On the Gauss Mapping for Hypersurfaces of Constant Mean Curvature in the Sphere*

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The proof of the Bernstein conjecture on minimal hypersurfaces in Euclidean space – for those dimensions in which it is known (see [1], [2], [4]) – raises the following interesting speculation on the geometry of minimal hypersurfaces in the Euclidean spheres:

If the Gauss image of a compact minimal hypersurface M^n in the Euclidean sphere S^{n+1} lies in a closed hemisphere of S^{n+1}, then M^n must be a great hypersphere in S^{n+1}.

E. de Giorgi [2] and J. Simons [4] have shown that the Gauss image of a minimal hypersurface other than a great hypersphere cannot lie in an open hemisphere. We prove here that the above speculation is indeed true and generalizes to hypersurfaces of constant mean curvature (Theorem 2).

To prove this result we first obtain a characterization of the hyperspheres (great or small) of S^{n+1} among all complete hypersurfaces of S^{n+1} in terms of their Gauss images (Theorem 1). With this preparation the main theorem follows more or less directly on using the standard integral formulas for hypersurfaces in the sphere.

We follow here the terminology and notations of Chapter VII, Volume II, of Kobayashi–Nomizu [3].

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§ 1. The Gauss Mapping

In the sequel M will be a complete orientable Riemannian manifold of dimension n and $f: M \to S^{n+1}$ is an isometric immersion of M into the unit sphere S^{n+1} in the Euclidean space $\mathbf{E}^{n+2}$ with centre at the origin. By a hypersphere Σ^n in S^{n+1} we will mean the intersection of S^{n+1} with a hyperplane in $\mathbf{E}^{n+2}$. Σ^n is called a great (equatorial) or small (non-equatorial) hypersphere according as the hyperplane passes through the origin of $\mathbf{E}^{n+2}$ or not. It may of course degenerate into a single point.

Since M is orientable we may choose a global field of unit vectors ξ, normal to M in S^{n+1} with respect to the immersion f. For vector fields X and Y on M the Riemannian connections $\tilde{\nabla}$ and ∇ of S^{n+1} and M, respectively, are related by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\,\xi,$$

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where g is the metric on M and A is the symmetric tensor field of type (1.1) on M defined by

$$\tilde{\nabla}_X \xi = - AX .$$

The Gauss mapping

$$\phi : M^n \to S^{n+1}$$

is defined by $\phi(p) = \xi_{f(p)} \in S^{n+1}$ for each $p \in M$. $\phi(M)$ is called the Gauss image of M. Depending, as it does, on the choice of ξ, the Gauss image is only determined to within the antipodal mapping of S^{n+1}. Thus the statement that the Gauss image of M is contained in a closed hemisphere (or in a hypersphere) of S^{n+1} is independent of the Gauss mapping selected. We remark that $\phi(\Sigma^n)$ is a point (resp. a small hypersphere) of S^{n+1} if Σ^n is a great (resp. small) hypersphere of S^{n+1}.

THEOREM 1. *Let M be a complete orientable Riemannian manifold of dimension $n \geqslant 2$ isometrically immersed in S^{n+1} and let ϕ be the associated Gauss mapping.*

i) *If $\phi(M)$ is contained in a great hypersphere of S^{n+1} then M is imbedded as a great hypersphere and so $\phi(M)$ is a single point.*

ii) *If $\phi(M)$ is contained in a small hypersphere of S^{n+1} but is not a single point, then M is imbedded as a small hypersphere and $\phi(M)$ is a full small hypersphere.*

