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On complex affine surfaces with $\mathbb{C}^+$-action

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0. Introduction

The subject of this paper is the classification of normal complex affine surfaces endowed with a nontrivial action of the additive group $\mathbb{C}^+$ as well as certain aspects of their topology. Such surfaces have already been studied by Miyanishi in [6] and [7]; on the one hand from an algebraic point of view by looking at iterative systems of higher order derivations (in arbitrary characteritic) and on the other side by investigating "cylinderlike" affine surfaces, i.e. surfaces which admit non-empty open subsets of the form $Z \times \mathbb{C}$.

One goal here is to complete that picture in the complex case: As a cylinderlike surface a normal affine $\mathbb{C}^+$-surface can be constructed from a product $Z \times \mathbb{C}$, Z a smooth affine curve, and $\mathbb{C}^+$ acting by translation on the second factor, by replacing in the fibration $pr_Z : Z \times \mathbb{C} \to Z$ a finite number of orbits by "exceptional fibres". Since there is no twisting over the affine curve Z, the resulting surface V is uniquely determined by the germs of $\mathbb{C}^+$-invariant neighbourhoods of the glued in exceptional fibres.

But in contrast to the reductive group $\mathbb{C}^*$ [1)], those fibres may be non-connected. In order to deal with non-connected fibres we replace the base curve Z with a nonseparated "connected" quotient X, i.e. the quotient morphism has connected fibres. Over X there is nontrivial twisting, and in fact we obtain already non-trivial affine $\mathbb{C}^+$-principal bundles over X, cf. Prop. 1.4: they are affine whenever they are separated. Surfaces of this type have been used by W. Danielewski, to construct his counterexample to the Zariski cancellation problem, cf. [1] and Remark 1.5. The next step is to investigate the structure near connected exceptional fibres of the connected quotient $\pi : V \to X$. A first distinction between such fibres $\pi^{-1}(x_0)$ uses two numerical invariants: the multiplicity $m \geq 1$ of $\pi^{-1}(x_0)$ as fibre of the morphism π, and its "fixed point order" $\mu \geq 0$, i.e. the vanishing order of the velocity

[1)] The situation for the multiplicative group $\mathbb{C}^*$ has been studied in the papers [2] and [3].

vector field associated to the $\mathbb{C}^+$-action along that fibre. For $m = 1$ the morphism π is near $\pi^{-1}(x_0)$ a projection, i.e. there is a neighbourhood U of x_0 such that $\pi^{-1}(U) \cong U \times \mathbb{C}$, and $\mu = 0$ means that this isomorphism is even equivariant.

For $m \geq 2$ we describe explicitly invariant neighbourhoods $\pi^{-1}(U)$ as quotients W/C_m, where W is a smooth affine surface without multiple fibres over a Galois cover Y of U with cyclic Galois group C_m and a ramification point y_0 of order m over x_0. The neighbourhoods $\pi^{-1}(U)$ are determined up to isomorphism by C_m-orbits in $Q(\mathcal{O}_{Y,y_0})/h^{-\mu}\mathcal{O}_{Y,y_0}$, such that the fixed point order μ is coprime to the order of the isotropy subgroup along that orbit, and the smooth case corresponds to principal orbits, i.e. those with m elements, while otherwise there is exactly one singular point. Here h denotes a generator of the maximal ideal $\mathfrak{m}_{y_0} \subset \mathcal{O}_{Y,y_0}$.

In particular for the description of invariant neighbourhoods of connected fibres one needs infinite-dimensional "moduli", another feature that distinguishes the additive group $\mathbb{C}^+$ from the multiplicative group $\mathbb{C}^*$.

Finally the case of nonconnected fibres is as simple as that of $\mathbb{C}^+$-principal bundles: different models V_i of invariant neighbourhoods $\pi^{-1}(U)$ of connected fibres can rather arbitrarily be patched together.

In the second section we construct a minimal equivariant compactification $\bar{V}$ for a smooth affine $\mathbb{C}^+$-surface V and use the information about the divisor at infinity $D := \bar{V} \setminus V$ to compute the singular homology of V, as well as the first homology group at infinity in the case that no multiple fibres occur. This allows us to distinguish the topological types of the Danielewski surfaces.

