymptotic to S at infinity.

Proof. The outline of the proof resembles that of Theorem 4 of [An], where an analogous theorem was proved for the case of S a k-manifold in $S^{n-1}(\infty)$. We choose $O \in H^n$ as an origin and view $M_j \subset S^{n-1}(j)$ via geodesic projection from O.

Let Σ_j be an integral $(n-1)$-current representing a solution to the Plateau problem with boundary M_j; thus we have $\partial\Sigma_j = M_j$ and

$$\mathbf{M}(\Sigma_j) \leq \mathbf{M}(\mathcal{S}),$$

for $\mathcal{S}$ any integral $(n-1)$-current with $\partial\mathcal{S} = M_j$. The proof is based on establishing the estimates

$$c_r \leq \mathbf{M}(\Sigma_j \llcorner B_r) \leq C_r \tag{2.2}$$

on the mass of Σ_j inside the geodesic r-ball B_r centered at O.

[A] *Existence of C_r*

We begin with

LEMMA 2.3. *Let Σ be an area-minimizing $(n-1)$ current in $B^n(s)$ with $\partial\Sigma = M$ a connected manifold in $S^{n-1}(s)$. Then* supp Σ *is connected and disconnects $B^n(s)$ into two components $\Omega^{\pm}$.*

Proof. We recall that supp Σ is an analytic submanifold outside a closed subset Z of Hausdorff dimension at most $n-8$. The work of Hardt–Simon [HS] on boundary regularity shows that $Z \cap \operatorname{supp} \partial\Sigma = \varnothing$. Thus the boundary of each component of supp Σ is M, and so it follows that supp Σ is connected. Since Z is of high codimension, it follows that $\pi_1(B^n(s) - Z) = 0$; see, e.g., [HP: Theorem 4.1b].

Suppose $B^n(s)\setminus \operatorname{supp}\Sigma$ were connected; choose a regular point $x\in\operatorname{supp}\Sigma$ and L a transverse curve so that $L\cap\operatorname{supp}\Sigma = x$. We may join the endpoints ∂L in $B^n(s)\setminus\operatorname{supp}\Sigma$ and obtain an embedding $f: S^1\to B^n(s)$ such that $f(S^1)\cap\operatorname{supp}\Sigma = x$. It follows that f extends to a map $f: D^2\to B^n(s)$; assume w.l.o.g. that f is transverse to $\operatorname{supp}(\Sigma - Z)$. Thus $f^{-1}(\operatorname{supp}(\Sigma-Z))$ is a 1-manifold with single boundary component x, a contradiction.

To see there are at most two components of $B^n(s)\setminus\operatorname{supp}\Sigma$, let x, L be as above and for any $y\in B^n(s)\setminus\operatorname{supp}\Sigma$, let τ_y be a shortest geodesic from y to $\operatorname{supp}\Sigma$. If p_y is the endpoint of τ_y, then p is regular and one may join p and x by a path γ in the regular set of Σ. By sliding γ in the direction normal to $\operatorname{supp}\Sigma$, one may join y to one endpoint of ∂L by a path in $B^n(s)\setminus\operatorname{supp}\Sigma$. ■

We apply Lemma 2.3 to the current Σ_j in $B^n(j)$ and see that $\operatorname{supp}\Sigma_j$ separates $B^n(j)$ into 2 components. The current Σ_j is of multiplicity 1, so that Σ_j represents a boundary of least area in $B^n(j)$; in other words, letting $B^n(j)\setminus\operatorname{supp}\Sigma_j = \Omega_j^+\cup\Omega_j^-$, we have $\Sigma_j = \partial\Omega_j^+$ and

$$\operatorname{vol}(\partial\Omega_j^+\cap K)\le\operatorname{vol}(\partial K\cap\Omega_j^+),$$

for any compact $K\subset B^n(j)$. Choosing $K = B^n(r)$, $r<j$, it follows that

$$\mathbf{M}(\Sigma_j \llcorner B_r)\le\tfrac{1}{2}\operatorname{vol} S(r), \tag{2.4}$$

for all j. This gives the upper bound $C_r = \frac{1}{2}\operatorname{vol} S(r)$.

[B] *Existence of* c_r

Recall that given a set $T\subset\overline{H^n}$ one may define the convex hull $\mathscr{C}(T)$ of T as the smallest geodesically convex set containing T. It is not difficult to prove that if Σ is a stationary p-current in H^n, then

$$\operatorname{supp}\Sigma\subset\mathscr{C}(\operatorname{supp}\partial\Sigma); \tag{2.5}$$

see e.g. [An], [AS]. We note also the useful fact that for $T\subset S^{n-1}(\infty)$

$$\mathscr{C}(T)\cap S^{n-1}(\infty) = \bar{T}. \tag{2.6}$$

Now choose points x, y in different components of $S^{n-1}(\infty)\setminus S$ and let γ be the unique geodesic asymptotic to x and y. For j sufficiently large, it is clear that the intersection $\gamma\cap S^{n-1}(j)$ consists of two points x_j, y_j with $x_j\to x$ and $y_j\to y$ as $j\to\infty$ and x_j, y_j lie in distinct components of $S^{n-1}(j)\setminus M_j$. Since, by Lemma 2.3

again, supp Σ_j separates $B^n(j)$ into two components, it follows that

$$\operatorname{supp} \Sigma_j \cap \gamma \neq \varnothing,$$

for all j sufficiently large. Since supp $\Sigma \subset \mathscr{C}(M_j)$ and $\mathscr{C}(M_j)$ converges to $\mathscr{C}(S)$ as $j \to \infty$, we see that the sequence

$$\{\operatorname{supp} \Sigma_j \cap \gamma\} \subset K,$$

for some compact set $K \subset \mathring{H}^n$. In particular, it follows that there is a $p \in \gamma$ and $R > 0$ such that

$$\operatorname{dist}(p, \operatorname{supp} \Sigma_j) < R, \quad \text{for all } j.$$

Thus, supp Σ_j intersects a fixed ball of radius R in H^n, for each j. The existence of the lower bound c_r now follows from standard monotonicity estimates on the mass of stationary currents in geodesic balls, see e.g., [An], [L_2].

The proof of Theorem 2.1 is now straightforward. The estimate (2.2) together with the compactness theorem for integral currents show that the sequence $\{\Sigma_j \llcorner B_i\}_{j=1}^{\infty}$ has a weakly convergent subsequence for each fixed i. Choosing such for each i and taking the diagonal subsequence, we find there is a subsequence $\{\Sigma_{j'}\}$ of $\{\Sigma_j\}$ and an integral $(n-1)$-current Σ such that

$$\Sigma_{j'} \to \Sigma$$

on any compact set, in the weak topology. The current Σ is absolutely area minimizing, being a limit of area-minimizing currents, and is easily seen to have asymptotic boundary S, using (2.5) and (2.6) again. ■

Remark 1. We note that these currents Σ are smoothly embedded complete submanifolds in case $n \leq 7$ and have singular set Z of Hausdorff codimension at least 8 in higher dimensions. As examples of boundaries S to which the theorem applies, we mention the following.