Proof. We first observe that either of the above conditions on the Gauss image gives rise to a unit vector a in $\mathbf{E}^{n+2}$ for which $\langle \xi, a \rangle$ is a constant on M – α say – with $0 \leqslant \alpha \leqslant 1$. Here $\langle\ ,\ \rangle$ denotes the Euclidean metric on $\mathbf{E}^{n+2}$. With the usual identification of tangent spaces under the immersion f of M into S^{n+1} we define a vector field Z on M by

$$Z_p = a - \langle \xi_{f(p)}, a \rangle\, \xi_{f(p)} - \langle x_{f(p)}, a \rangle\, x_{f(p)}, \tag{1}$$

where $x_{f(p)}$ is the position vector of $f(p) \in S^{n+1}$ in $\mathbf{E}^{n+2}$. Denoting the connection on $\mathbf{E}^{n+2}$ by D and differentiating the equation $\langle \xi, a \rangle = \alpha$ on M, we obtain, for $X \in T(M)$:

$$\begin{aligned} 0 &= \langle D_X \xi, a \rangle \\ &= \langle \tilde{\nabla}_X \xi - \langle X, \xi \rangle\, x, a \rangle \\ &= \langle - AX, a \rangle \end{aligned}$$

since $\tilde{\nabla}_X \xi = -AX$ and $\langle X, \xi \rangle = 0$. In other words $g(AX, Z) = 0$ for all $X \in T(M)$, so that

$$Z \in \operatorname{Ker} A \tag{2}$$

by the symmetry of A. Moreover

$$\left.\begin{aligned} \nabla_X Z &= \tilde{\nabla}_X Z - g(AX, Z)\,\xi \\ &= \tilde{\nabla}_X Z \quad \text{by (2)}, \\ &= D_X Z + g(X, Z)\,x \\ &= -\langle \xi, a\rangle D_X \xi - \langle X, a\rangle x - \langle x, a\rangle X + g(X, Z)\,x \quad \text{by (1)}, \\ &= \langle \xi, a\rangle AX - \langle x, a\rangle X \\ &= (\alpha A - \beta I)\,X, \end{aligned}\right\} \tag{3}$$

where I is the identity transformation and the function β on M is given by $\beta(p) = \langle x_{f(p)}, a\rangle$. By reason of Codazzi's equation and (2) we have

$$\begin{aligned} (\nabla_Z A)\,X &= (\nabla_X A)\,Z \\ &= \nabla_X (AZ) - A\nabla_X Z \\ &= (\beta A - \alpha A^2)\,X \end{aligned}$$

for each $X \in T(M)$, that is,

$$\nabla_Z A = \beta A - \alpha A^2. \tag{4}$$

In particular

$$Z(\operatorname{Tr} A) = \operatorname{Tr}(\nabla_Z A) = \beta \operatorname{Tr} A - \alpha \operatorname{Tr} A^2 \tag{5}$$

where Tr denotes the trace.

The zeroes of the vector field Z occur at those points p of M where a is orthogonal to $f_*(T_p(M))$. If $Z \equiv 0$ on $M, f(M)$ lies in one of the hyperspheres determined by the system of hyperplanes in $\mathbf{E}^{n+2}$ orthogonal to a, and by completeness of M, the set $f(M)$ is a full hypersphere in S^{n+1}. In particular, when $\alpha = 1$ (i.e. $\xi = a$) we have $Z \equiv 0$ and $\langle x, a\rangle \equiv 0$, so that $f(M)$ is a full great hypersphere.

We therefore suppose henceforth that $Z \not\equiv 0$ on M and as remarked above we must then have $0 \leqslant \alpha < 1$. It will be shown that $f(M)$ is then a full great hypersphere, a separate argument being necessary for the case $\alpha = 0$.

By virtue of (2) and (3), $\nabla_Z Z = -\beta Z$ on M and therefore $Z/\|Z\|$ is a geodesic vector field on the open submanifold

$$M' = \{p \in M;\, Z_p \neq 0\}$$

of M, where $\|Z\|$ denotes the length of Z. Fixing $p_0 \in M'$, let γ be the geodesic (parametrized by arc length s and extended indefinitely in both directions along M) which emanates from p_0 tangent to Z_{p_0}. By virtue of the above remarks, the vector field Z is tangent to γ along γ. Consider the real function h defined on $\mathbf{R}$ by