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1. Free $\mathbb{C}^+$-actions on normal affine surfaces

Let $V = \operatorname{Sp}(A)$ be a connected normal affine surface. Algebraic $\mathbb{C}^+$-actions

$$\mathbb{C} \times V \to V, \qquad (t, v) \mapsto t * v$$

on V are in one-to-one correspondence with locally nilpotent derivations $D : A \to A$: The comorphisms $\mu : A \to A[T]$ associated to a $\mathbb{C}^+$-action are exactly those of the form

$$\mu(a) = \sum_{n=0}^{\infty} \frac{D^n a}{n!} T^n$$

with D as above, cf. [7]. The kernel of D is the subalgebra $A_0 := A^{\mathbb{C}^+}$ of invariant regular functions, while D^2 kills exactly those functions f, which are affine linear on every orbit, i.e. $f(t * v) = f_1(v)t + f_0(v)$ for every $v \in V$ and $t \in \mathbb{C}^+$ with $f_n = D^n f$, $n = 0, 1$.

A normal affine surface together with a nontrivial algebraic $\mathbb{C}^+$-action we shall also call an affine $\mathbb{C}^+$-surface. Note that each orbit is either a fixed point or the complex line; the latter being maximal affine it follows that all orbits are closed.

1.1. LEMMA. *For an affine $\mathbb{C}^+$-surface $V = \mathrm{Sp}\,(A)$ the algebra A_0 of invariant functions is finitely generated, and the natural "quotient" morphism $q : V = \mathrm{Sp}\,(A) \to Z := \mathrm{Sp}\,(A_0)$ is a surjection onto a smooth curve; furthermore over a nonempty open subset $Z^* \subset Z$ there is an equivariant isomorphism $q^{-1}(Z^*) \cong Z^* \times \mathbb{C}$, where $\mathbb{C}^+$ acts by translation on the second factor.*

Proof. Since the action is nontrivial, we can find a function $f \in \mathrm{Ker}\,(D^2) \backslash \mathrm{Ker}\,(D)$. Let $V^* := V_{Df}$ be the (invariant) special open set where Df does not vanish; set $S := \{v \in V^*; f(v) = 0\}$. Consider the map $\mathbb{C} \times S \to V^*$, $(t, v) \mapsto t * v$. It is obviously bijective and even an isomorphism, since V^* is normal. In particular S is smooth; so we may consider the smooth projective closure $\bar{S} \subset \mathbb{P}_n$ of S and interpret the projection $pr_S : V^* \to S$ as rational map from $q : V \to \bar{S}$. We want to show that it is in fact a morphism: Otherwise it lifts to a morphism $\tilde{V} \to \bar{S}$ with a suitable modification $\tilde{V}$ of V with centre in $V \backslash V^*$; and this lifting restricts on some irreducible component E of the exceptional fibres to a finite surjective map $E \to \bar{S}$.

Every orbit $\mathbb{C} * v$, $v \in S$, is closed in V and $\tilde{V}$. Consequently the generic point in E lies isolated in its fibre, which is impossible.

Now let $Z := q(V)$. Suppose $Z = \bar{S}$. Then $A_0 \cong \mathcal{O}(Z) = \mathbb{C}$; in particular Df is a constant, whence $V = V^* = Z \times \mathbb{C}$, a contradiction. Consequently Z as proper subset of $\bar{S}$ is affine and $A_0 \cong \mathcal{O}(Z)$ finitely generated. Finally set $Z^* := S \subset Z$. □

Denote by $z_1, \ldots, z_s$ the points $z_j \in Z$ near which the map $q : V \to Z$ is not equivariantly locally trivial. By replacing each point z_j by as many points $x_{1j}, \ldots, x_{r_j j}$ as there are connected components in $q^{-1}(z_j)$, we obtain an (in general non-separated) smooth prevariety X as well as a factorization $q = p \circ \pi$, where $\pi : V \to X$ is $\mathbb{C}^+$-invariant with connected fibres and the "separation morphism" $p : X \to Z$ is induced by the isomorphism $\mathcal{O}(Z) \cong \mathcal{O}(X)$. We shall call $\pi : V \to X$ also the connected quotient morphism of V and Z resp. $q : V \to Z$ the separated quotient (morphism).