EXAMPLE 1. In dimension 3, we may choose S to be an arbitrary Jordan curve (not necessarily rectifiable) on $S^2(\infty)$. This follows from the fact that any Jordan curve may be approximated, in the Hausdorff distance, by inscribed polygons.

EXAMPLE 2. In higher dimensions, let S be the image of the equator $S^{n-2} \subset S^{n-1}$ under a homeomorphism h of S^{n-1}. Then S satisfies the hypothesis of the theorem. In fact, for any $\varepsilon>0$, let $T=\{x: d(x, S^{n-2})<\varepsilon\}$ be the ε-tubular neighborhood of the equator S^{n-2}. Define

$$f: T\to\mathbb{R} \quad\text{by}\quad f(x)=\frac{1}{\varepsilon^2-d(x, S^{n-2})^2}.$$

Then $f_h=f\circ h^{-1}: h(T)\to\mathbb{R}$ is a proper exhaustion function of $h(T)$. We may choose a uniform approximation to f_h by a C^∞ function $\tilde{f}_h$ and, for any regular value q, define

$$M_q=\tilde{f}_h^{-1}(q).$$

Thus, $\rho(S, M_q)<\varepsilon$, as desired.

Remark 2. It is unknown whether a result analogous to Theorem 2.1 holds in higher codimension; the estimation (2.4) is no longer valid.

Remark 3. We note that a result analogous to Theorem 2.1 holds in complete manifolds of curvature $c_2\leq K_N\leq c_1<0$; the proof will appear elsewhere.

§3. Kleinian groups and invariant solutions

In this section, we will study the existence of area-minimizing hypersurfaces invariant under a discrete group of isometries acting on H^n.

Let Γ be a discrete subgroup of $O^+(n, 1)$, the group of orientation-preserving isometries of H^n. The limit set Λ_Γ of Γ is the set of accumulation points of an orbit Γ_x, $x\in H^n$ on $S^{n-1}(\infty)$; this turns out to be independent of the choice of $x\in H^n$. Λ_Γ is a closed set, minimal under the conformal action of Γ on $S^{n-1}(\infty)$; we have

$$S^{n-1}(\infty)=\Omega_\Gamma\cup\Lambda_\Gamma,$$

where Ω_Γ is the 'domain of discontinuity' of Γ; Γ acts properly discontinuously on Ω_Γ. Ω_Γ may be empty, or have one, two or infinitely many components. We will call Γ *quasi-Fuchsian* if Ω_Γ has exactly two components. In case Γ acts freely (Γ is torsion free), we see that Γ is quasi-Fuchsian if and only if the quotient manifold $\mathscr{C}^n$

$$\mathscr{C}^n=\frac{\mathscr{C}(\Lambda_\Gamma)}{\Gamma}\subset M^n=\frac{H^n\cup\Omega_\Gamma}{\Gamma}$$

is a 'convex' hyperbolic manifold with two boundary components strictly contained in $\mathring{M}^n$; we note that

$$\pi_1(M^n) \cong \pi_1(\partial M).$$

[A manifold N is convex if any path in N is homotopic to a geodesic in N, relative to the endpoints.] In H^3, Maskit [M] has shown that if Γ is finitely generated and torsion free, then Γ is quasi-Fuchsian if and only if Γ is a quasi-conformal deformation of a Fuchsian group, i.e. a discrete subgroup of Isom (H^2); in this case, Λ_Γ is the image of a circle S^1 under a quasi-conformal homeomorphism of S^2.

Remark. In dimension 3, if Γ is a surface group, i.e. $\Gamma \cong \pi_1(\Sigma)$ where Γ is a (punctured) surface, Ω_Γ has either 0, 1 or 2 components; it is conjectured that the 'degenerate' groups with Ω_Γ having 0 or 1 component are suitable limits of quasi-Fuchsian groups. Thus quasi-Fuchsian groups play a central role in dimension 3.

The main result of this section is the following.

THEOREM 3.1. *Let Γ be a quasi-Fuchsian group acting on H^n. Then there exist complete Γ-invariant absolutely area-minimizing $(n-1)$-currents Σ_Γ in H^n.*

Proof. Let $\mathscr{C}(\Lambda_\Gamma)$ be the convex hull of Λ_Γ and let M_j be a sequence of smooth manifolds in the interior of $\mathscr{C}(\Lambda_\Gamma)$ eventually lying outside any compact set in H^n. We may apply Theorem 2.1, since $S^{n-1}(\infty)\setminus\Lambda_\Gamma$ has exactly two components; let Σ be a complete area-minimizing hypersurface in H^n asymptotic to Λ_Γ. We may assume that supp Σ is connected, since we may replace it by a component of supp Σ. Then, by Lemma 2.3, $\overline{H^n\setminus \operatorname{supp}\Sigma}$ has two components $\Omega^\pm$ such that $\Omega^\pm \cap S^{n-1}(\infty)$ are the two components of $S^{n-1}(\infty)\setminus\Lambda_\Gamma$; we note these latter are Γ-invariant. Consider the currents $g\Sigma$ defined by

$$(g\Sigma)(\omega) = \Sigma(g^*\omega), \quad \text{for} \quad g \in \Gamma.$$

Each $g\Sigma$ is a minimizing integral $(n-1)$-current; in fact $g\Sigma$ is a boundary of least area;

$$\partial(g\Omega^+) = g\Sigma,$$

where $g\Omega^\pm$ are the components of $H^n\setminus \operatorname{supp}(g\Sigma)$. Consider

$$\Omega_1 = \bigcap_{g\in\Gamma} g\Omega^+.$$

It is clear that Ω_1 is Γ-invariant and so it follows that $\partial\Omega_1$ is also Γ-invariant. If $\partial\Omega_1$ is a boundary of least area, we are done. If not, then we proceed to solve the Plateau problem in Ω_1 as follows. Let B_i be a sequence of smooth connected $(n-2)$ manifolds in $\Omega_1 \cap \mathscr{C}(\Lambda_\Gamma)$, eventually lying outside any compact set $K \subset H^n$. Let $\mathscr{S}_i$ be a solution to the Plateau problem with boundary B_i. We now claim that $\mathscr{S}_i \subset \Omega_1$, for all i. To see this, one has $B_i \subset \Omega_1$, so that in particular $B_i \subset g\Omega^+$, for any $g \in \Gamma$. Since $g\Omega^+$ has a boundary of least area, it follows that $\mathscr{S}_i \subset g\Omega^+$, for any g; this gives the claim. Thus there is a sequence of boundaries of least area $\{\mathscr{S}_i\}$ in Ω_1, with $\{\partial\mathscr{S}_i\}$ converging to Λ_Γ in the sense of Hausdorff distance. Apply the proof of Theorem 2.1 to $\{\mathscr{S}_i\}$; it follows there is a convergent subsequence, call it $\{\mathscr{S}_i\}$ again, such that

$$\mathscr{S}_i \to \mathscr{S}^1 \text{ weakly,}$$

with $\operatorname{supp} \mathscr{S}^1 \subset \Omega_1$. Now $\mathscr{S}^1$ is a boundary of least area with support 'above' all $g\Sigma$, $g \in \Gamma$. In other words, one may define an ordering $<$ on the set of complete minimal currents asymptotic to Λ_Γ by