$$h(s) = g(\dot{\gamma}(s), Z_{\gamma(s)})$$

where $\vec{\gamma}(s)$ is the velocity vector of $\gamma(s)$. Let (a, b) be the maximal interval (possibly semi-infinite or infinite) containing 0 for which $\gamma((a, b))$ lies in the connected component of M' containing p_0. Then

$$\left.\begin{aligned}\frac{dh}{ds} &= \vec{\gamma}(s)\, g(\vec{\gamma}(s), Z_{\gamma(s)})\\ &= g(\vec{\gamma}(s), \nabla_{\vec{\gamma}(s)} Z_{\gamma(s)})\\ &= g(\vec{\gamma}(s), (\alpha A - \beta I)\,\vec{\gamma}(s)) \quad \text{by (3)},\\ &= -\beta\circ\gamma(s), \quad s\in(a, b),\end{aligned}\right\} \tag{6}$$

since $\vec{\gamma}(s)$ is a multiple of Z when $s\in(a, b)$ and $Z\in\operatorname{Ker} A$ by (2). Thus

$$\left.\begin{aligned}\frac{d^2h}{ds^2} &= -\frac{d}{ds}\langle x_{\gamma(s)}, a\rangle\\ &= -\langle\vec{\gamma}(s), a\rangle\\ &= -h(s), \quad s\in(a, b).\end{aligned}\right\} \tag{7}$$

The solution of this differential equation with initial conditions $dh/ds(0) = -\beta\circ\gamma(0) = -\beta_0$ and $h(0) = \sqrt{1-\alpha^2-\beta_0^2}$ is

$$h(s) = \sqrt{1-\alpha^2}\cos(s+s_0), \quad s\in(a, b), \tag{8}$$

where $s_0\in(-\pi/2, \pi/2)$ is determined by $\sin s_0 = \beta_0/\sqrt{1-\alpha^2}$. Furthermore, it follows from (6) that

$$\beta\circ\gamma(s) = \sqrt{1-\alpha^2}\sin(s+s_0), \quad s\in(a, b), \tag{9}$$

and from (8) that

$$Z_{\gamma(s)} = \sqrt{1-\alpha^2}\cos(s+s_0)\,\vec{\gamma}(s), \quad s\in(a, b). \tag{10}$$

$h(0)$ being positive, it follows that h is positive on (a, b) and we infer from (8) that (a, b) is a finite interval. The maximality condition on the interval (a, b) implies that $Z_{\gamma(a)}=0$ and $Z_{\gamma(b)}=0$ which means, by virtue of (10) and continuity, that

$$\cos(a+s_0) = \cos(b+s_0) = 0 \tag{11}$$

Letting $k(s) = (\operatorname{Tr} A)\circ\gamma(s)$ we may rewrite (5) as

$$\sqrt{1-\alpha^2}\cos(s+s_0)\frac{dk}{ds} = \sqrt{1-\alpha^2}\sin(s+s_0)\,k(s) - \alpha(\operatorname{Tr} A^2)_{\gamma(s)}$$

on (a, b), that is,

$$\sqrt{1-\alpha^2}\frac{d}{ds}(\cos(s+s_0)\,k(s)) = -\alpha(\operatorname{Tr} A^2)_{\gamma(s)} \tag{12}$$

on (a, b).

Consequently the function $\cos(s+s_0)\,k(s)$ is monotone decreasing on (a, b) and vanishes at $s=a, b$. Thus $k=0$ along (a, b) and it follows from (12) that $\operatorname{Tr} A^2=0$ along $\gamma((a, b))$, if $\alpha\neq 0$, and in particular $A=0$ at $p_0=\gamma(0)$. Assuming $\alpha\neq 0$ we have therefore proved that $A\equiv 0$ on M'. However $Z=0$ and $\beta^2=1-\alpha^2$ on the open set $M-\bar{M}'$, so that $A\equiv((1-\alpha^2/\alpha)\,I$ there, by virtue of (3). Since M is connected and M' is non empty, $A\equiv 0$ on M. The completeness of M now implies that $f(M)$ is a full great hypersphere.