More precisely, X is constructed in the following manner: denote by $Z_j \subset Z$ an open neighbourhood of z_j containing none of the remaining points $z_k, k \neq j$, consider then copies $X_{ij} \cong Z_j$, $1 \leq i \leq r_j$, and glue them together along $X_{ij}^* \cong Z_j^* :=$

$Z_j \backslash \{z_j\}$. The resulting spaces X_j project onto Z_j via the map p_j, say; now identify $p_j^{-1}(Z_j \cap Z_k)$ and $p_k^{-1}(Z_j \cap Z_k)$ in the obvious manner.

1.2. LEMMA. *If the fibre $\pi^{-1}(x_0)$ of $x_0 \in X$ is reduced and $h \in \mathbf{m}_{x_0}$ is a generator of the maximal ideal $\mathbf{m}_{x_0} \subset \mathcal{O}_{X,x_0}$, then there exists a $\mu \in \mathbb{N}$ and an affine open neighbourhood U of x_0, such that $h \in \mathcal{O}(U)$ and $\pi^{-1}(U)$ is equivariantly isomorphic to $U \times \mathbb{C}$ with $\mathbb{C}^+$ acting by $t * (x, u) := (x, u + th(x)^\mu)$.*

1.3. DEFINITION. For a normal surface V with nontrivial algebraic $\mathbb{C}^+$-action $\mathbb{C} \times V \to V$ and an invariant irreducible curve $C \hookrightarrow V$ we define the "fixed point order" $\mu := \mu(C)$ as the maximal number $n \in \mathbb{N}$ such that, over the regular part of V, the velocity vectorfield associated to the $\mathbb{C}^+$-action determines a section in $\mathcal{I}_C^n \Theta$, where $\mathcal{I}_C$ denotes the ideal sheaf of C and Θ the sheaf of algebraic vectorfields.

Proof. Let us use the notation of the proof of 1.1, set $X^* := p^{-1}(Z^*)$. We choose an affine neighbourhood $U \subset X^* \cup \{x_0\}$ of x_0 such that $h \in \mathcal{O}(U)$ and x_0 is the only zero of h in U and consider its inverse image $\pi^{-1}(U) \cong \mathrm{Sp}\,(B)$.

We use for the induced derivation on B also the symbol D: Let μ be the biggest natural number n such that $D(B) \subset h^n B$, where we identify h and $h \circ \pi$. Obviously the derivation $\tilde{D} := h^{-\mu} D : B \to B$ is also locally nilpotent and thus has an associated $\mathbb{C}^+$-action $\mathbb{C}^+ \times \pi^{-1}(U) \to \pi^{-1}(U)$, $(t, v) \mapsto t \circ v$ – note that $t \circ v = (th(\pi(v))^{-\mu}) * v$ for $v \notin \pi^{-1}(x_0)$. Since $\pi^{-1}(U)$ is normal and $\pi^{-1}(x_0)$ reduced, h (or rather $h \circ \pi$) generates the ideal of the fibre $\pi^{-1}(x_0)$. Consequently $\tilde{D}(B)$ contains functions which do not vanish identically on the fibre $\pi^{-1}(x_0)$. Hence it contains or, being connected, rather equals a nontrivial orbit. In particular, the action $\circ$ is free and $\pi^{-1}(U)$ thus is smooth.

It remains to prove that there is an equivariant isomorphism $\pi^{-1}(U) \cong U \times \mathbb{C}$, where on the left hand side we consider the action "$\circ$" and on the right hand side translation on the second factor.

This is a well known fact, but because of lack of a good reference we sketch the argument: We may assume that $X = U$. It suffices to construct a section of π. Over X^* a section σ is defined by $S \hookrightarrow V^* := \pi^{-1}(X^*)$. Now choose a function $a \in A = \mathcal{O}(V)$ which restricts to a coordinate function on $\pi^{-1}(x_0) \cong \mathbb{C}$ with $y_0 \in \pi^{-1}(x_0)$ as origin, let $Y \hookrightarrow V$ denote its set of zeros on V. The condition $g(y) \circ \sigma(\pi(y)) = y$ defines a regular function $g \in \mathcal{O}(Y^*)$ with $Y^* := Y \cap V^*$. Since the fibre $\pi^{-1}(x_0)$ is reduced, $\pi|_Y$ is étale at y_0, and we have a surjection $Q(\mathcal{O}_{X,x_0}) \to Q(\hat{\mathcal{O}}_{X,x_0})/\hat{\mathcal{O}}_{X,x_0} \cong Q(\hat{\mathcal{O}}_{Y,y_0})/\hat{\mathcal{O}}_{Y,y_0}$. We may assume that a preimage b of the residue class of $g \in Q(\hat{\mathcal{O}}_{Y,y_0})$ is regular in X^*; then the section $X^* \to V^*$, $x \mapsto b(x) \circ \sigma(x)$ extends to a section on the whole of X. $\square$