$$\Sigma_1 < \Sigma_2 \Leftrightarrow \Omega_1^+ \supset \Omega_2^+,$$

where $\Omega_i^+ \cap S^{n-1}(\infty)$ is the + component of $S^{n-1}(\infty) \setminus \Lambda_\Gamma$. We thus have

$$g\Sigma < \mathscr{S}^1, \quad \text{for all} \quad g \in \Gamma.$$

Now repeat on $\mathscr{S}^1$ the process above. If $\mathscr{S}^1$ is not Γ-invariant, let

$$\Omega_2 = \bigcap_{g \in \Gamma} g(\Omega_1)^+$$

where $(\Omega_1)^+$ gives the positive component of $S^{n-1}(\infty) \setminus \Lambda_\Gamma$. Continuing in this fashion, we produce a sequence of boundaries of least area $\mathscr{S}^i$ such that

$$\Sigma = \mathscr{S}^0 < \mathscr{S}^1 < \cdots < \mathscr{S}^k < \cdots.$$

and also

$$g\mathscr{S}^i < \mathscr{S}^{i+1},$$

for all $g \in \Gamma$, and for all i. Each $\mathscr{S}^i$ is a complete area-minimizing $(n-1)$ current asymptotic to Λ_Γ satisfying

$$\operatorname{supp} \mathscr{S}^i \subset \mathscr{C}(\Lambda_\Gamma).$$

One may again apply the proof of Theorem 2.1 to obtain a convergent subsequence $\{\mathcal{S}^{k'}\}\subset\{\mathcal{S}^k\}$ with

$$\mathcal{S}^{k'}\to\Sigma_\Gamma \quad \text{as} \quad k\to\infty, \quad \text{weakly.}$$

It is now clear that Σ_Γ is a complete area-minimizing integral $(n-1)$-current asymptotic to Λ_Γ. To see that Σ_Γ is Γ-invariant, note that $\Sigma_\Gamma=\lim_{k\to\infty}\mathcal{S}^k$ so that $g\Sigma_\Gamma=\lim_{k\to\infty}g\mathcal{S}^k$; by construction, $g\mathcal{S}^k<\mathcal{S}^{k+1}$ so that

$$g\Sigma_\Gamma\leq\Sigma_\Gamma,$$

for any $g\in\Gamma$. Replacing g by g^{-1}, it follows that $g\Sigma_\Gamma=\Sigma_\Gamma$, for all $g\in\Gamma$. ∎

We now discuss some applications to closed minimal hypersurfaces in hyperbolic manifolds. The theory is most complete for surfaces in 3-manifolds, so we begin with this.

Let Γ be an arbitrary quasi-Fuchsian group (not necessarily finitely generated). The orbit space

$$M^3=H^3/\Gamma$$

is a 3-manifold with boundary equal to Ω_Γ/Γ; note that $M^3\approx(\Omega_\Gamma^+/\Gamma)\times I$, where $\Omega_\Gamma^\pm$ are the components of Ω_Γ. Conversely, recall the simultaneous uniformization theorem of Bers [B] which states that, given any pair of homeomorphic Riemann surfaces Σ_1, Σ_2 (possibly having punctures and branch points), there is a quasi-Fuchsian group Γ such that $\Omega_\Gamma/\Gamma=\Sigma_1\cup\Sigma_2$; Γ is unique up to conjugation in $\mathrm{PSL}(2,\mathbb{C})$. In case that Γ acts freely, M^3, and its boundary Ω_Γ/Γ, inherit complete hyperbolic metrics. On the other hand, there are, for example, groups Γ with $M^3/\Gamma\approx S^2\times I$ topologically; clearly Γ does not act freely, since $S^2\times I$ does not admit any complete hyperbolic metric.

The following is a simple consequence of Theorem 3.1.

COROLLARY 3.2. *Let $M^3=H^3/\Gamma$ be a quasi-Fuchsian 3-manifold. Then M^3 contains a branched minimal embedding of a Riemann surface S satisfying*

$$\pi_1(S)\to\pi_1(M^3)\to 0.$$

In case Γ acts freely, S is a smoothly embedded complete stable minimal surface *with*

$$0 \to \pi_1(\tilde{S}) \to \pi_1(S) \to \Gamma \to 0,$$

where $\tilde{S}$ *is the* Γ*-covering of* S *in* H^3.

Proof. The first statement follows from Theorem 3.1 by passing to the orbit space H^3/Γ; in this context, minimality means vanishing of the mean curvature away from the branch points of S. The second statement follows similarly; it a consequence of [F-CS, Theorem 1] that the embedded surface S is stable when Γ acts freely.

Remark 1. It is not necessarily true that $\pi_1(\tilde{S}) = 0$; in §4 and §5, we will prove the existence of smoothly embedded minimal surfaces S in certain $M^3 = H^3/\Gamma$ with $\pi_1(\tilde{S}) \neq 0$; in particular, these surfaces are not incompressible. On the other hand, in §4 (see Theorem 4.4), we will also show the existence of embedded minimal surfaces S in M^3 with $\pi_1(S) = \pi_1(M^3) = \Gamma$, for every torsion-free quasi-Fuchsian group Γ.

Remark 2. In case Γ acts freely and represents a compact surface, $\Gamma \cong \pi_1(\Sigma_g)$, Schoen–Yau [SY] and Sachs–Uhlenbeck [SU] have obtained very strong results on the existence of incompressible minimal surfaces in Riemannian manifolds. It is clear however that in general, the surfaces produced above are inequivalent; in particular, the lifts of incompressible minimal surfaces in compact 3 manifolds to H^3 are not necessarily area-minimizing. Further, our constructions apply to surfaces having cusps and branch points, as well as infinitely generated π_1.

For higher dimensions, one obtains the following.

COROLLARY 3.3. *Let* N^n *be a compact convex hyperbolic* n*-manifold with exactly two boundary components. Then there is a closed minimal hypersurface (integral* $(n-1)$*-current)* Σ *satisfying*

$$0 \to \pi_1(\text{supp}\, \Sigma_\Gamma) \to \pi_1(\text{supp}\, \Sigma) \to \pi_1(N^n) \to 0$$

where Σ_Γ *is the* Γ*-lift of* Σ *to* H^n*. In case* $n \leq 7$, Σ *is a smoothly embedded stable submanifold.*

Proof. Theorem 3.1 gives the existence of complete area-minimizing integral $(n-1)$-currents Σ_Γ invariant under the action of $\Gamma = \pi_1(N^n)$ on H^n. Passing to the orbit space gives the desired current Σ; stability follows as in Corollary 3.2.

Remark 3. Corollary 3.3 proves the existence of closed stable minimal hypersurfaces Σ in compact hyperbolic n-manifolds N^n which are covered by a hyperbolic manifold $\tilde{N}^n$ having two ends and compact convex hull; furthermore, we have

$$\pi_1(\operatorname{supp} \Sigma) \to \pi_1(\tilde{N}) \subset \pi_1(N).$$

A similar result holds for N^n of pinched negative curvature. However, the class of manifolds satisfying the above conditions is not well understood.