It remains to attend to the case where $Z\not\equiv 0$ and $\alpha=0$. Here the equation essential to our proof is

$$Z(\operatorname{Tr} A^2) = \operatorname{Tr} \nabla_Z A^2 = 2\beta \operatorname{Tr} A^2, \tag{13}$$

which is an easy consequence of (4). Since $\alpha=0$ it is readily verified that the equations (6)–(10) are valid for all $s\in\mathbf{R}$. Using these equations and setting $l(s)=(\operatorname{Tr} A^2)\circ\gamma(s)$, (13) reduces to

$$\cos(s+s_0)\frac{dl}{ds} = 2\sin(s+s_0)\,l(s).$$

Thus $l(s)=c/\cos^2(s+s_0)$ on $-\pi/2<s+s_0<\pi/2$ for some constant c, and we have a contradiction unless c – and therefore l – is zero; thus $A=0$ on M'. Since $\alpha=0$ and $Z=0$ on $M-M'$, we have $\beta^2=1$ on $M-M'$; by virtue of (4), $A=0$ on $M-M'$. It now follows as before that $f(M)$ is a full great hypersphere.

In every case it has been shown that f immerses M on a full hypersphere Σ^n in S^{n+1}. The completeness of M then implies that $f\colon M^n\to\Sigma^n$ is a covering map (p. 176, Volume I, [3]) and since Σ^n is simply connected if $n\geqslant 2$, f is an imbedding if $n\geqslant 2$. This completes the proof of the theorem.

Remark. Theorem 1 remains valid of course if $n=1$, except that f is no longer an imbedding in general.

It seems appropriate at this point to emphasise that Theorem 1 is a global result, that is to say that there is no local analogue if the assumption of completeness is dropped. Indeed the example which follows serves to construct a large class of hypersurfaces in S^{n+1} whose Gauss images lie in a great hypersphere. There is even a large class of minimal hypersurfaces having this property.

Example. Let ψ be an immersion of a connected orientable $(n-1)$-dimensional manifold N into a great hypersphere S^n in S^{n+1} With e_{n+2} denoting the unit vector orthogonal to the hyperplane of S^n in $\mathbf{E}^{n+2}$ and angle θ as coordinate on the unit circle S^1, the suspension $f\colon N\times S^1\to S^{n+1}$ of the immersion ψ by geodesics from the north and south poles of S^{n+1} is defined as

$$f(p, \theta) = \cos\theta\,\psi(p) + \sin\theta\, e_{n+2},$$

where p is any point of N. Choosing local coordinates $(x^1, \dots, x^{n+1})$ on N we see that

$$f_*\left(\frac{\partial}{\partial x^i}\right) = \cos\theta \frac{\partial \psi}{\partial x^i}, \quad 1 \leqslant i \leqslant n-1,$$

$$f_*\left(\frac{\partial}{\partial \theta}\right) = \sin\theta\, \psi + \cos\theta\, e_{n+2}.$$

Thus f immerses $N' = \{(p, \theta) \in N \times S^1;\ \theta \neq \text{odd multiple of } \pi/2\}$ in S^{n+1}. We denote by M one of the two connected components of N'.

Let η be a unit vector field normal to N in S^n and let B be the matrix of the second fundamental form in the coordinates $(x^1, \dots, x^{n+1})$. If ξ is a unit vector field normal to M in S^{n+1} we observe that ξ is orthogonal to $f(p, \theta)$, $f_*(\partial/\partial x^i)$ and $f_*(\partial/\partial\theta)$ and therefore to $\psi(p)$, e_{n+2} and $\partial\psi/\partial x^i$. Consequently, choosing the direction of ξ suitably we have $\xi_{f(p,\theta)} = \eta_{\psi(p)}$ for all $(p, \theta) \in M$. In particular $\langle \xi, e_{n+2} \rangle \equiv 0$ on M, that is, *the Gauss image of M lies in a great hypersphere of S^{n+1}*. On the other hand it is easily seen that