§4. Minimal surfaces in hyperbolic 3-manifolds

In this section, we will work exclusively with hyperbolic 3-manifolds. Almgren–Simon in [AS] have proved the existence of embedded minimal discs in Riemannian 3-manifolds provided the boundary is constrained to lie on a convex set. More precisely, given a C^2 Jordan curve $\gamma \subset \partial C$, for C a convex set, consider the space $\mathcal{M}_0$ of smooth embeddings

$$f: D^2 \to M^3 \quad \text{such that} \quad f|_{S^1} = \gamma.$$

They show that the area functional achieves a minimum on $\mathcal{M}_0$ giving the existence of an embedded minimal disc $f_0(D^2)$ in M^3 with boundary γ. The work of Meeks–Yau [MY] actually shows that $f_0(D^2)$ realizes the minimum area over all branched immersions $D^2 \to M^3$; however, we shall not be using their techniques here.

We begin by using the result and method of proof of Almgren-Simon to construct complete embedded minimal discs in H^3.

THEOREM 4.1. *Let γ be a Jordan curve on $S^2(\infty)$. Then there exists a complete embedded minimal surface D in H^3 of the topological type of the disc, asymptotic to γ. Further, D minimizes area in the category of embedded discs.*

Proof. Let $\gamma_i \subset S^2(i)$ be a sequence of C^2-Jordan curves in H^3 whose limit is γ, in the sense of Hausdorff distance (see §1, Example 1). Then the work of [AS] gives existence of smoothly embedded minimal discs D_i with $\partial D_i = \gamma_i$. We apply the proof of Theorem 2.1 to $\{D_i\}$ (in place of $\{\Sigma_j\}$ there). The estimate

$$\mathbf{M}(D_i \llcorner B_r) \leq \tfrac{1}{2} \operatorname{vol} S(r), \tag{4.2}$$

will follow easily from the following lemma.

LEMMA 4.2. *Let D^2 be a minimally embedded disc with $\partial D^2 \subset S^2(r)$. Then $D^2 \cap B^3(s)$ is a disjoint union of discs, for almost all $s \leq r$.*

Proof. Let $j: D^2 \to H^3$ be the inclusion and let s be a regular value of $d \circ j: D^2 \to \mathbb{R}$, where d is the distance function from 0. Then $j^{-1}(S^2(s))$ is a disjoint collection of circles $\{S_\alpha\}$ in $\mathring{D}^2$. Consider $j^{-1}(B(r)\setminus\mathring{B}(s)) \subset D^2$: this is a compact set K in D^2 with boundary equal to $\partial D^2 \cup \bigcup S_\alpha$. It follows easily from the convex hull property (2.5) that K is connected; thus none of the curves S_α are nested and so the complement $j^{-1}(B(s))$ is a union of discs. ■

Returning to the proof of Theorem 4.1, we now have by Lemma 4.2 that $D_i \llcorner B_r$ is a finite collection of discs. The area-minimizing property of D_i among embedded discs then gives (4.2) immediately. We may now copy the proof of Theorem 2.1 for $\{D_i\}$ and produce a stationary integral 2-current $\tilde{D}$ such that a subsequence converges

$$D_{i'} \to \tilde{D} \quad \text{weakly on compact sets.}$$

One sees that $\tilde{D}$ is a complete stationary integral 2-current asymptotic to γ and area minimizing among comparision discs in the following sense: if $\gamma \subset \operatorname{supp} \tilde{D}$ is a smooth Jordan curve wtih $\partial T = \gamma$, where T is a stationary 2-current and $\operatorname{supp} T \subset \operatorname{supp} \tilde{D}$, then

$$\mathbf{M}(T) \leq \operatorname{vol}(V),$$

where V is any embedded disc in H^3, $\partial V = \gamma$.

Our aim is to prove that $\tilde{D}$ is in fact a smoothly embedded disc. Thus, consider $x \in \operatorname{supp} \tilde{D}$. The slices $\partial(\tilde{D} \llcorner B_x(\varepsilon))$ are closed rectifiable 1-currents, for almost all $\varepsilon > 0$. Similarly, by means of Sard's theorem, the restriction $D_i \cap B_x(\varepsilon)$ is a union of smoothly embedded submanifolds with smooth Jordan curves as boundary, for almost all $\varepsilon > 0$. By Lemma 4.2, each component of $D_i \cap B_x(\varepsilon)$ is in fact a smooth embedded disc.

We claim there is a $\delta > 0$, with perhaps $\delta \ll \varepsilon$, such that at most four components of $D_i \cap B_x(\varepsilon)$ intersect $B_x(\delta)$, for all i. To see this, let C_i^j, $j = 1, 2, \ldots, K_i$ denote the components of $D_i \cap B_x(\varepsilon)$ intersecting $B_x(\delta)$; by a simple area comparison, we have

$$\sum_{j=1}^{K_i} \mathbf{M}(C_i^j) < \mathbf{M}(\partial(B_x(\varepsilon))).$$

Further, by the local monotonicity of stationary currents (see [An], [L_2]), it follows that

$$\mathbf{M}(C_i^j) \geq 1 \cdot \mathrm{vol}\,(B^2(\varepsilon-\delta)), \quad \text{for each } i, j.$$

Thus we find

$$K_i \cdot \mathrm{vol}\, B^2(\varepsilon-\delta) < \mathbf{M}(\partial(B_x(\varepsilon))) \approx 4\pi\varepsilon^2,$$

for ε sufficiently small. Since $\mathrm{vol}\, B^2(\varepsilon-\delta) \approx \pi(\varepsilon-\delta)^2$, we see that

$$K_i \leq 4, \quad \text{for any } i.$$

Thus the limiting current $\tilde{D} \llcorner B_x(\delta)$ is the limit of regular currents $D_i \llcorner B_x(\delta)$ having at most four components, each a smoothly embedded disc. By relabelling and passing to a subsequence, we may assume the sequence of components $\{C_i^j\}_{i=1}^{\infty}$ converges weakly to a current W^j

$$\tilde{D} \llcorner B_x(\delta) = \sum_j W^j.$$

The regularity of the current $\tilde{D}$ follows from the methods of Almgren–Simon. In fact, let $\tilde{T}_x$ be the (varifold) tangent cone to $\tilde{D}$ at x: it is known that $\tilde{T}_x$ either has support contained in a plane or is locally a union of half-discs with common diameter L (see [AS, Corollary 2]). Let T_x^j denote the varifold tangent cones to W^j at x; we then have

$$\sum_j T_x^j = \tilde{T}_x.$$

For fixed j, we choose a sequence $r_k \to \infty$ so that the expansions $N_k \equiv \mu_{r_k}(C_k^j)$ converge to the varifold tangent

$$\mu_{r_k}(C_k^j) \to T_x^j \text{ weakly}, \quad \text{as} \quad k \to \infty;$$

here μ_{r_k} denotes geodesic dilation of the ambient space H^3 centered at x_k, $x_k \to x$ as $k \to \infty$. Now the interior regularity results, Theorems 2 and 3 of [AS], applied to the sequence $\{N_k\}$, show that T_x^j is a plane (with multiplicity 1), for each j. Since the components C_k^j for fixed k are disjoint, it follows that the tangent planes T_x^j are identical. We apply the basic regularity theorem of Allard [Al, §8] to

$(\tilde{D}, \tilde{T}_x)$ and find that $\tilde{D}$ is a regular varifold in a neighborhood of x:

$$\tilde{D} \llcorner B(x, \delta') = \rho \cdot [S],$$

for some integer $\rho \leq 4$, where S is an analytic, embedded minimal surface in $\overline{B(x, \delta')}$. Clearly, ρ is independent of x and we now see that $\tilde{D}$ is a regularly embedded minimal surface in H^3, asymptotic to γ.

The Allard regularity result also shows that the convergence $D_i \to D$ is smooth. Since for almost all r, $D_i \cap B(r)$ is a disjoint union of discs, it follows that $\tilde{D} \cap B(r)$ is as well; we thus find that $\tilde{D}$ is a complete embedded disc. ■

COROLLARY 4.3. *Let Γ be a quasi-Fuchsian group acting on H^3. Then there is a complete smoothly embedded Γ-invariant minimal disc $\tilde{D}$ in H^3. As above, $\tilde{D}$ minimizes area among embedded discs.*

Proof. Let Λ_Γ be the limit set of Γ on $S^2(\infty)$; since Γ is quasi-Fuchsian, Λ_Γ is a Jordan curve. By Theorem 4.1, there exists a complete embedded minimal disc D asymptotic to Λ_Γ. We now use the proof of Theorem 3.1 to construct a Γ-invariant minimal disc. If D is not Γ-invariant define gD as in Theorem 3.1 by

$$(gD)(\omega) = D(g^*\omega), \qquad g \in \Gamma.$$

Then each gD is a smoothly embedded minimal disc; let

$$\Omega_1 = \bigcap_{g \in \Gamma} g\Omega^+,$$

where $g\Omega^+$ is the component of $H^n \setminus \operatorname{supp}(gD)$ containing the positive component of $S^{n-1}(\infty) \setminus \Lambda_\Gamma$ in its closure. We see that Ω_1 and $\partial\Omega_1$ are Γ-invariant currents; if $\partial\Omega_1$ is a smoothly embedded disc, we are done; if not, choose extreme C^2 Jordan curves γ_i in $\Omega_1 \cap \mathscr{C}(\Lambda_\Gamma)$ eventually lying outside any compact set in H^3. Let $\mathscr{S}_i$ be an Almgren-Simon solution with boundary γ_i: thus $\mathscr{S}_i$ is a smoothly embedded minimal disc with boundary γ_i, area-minimizing among embedded discs with the same boundary. We see as before that $\mathscr{S}_i \subset g\Omega^+$, for all $g \in \Gamma$, so that $\mathscr{S}_i \subset \Omega_1$, for all i. Now repeat the process carried out in Theorem 3.1, using the regularity results of Theorem 4.1. In fact, we see that $\{\mathscr{S}_i\}$ subconverges to a stationary integral 2-current $\mathscr{S}^1$; by the proof of Theorem 4.1, $\mathscr{S}^1$ is a smooth embedded disc, asymptotic to Λ_Γ. One thus obtains a sequence $\{\mathscr{S}^i\}$ by repetition of the above argument. It follows that $\{\mathscr{S}^i\}$ will subconverge to a Γ-invariant stationary integral 2-current $\tilde{D}$; the fact that $\tilde{D}$ is a complete smoothly embedded minimizing disc follows from Theorem 4.1. ■

Remark 1. In connection with Remark 1 of §3, Corollary 4.3 produces complete embedded incompressible minimal surfaces Σ in quasi-Fuchsian 3-manifolds $M^3 \cong H^3/\Gamma$, $\Gamma \cong \pi_1(\Sigma)$. For example, there are complete minimal embeddings of a k-fold punctured S^2 in certain quasi-Fuchsian 3-manifolds, for any $k > 3$. As far as the author knows, these give the first non-trivial examples of non-compact complete minimal surfaces of finite volume.

The complete minimal discs constructed in Theorem 4.1 and Corollary 4.3 need not be absolutely area-minimizing. In case they are not, one may construct surfaces in H^3 of higher genus. To begin, we recall the results of Almgren-Simon [AS] in the compact case. Let γ be an extreme C^2-Jordan curve in H^3. Let $\mathcal{M}_g(\gamma)$ be the space of connected, oriented embedded C^2-surfaces $M \subset H^3$ with boundary γ, with genus $M = g$. Let

$$\begin{aligned}\alpha_g(\gamma) &= \inf\{\text{area}(M) : M \in \mathcal{M}_g(\gamma)\} \\ &= \inf\{\text{area}(M) : M \in \mathcal{M}_h(\gamma) : h \le g\}.\end{aligned}$$

Then it is proved in [AS] that if $\alpha_g(\gamma) < \alpha_{g-1}(\gamma)$, there is a surface $M \in \mathcal{M}_g(\gamma)$ with area $(M) = \alpha_g(\gamma)$.

For complete surfaces in H^3, we then prove:

THEOREM 4.4. *Let Σ_g be a complete embedded minimal surface of genus $\le g$ in H^3 asymptotic to γ and area-minimizing among embedded surfaces of genus $\le g$. If Σ_g is not absolutely area-minimizing, then there exists a complete embedded minimal surface $\Sigma_{g'}$ in H^3, of genus $\le g'$, for some finite $g' > g$, asymptotic to γ. Further, $\Sigma_{g'}$ is area-minimizing among comparison surfaces of genus $h \le g'$.*

Proof. As in the proof of Theorem 4.1, let γ_i be a sequence of extreme C^2-Jordan curves on Σ_g, with $\gamma_i \to \gamma$ as $i \to \infty$. By hypothesis, there is an i_0 and $g' > g$ such that

$$\alpha_{g'}(\gamma_{i_0}) < \alpha_g(\gamma_{i_0}).$$

Since we may assume that Σ_g is an annulus outside of γ_{i_0}, it is clear that $\alpha_{g'}(\gamma_i) < \alpha_g(\gamma_i)$, for all $i \ge i_0$. By [AS, Theorem 8], there exists smoothly embedded surfaces S_i, all of genus g', satisfying $\partial S_i = \gamma_i$ and area $(S_i) = \alpha_{g'}(\gamma_i)$. Consider the sequence of integral 2-currents $\{S_i\}$. The proof of Theorem 2.1 applies and gives, after passage to a subsequence, a weak limit

$$S_i \to \Sigma_{g'},$$

where $\Sigma_{g'}$ is a complete stationary integral 2-current asymptotic to γ. The regularity arguments of Theorem 4.1 apply here and prove that $\Sigma_{g'}$ is a smoothly embedded submanifold. Since $S_i \to \Sigma_{g'}$ in the C^2-topology and S_i has genus g', it follows genus $\Sigma_{g'} \leq g'$. Finally, the area-minimizing properties of $\Sigma_{g'}$ follow from those of $\{S_i\}$.

Remark 2. Of course, the surfaces Σ_g and $\Sigma_{g'}$ constructed above are geometrically distinct, since they have distinct area-minimizing properties.

Remark 3. The proof above does not show that genus $\Sigma_{g'} = g'$, or even genus $\Sigma_g \geq$ genus Σ_g, although it is likely that one can find surfaces with these properties.

In order to show that such a 'hierarchy' of complete minimal surfaces actually occurs, we use the following Proposition.