$$\frac{\partial^2 f}{\partial x^i\, \partial x^j} = \cos\theta \frac{\partial^2 \psi}{\partial x^i\, \partial x^j},$$

$$\frac{\partial^2 f}{\partial x^i\, \partial \theta} = -\sin\theta \frac{\partial \psi}{\partial x^i},$$

$$\frac{\partial^2 f}{\partial \theta^2} = -\cos\theta\, \psi - \sin\theta\, e_{n+2},$$

from which it follows that the matrix of the second fundamental form of M in the coordinates $(x^1, \dots, x^{n+1}, \theta)$ is given by

$$A = \frac{1}{\cos\theta}\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}.$$

Consequently, *M is totally geodesic (minimal) if and only if N is totally geodesic (minimal).*

§ 2. The Main Theorem

On an n-dimensional orientable Riemannian manifold isometrically immersed in S^{n+1}, the Laplacians of the functions $\langle x, a \rangle$ and $\langle \xi, a \rangle$ restricted to M are easily computed as

$$\Delta \langle x, a \rangle = \operatorname{Tr} A \langle \xi, a \rangle - n \langle x, a \rangle, \tag{14}$$

$$\Delta \langle \xi, a \rangle = -\langle \operatorname{grad}(\operatorname{Tr} A), a \rangle - \operatorname{Tr} A^2 \langle \xi, a \rangle + \operatorname{Tr} A \langle x, a \rangle, \tag{15}$$

a being any constant unit vector in $\mathbf{E}^{n+2}$. Since we will now be concerned only with hypersurfaces of constant mean curvature (i.e. $\operatorname{Tr} A = \text{constant}$ on M), we rewrite (15) as

$$\Delta \langle \xi, a \rangle = -\operatorname{Tr} A^2 \langle \xi, a \rangle + \operatorname{Tr} A \langle x, a \rangle. \tag{16}$$

Combining (14) and (16) we obtain

$$\left.\begin{aligned} \Delta \langle n\xi + \quad \mathrm{r}\, Ax, a \rangle &= -\{n \operatorname{Tr} A^2 - (\operatorname{Tr} A)^2\} \langle \xi, a \rangle \\ &= -\sum_{i<j} (\lambda_i - \lambda_j)^2 \langle \xi, a \rangle, \end{aligned}\right\} \tag{17}$$

where $\lambda_1, \ldots, \lambda_n$ denote the characteristic roots of A.

The following result sharpens and generalizes Theorem 5.2.1 of Simons [4].

THEOREM 2. *Let M be any compact connected orientable manifold of dimension $n \geqslant 2$ immersed in the sphere S^{n+1} with constant mean curvature. If the Gauss image of M lies in a closed hemisphere of S^{n+1}, then M imbeds onto a hypersphere in S^{n+1}.*

Proof. The assumption on the Gauss image of M is equivalent to the existence of a constant unit vector a in $\mathbf{E}^{n+2}$ for which $\langle \xi, a \rangle \geqslant 0$ on M. By virtue of (17), we have $\Delta \langle n\xi + \operatorname{Tr} Ax, a \rangle \leqslant 0$ and E. Hopf's lemma implies that $\langle n\xi + \operatorname{Tr} Ax, a \rangle$ is constant on M. If M is minimal $\langle \xi, a \rangle$ is constant on M and the result follows from Theorem 1. We now assume that $\operatorname{Tr} A \neq 0$. By (17) every point of $W = \{p \in M;\ \langle \xi_{f(p)}, a \rangle > 0\}$ is an umbilic. However $\langle n\xi + \operatorname{Tr} Ax, a \rangle$ being constant on M, it is clear that $\langle x, a \rangle$ is constant on $M - \bar{W}$. Therefore $M - \bar{W}$ immerses into a hypersphere of S^{n+1} so that $M - \bar{W}$ is also totally umbilic. Thus M immerses totally umbilically in S^{n+1} and must therefore be an imbedded hypersphere.

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