PROPOSITION 4.5. *There exist Jordan curves* γ *on* $S^2(\infty)$ *such that any absolutely area-minimizing surface* Σ *asymptotic to* γ *has genus* $g \geq g_0$, *for any prescribed* $g_0 \geq 0$.

Proof. The proof is a simple modification of work in [AS]; the case $g_0 = 1$ is given below. Let γ_0 be the curve consisting of two concentric circles of radii r_1, r_2 centered at the origin in $\mathbb{R}^2$, viewed as infinity in the upper half space model of H^3. It is not difficult to see that for $\frac{1}{2}r_2 \leq r_1 \leq r_2$, any area-minimizing surface Σ_0 asymptotic to γ does not intersect the line $l_1 = \{x = y = 0\}$. To justify this, we note that any area-minimizing surface asymptotic to γ_0 is invariant under rotation about l_1; if Σ_0 intersects l_1, it follows Σ_0 is the union of two totally geodesic hyperplanes asymptotic to γ_0. Now simple area-comparision with an annulus spanning γ_0 shows that Σ_0 cannot be area-minimizing, given the bounds on r_1, r_2 above.

Let γ_ε be the oriented Jordan curve obtained by joining the circles of γ_0 by line segments of Euclidean separation ε, and let Σ_ε be an area-minimizing surface asymptotic to γ_ε (see Figure 1). As $\varepsilon \to 0$, Σ_ε converges to Σ_0 in the weak topology on varifolds.

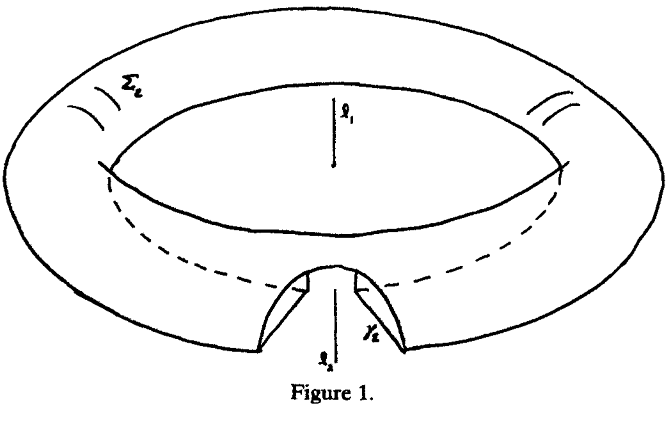

Figure 1.

Let l_2 be the ray consisting of the negative x-axis; we assume $\gamma_\varepsilon \cap l_2 = \emptyset$. Choose a ball B such that B is tangent to the plane $\{z = 0\}$ at a point on l_2 but $B \cap \Sigma_0 = \emptyset$. It follows from the area-minimizing property and the convergence $\Sigma_\varepsilon \to \Sigma_0$ that for all ε sufficiently small, $B \cap \Sigma_\varepsilon = \emptyset$. Thus there is a loop σ in $H^3 - \Sigma_\varepsilon$ such that σ does not bound in $H^3 - \Sigma_\varepsilon$. It follows that Σ_ε is not a disc for ε sufficiently small. ■

Remark 4. We note that the curves γ satisfying the above Proposition are stable under small perturbations; thus, if $\gamma \subset S^2(\infty)$ has only absolutely area-minimizing solutions of genus $\geq g_0$, then any Jordan curve γ' sufficiently close to γ in the Euclidean flat topology (or Hausdorff distance) also has only least area solutions of genus $\geq g_0$. One proves this by contradiction: if $\{\gamma_i\} \subset S^2(\infty)$ converge to γ in the flat topology, then after passing to a subsequence, any least area solutions Σ_i asymptotic to γ_i will converge smoothly to a least area solution Σ asymptotic to γ; thus for i sufficiently large, genus $\Sigma_i \geq$ genus $\Sigma \geq g_0$.

§5. Non-uniqueness, finiteness, and non-finiteness

In this section, we will continue the study of minimal surfaces in hyperbolic 3-manifolds, using the results of §4 in particular. We begin by using Proposition 4.5 to show that complete area-minimizing surfaces of infinite genus arise naturally in H^3.

THEOREM 5.1. *There exist torsion-free quasi-Fuchsian groups Γ_g such that any complete absolutely area-minimizing Γ_g-invariant surface in H^3 has infinite genus.*

Proof. Let γ be a curve as in Proposition 4.5, given explicitly as in Figure 2. Then there is a band B around γ, given as in Figure 2 also, with the following property: if Σ is any area-minimizing surface asymptotic to a Jordan curve $\gamma' \subset B$, then genus $\Sigma \geq 1$. This follows by using the arguments of Proposition 4.5.

Now inscribe successively, within the band B, N Euclidean circles C_i so that C_i intersects C_{i+1} at an angle of $\pi/2$ and $C_i \cap C_{i+k} = \emptyset$, for all $k \geq 2$. It is not difficult to see that this can be done for any $N \geq N_0 = 30$, for example.

Let Γ' be the Kleinian group acting on H^3 generated by reflections through hyperplanes asymptotic to C_i and let $\Gamma_0 \subset \Gamma'$ be the subgroup of orientation preserving mappings. It is well known that Λ_{Γ_0} is a Jordan curve lying inside the circles C_i (see [B]): in particular $\Lambda_{\Gamma_0} \subset B$ and Γ_0 is quasi-Fuchsian. We now claim

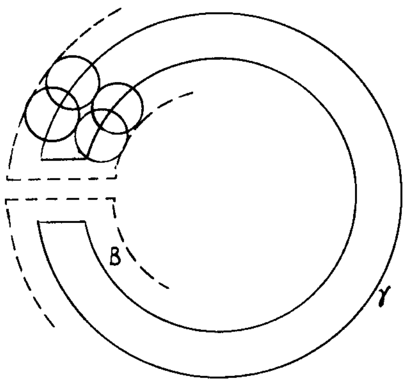

Figure 2.

that Γ_0 has a torsion-free surface subgroup Γ of index 2 such that

$$\Gamma = \pi_1(\Sigma_g),\ \Sigma_g \text{ a surface of genus g, where } g = N.$$

To see this, we note that $M^3 = H^3/\Gamma_0$ is a 3-orbifold in the sense of Thurston [T]; topologically $M^3 \approx S^2 \times I$ where S^2 has $2N$ elliptic points (branch points) with group $\mathbb{Z}_2$ determined by the circle intersections at infinity in H^3. In fact, the action of Γ' on $S^2(\infty)$ has a disc with $2N$ corner angles of $\pi/2$ on the boundary as fundamental domain; passing to Γ_0, its fundamental domain is two copies of this disc glued together along the boundary to give the desired S^2. Now such orbifolds have a surface Σ_g of genus $g = N$ as 2-fold orbifold covers. In fact, embed Σ_g in $\mathbb{R}^3$ in such a way that the z-axis L passes through all the "holes" of Σ_g and $\Sigma_g \cap L$ consists of $2N+2$ points; assume that Σ_g is invariant under rotation by 180° in the z-axis. Under this $\mathbb{Z}_2$ action on Σ_g, the quotient space $\Sigma_g/\mathbb{Z}_2$ is easily seen to be an S^2 with $2N$ elliptic points with group $\mathbb{Z}_2$.

The quasi-Fuchsian group Γ has limit set $\Lambda_\Gamma = \Lambda_{\Gamma_0}$, since Γ is normal in Γ_0 ([T:8.1.3]). Applying Theorem 3.1, we may construct complete, Γ-invariant area-minimizing surfaces $\tilde{\Sigma}$ in H^3; it is clear that such surfaces have genus either 0 or ∞. Since $\tilde{\Sigma}$ is asymptotic to $\Lambda_\Gamma \subset B$, $\tilde{\Sigma}$ cannot have genus 0. ■

Remark 1. Define the Bers isomorphism

$$T(\Sigma_g) \times T(\Sigma_g) \xrightarrow{\cong} QF_g$$

by associating to any pair of points in the Teichmüller space of a surface of genus g the associated quasi-Fuchsian group. Then we have shown that for any $g \geq 30$, e.g., there are quasi-Fuchsian groups Γ_g having Γ_g-invariant area-minimizing surfaces of infinite genus. In the orbit space $M^3 = H^3/\Gamma_g$, these surfaces descend to compact embeded stable minimal surfaces $\Sigma_{\bar{g}}$ of genus $\bar{g} > g$; clearly, these surfaces are not incompressible. Further examination of the proof shows that for any g, there is a lower bound $N(g)$ on the number of quasi-Fuchisan groups of genus g having such surfaces: we have $N(g) \to \infty$ as $g \to \infty$. In the other direction, fixing the genus g, if one takes a sequence in $T(\Sigma_g) \times T(\Sigma_g)$ tending to "infinity" in both factors (but not diagonally), it seems likely that again the number of area-minimizing surfaces of infinite genus becomes unbounded; see the discussion in [U] and [T, §9].

Let $\mathcal{G}$ be the class of quasi-Fuchsian groups such that any Γ-invariant area-minimizing surface is of infinite genus, $\mathcal{G}_g$ the subset of $\Gamma \in \mathcal{G}$ such that $\pi_1(H^3/\Gamma) = \pi_1(\Sigma_g)$. Thus the above Remark shows that the cardinality of $\mathcal{G}_g$ is unbounded in g.

We may use these surfaces to construct infinitely many geometrically distinct complete minimal surfaces asymptotic to a given boundary.

THEOREM 5.2. *Let Λ_Γ be the limit circle of a quasi-Fuchsian group $\Gamma \in \mathcal{G}$. Then there exist infinitely many complete, smoothly embedded minimal surfaces asymptotic to Λ_Γ; furthermore, there is a finite bound on the maximal normal distance between these surfaces.*

Proof. By Theorem 4.1, we know there is a complete Γ-invariant embedded minimal disc Σ_0. By definition of Λ_Γ, Σ_0 is not absolutely area-minimizing; thus we may choose extreme Jordan curves γ_i on Σ_0 and embedded minimal surfaces Σ_i of fixed genus $g_1 > 0$ with $\partial\Sigma_i = \gamma_i$. By the techniques of Theorems 4.1 and 4.4, $\{\Sigma_i\}$ will subconverge to a smoothly embedded surface Σ_{g_1} of genus $\leq g_1$. If Σ_{g_1} happens to be area-minimizing, the translates $h \cdot (\Sigma_{g_1})$, for $h \in \Gamma$ give an infinite family of distinct (but isometric) minimal surfaces asymptotic to Λ_Γ. If Σ_{g_1} is not area-minimizing, we may repeat the process on Σ_{g_1}: in either case we obtain an infinite family of distinct surfaces.

To verify the second statement, note that all surfaces are contained in the convex hull $\mathscr{C}(\Lambda_\Gamma)$: note also that the diameter of $\mathscr{C}(\Lambda_\Gamma)$

$$d_\Gamma = \sup_{x \in \partial(\mathscr{C}(\Lambda_\Gamma))} \{\text{dist}\,(x, \partial(\mathscr{C}(\Lambda_\Gamma))\} < \infty;$$

in particular, there is an upper bound to the distances of all minimal surfaces asymptotic to Λ_Γ. ■

Note. One expects that Σ_{g_1} constructed above is not area-minimizing; this would then give an infinite sequence of isometrically distinct surfaces.

Next we prove a non-uniqueness result for incompressible minimal surfaces in a given homotopy class in hyperbolic 3-manifolds.

THEOREM 5.3. *Let Γ be a quasi-Fuchsian group in $\mathcal{G}_g$, so $\pi_1(\Sigma_g) = \Gamma$. Then in the homotopy class of the inclusion*

$$\Sigma_g \xrightarrow{\sim} M^3 = H^3/\Gamma,$$

there are at least two geometrically distinct compact stable embedded minimal surfaces of genus g.

Proof. Let Σ_∞ be a Γ-invariant area-minimizing surface of infinite genus in H^3 and let $\Omega^\pm$ be the Γ-invariant components of $H^3 \backslash \Sigma_\infty$: we will construct Γ-invariant stably embedded minimal discs in $\Omega^\pm$. It suffices to work in Ω^+: let γ_i be a sequence of smooth extreme Jordan curves in $\Omega^+ \cap \mathscr{C}(\Lambda_\Gamma)$ converging to Λ_Γ as $i \to \infty$. It is well known one may solve the Plateau problem for minimal discs in Ω^+, see e.g. [MY]. By the work of [AS] or [MY], any solution S_i is an embedded minimal disc, area-minimizing among embedded minimal discs in Ω^+. Letting $i \to \infty$, the techniques of Theorem 4.1 show that $\{S_i\}$ subconverges to a complete embedded minimal disc D^+ in Ω^+ asymptotic to Λ_Γ. If D^+ is not Γ-invariant, we may use the methods of Corollary 4.4 to produce a Γ-invariant minimal disc, call it again D^+ in Ω^+. (In fact there are at least two such in Ω^+ if D^+ was not Γ-invariant to begin with.) The quotient surfaces D^+/Γ, D^-/Γ are then stable minimal surfaces embedded in M^3, inducing an isomorphism on π_1. ■

Remark 2. This result contrasts with the result that harmonic maps $f: M \to N$ are unique in their homotopy class, provided $K_N < 0$ and N is compact. Thurston has shown that there are infinitely many (isometric) minimal surfaces in $M^3 = H^3/\Gamma$, where Γ is a "doubly degenerate group", i.e. $\Gamma = \pi_1(\Sigma_g)$, where $\Sigma_g \to N^3 \to S^1$ is a smooth fibration over S^1, N^3 having a hyperbolic structure. In this case, Γ is a suitable 'limit' of quasi-Fuchsian groups: see [T§9].

The following theorem shows that least area incompressible surfaces constructed by Schoen–Yau [SY] are not necessarily area-minimizing in their homology class.

THEOREM 5.4. *Let $\Sigma_g \overset{i}{\hookrightarrow} M^3$ be a least area incompressible surface in M^3, where $M^3 = H^3/\Gamma$, $\pi_1(\Sigma_g) \approx \Gamma$. Then if $\Gamma \in \mathcal{G}_g$, there exists $\Gamma' \in \mathcal{G}_{g'}$, with $\Gamma' \lhd \Gamma$ of finite index such that the lift*

$$H^2/\Gamma' \approx \Sigma_{g'} \overset{i'}{\hookrightarrow} M' = H^3/\Gamma'$$

covering i is a least area incompressible embedding, but $[\Sigma_{g'}] \in H_2(M', \mathbb{Z})$ is not of least area in its homology claass.

Proof. Since $\Sigma_g \hookrightarrow M^3$ is incompressible, the lift $\tilde{\Sigma}_g \hookrightarrow H^3$ is a complete (embedded) disc, asymptotic to Λ_Γ. Since $\Gamma \in \mathcal{G}_g$, $\tilde{\Sigma}_g$ is not absolutely area minimizing. Let D be a domain in $\tilde{\Sigma}_g$ such that D is not area-minimizing w.r.t. ∂D. Now choose $\Gamma' \lhd \Gamma$ such that D is contained in a fundamental domain of Γ'; this is possible since Γ is residually finite [H]. Let $D' = \tilde{\Sigma}_g \cap \Gamma'$ and let S' be an area minimizing surface in H^3 with $\partial S' = \partial D'$; clearly D' and S' are homologous in H^3. It follows that $D'/\Gamma' = \Sigma_{g'}$ and S'/Γ' are homologous in $M' = H^3/\Gamma'$ and since area $(S') <$ area (D'), area $(S'/\Gamma) <$ area (Σ'_g). On the other hand, it is not difficult to see that $\Sigma_{g'}$ is of least area in its homotopy class; see [FHS] (Lemma 3.3) for the details.

Finally we prove a general finiteness result for stable minimal surfaces in compact Riemannian 3-manifolds; this will show in particular that "most" hyperbolic 3-manifolds admit only finitely many stable minimal surfaces of a given genus.

Define a surface S in N^3 to be R-locally area-minimizing if for any geodesic R-ball $B(x, R)$ in N^3, the surface $S \cap B(x, R)$ is area-minimizing with respect to its boundary.

THEOREM 5.5. *Let N^3 be a compact oriented 3-manifold with an analytic Riemannian metric. Then for any given $R > 0$, either*

(1) N^3 contains only finitely many compact stable, oriented, R-locally minimizing surfaces of uniformly bounded area, or

(2) N^3 fibres over S^1 with fibres smooth compact minimal surfaces.

We expect the added condition of R-locally minimizing may be dropped, but have not been able to do so.

Proof. The proof is based on the method of Tomi [To] on the finite solvability of the Plateau problem in $\mathbb{R}^3$. We suppose (1) does not hold; let $\{M_i\}$ be a sequence of R-locally area-minimizing surfaces in N^3 with area $(M_i) < K$. The compactness theorem for integral currents implies that $\{M_i\}$ converges, after passing to a subsequence, to an R-locally minimizing integral 2-current $\mathcal{M}$. Since each M_i is stable, the regularity theorem of Schoen–Simon [SS] implies that $\mathcal{M}$ is a smooth stable minimal surface; furthermore the fact that $\mathcal{M}$ is R-locally area-minimizing implies that $\mathcal{M}$ and M_i may be locally graphed over the tangent planes of $\mathcal{M}$, for i sufficiently large, see e.g. [P]. Thus, in sufficiently small geodesic balls, $\mathcal{M}$ and M_i are embedded discs D, D_i, and the convergence $D_i \to D$ is C^2 (in fact analytic).

These results show that each M_i may be graphed globally over $\mathcal{M}$ in the following sense: for any $f \in C^{2,\alpha}(\mathcal{M})$, define M_f to be the graph of f over $\mathcal{M}$, i.e.,

$$M_f = \{y : y = \exp_x f(x) \cdot E_0\},$$

where E_0 is the unit normal to $\mathcal{M}$ in N. Thus M_i defines a unique function $f_i \in C^{2,\alpha}$ such that $M_i = M_{f_i}$, where $f_i \to 0$ as $i \to \infty$ and $\mathcal{M} = M_0$. Define

$$H : C^{2,\alpha}(\mathcal{M}) \to C^{0,\alpha}(\mathcal{M}) \quad \text{by}$$
$$H(f) = \text{mean curvature function of } M_f.$$

Using the fact that $N, \mathcal{M}$ and the normal exponential map of $\mathcal{M}$ in N are analytic, it is a straightforward, but lengthy, computation to show that H is an analytic mapping in a neighborhood of $0 \in C^{2,\alpha}(\mathcal{M})$.

The arguments of Tomi [To] then show that $H^{-1}(0)$ is an analytic 1-manifold V in a neighborhood of $\mathcal{M}$. Using the compactness theorem again, we see V is a compact analytic 1-manifold (diffeomorphic to S^1) parametrizing diffeomorphic stable minimal surfaces M_t in N^3. It now follows that the natural projection

$$\pi : N^3 \to V$$
$$\pi(x) = t, \quad \text{where} \quad x \in M_t$$

gives the desired fibration. □

COROLLARY 5.6. *A quasi-Fuchsian 3-manifold $M = H^3/\Gamma$ has only finitely many stable, locally area-minimizing compact surfaces of a given genus.*

Proof. The convex hull property shows that all compact minimal surfaces in M^3 are contained in the convex part of M: since this latter does not fiber over S^1 isometrically, it follows from the proof of Theorem 5.5 that M contains only finitely many R-locally area-minimizing surfaces of bounded area. Now we have, for Σ_g a minimally immersed surface of genus g in a hyperbolic manifold that,

$$\text{vol}(\Sigma_g) = \int_{\Sigma_g} 1 \le -\int_{\Sigma_g} K = -2\pi\chi(\Sigma_g),$$

where K is the Gaussian curvature of Σ_g, $\chi(\Sigma_g) = (2-2g)$ is the Euler characteristic. Thus a bound on genus gives a bound on area, proving the corollary. ■

Remark 3. As noted above in Remark 2, the Corollary is false if we drop the assumption that Γ is quasi-Fuchsian. On the other hand, it does hold for any compact hyperbolic 3-manifold which does not fibre over S^1 with fibres being minimal surfaces. We conjecture that no hyperbolic 3-manifold has this property: more generally, we conjecture that if M^3 is a closed hyperbolic 3-manifold, then there does not exist a local 1-parameter family of closed minimal surfaces in M^3.

A result of this type, together with Theorem 3.4 would provide a good basis in understanding the moduli spaces of minimal surfaces in negatively curved 3-manifolds.

Finally, one obtains a purely topological result from Theorem 5.5.

COROLLARY 5.7. *Let N^3 be a compact 3-manifold admitting a metric of curvature $K_N \le c < 0$. Then for any given g, the set of homotopy classes $[\Sigma_g, N^3]_i$ of incompressible surfaces in N^3 is finite, up to conjugacy.*

Proof. It follows from [SY] that in any class of $[\Sigma_g, N^3]_i$, there is an immersed least area incompressible surface. By the estimate in Corollary 5.6, any such surface has a bound on its area. If there was infinitely many such homotopy classes, the proof of Theorem 5.5 implies the least area surfaces must subconverge to a limiting surface; thus all surfaces will eventually be homotopic. ■

Corollary 5.7 has been proved by Thurston [T:8.8.6] by means of pleated surfaces.

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