Let us now consider the case that $W = V$ is a smooth affine $\mathbb{C}^+$-surface such that the morphism $\pi : W \to X$ is a submersion. Lemma 1.2 tells us what W looks like locally over X, and it remains the question, which equivariant gluing procedures of the local models yield an affine surface. This is a local problem with respect to Z; hence we may assume $s = 1$. We write $x_i = x_{i1}$, $1 \le i \le r := r_1$. Let $h \in \mathcal{O}(Z)$ be a regular function which vanishes of first order at z_1 and nowhere else. Furthermore let $\mathbb{C}^+$ act on $X_i \times \mathbb{C} \cong Z \times \mathbb{C}$ by $t * (x, u) = (x, u + th(p(x))^{\mu_i})$ with natural numbers $\mu_i \in \mathbb{N}$. Fix functions $f_{ij} \in \mathcal{O}(Z^*)$ such that the cocycle relations

$$f_{ii} = 0, \qquad f_{ik} = h^{\mu_k - \mu_j} f_{ij} + f_{jk}$$

are satisfied. Consider then

$$W = \dot{\bigcup_{i=1}^{r}} X_i \times \mathbb{C}/\sim$$

with the identification

$$X_i \times \mathbb{C} \ni (x, u) \sim (x', u') \in X_j \times \mathbb{C} \Leftrightarrow x = x' \quad \text{and} \quad u' = h(p(x))^{\mu_j - \mu_i} u + f_{ij}(p(x)).$$

1.4. PROPOSITION. *For a surface W as above the following statements are equivalent*:

(i) *W is affine.*

(ii) *W is separated.*

(iii) $n_{ij} := -\operatorname{ord}_{z_1}(f_{ij}) > 0$ *for* $i \neq j$, $1 \le i, j \le r$.

REMARK. The third condition is equivalent to the fact that none of the maps $\Psi_{ij} : X_i^* \times \mathbb{C} \to \mathbf{X}_j^* \times \mathbb{C}$, $(x, u) \mapsto (x, h(p(x))^{\mu_j - \mu_i} u + f_{ij}(p(x)))$ can be extended to a morphism $X_i \times \mathbb{C} \to \mathbf{X}_j \times \mathbb{C}$.

Proof. (i) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (iii): Suppose W is separated and $\operatorname{ord}_{z_1}(f_{ij}) \ge 0$ for some indices i and j. Then the point $(0, f_{ij}(0)) \in X_j \times \mathbb{C}$ lies in the closure $Y := \overline{X_i \times \{0\}}$; this is a contradiction, since $q|_Y : Y \to Z$ as a birational morphism onto a smooth curve is an isomorphism.

(iii) $\Rightarrow$ (i): In order to prove that W is affine we use induction on r. The case $r = 1$ being trivial we may assume $r > 1$ as well as $\mu_1 \ge \mu_2 \ge \cdots \mu_r$. Let $n := \max\{n_{i1}; 2 \le i \le r\}$. Consider the regular funcion $g \in \mathcal{O}(W)$ with

$$g(x, u) = h(p(x))^n (h(p(x))^{\mu_1 - \mu_i} u + f_{i1}(p(x))) \qquad \text{for } (x, u) \in X_i \times \mathbb{C}.$$

We have $g|_{\pi^{-1}(x_1)} \equiv 0$, and $g|_{\pi^{-1}(x_{i_0})} \equiv a$ for some $a \in \mathbb{C}^*$, if we choose i_0 such that $n = n_{i_0 1}$. It suffices to show that $g : W \to \mathbb{C}$ is an affine morphism. But this is clear, since by induction hypothesis the union of at most $r-1$ open subspaces $X_i \times \mathbb{C} \subset W$ is affine and

$$g^{-1}(\mathbb{C}^*) = \left(\bigcup_{i=2}^{r} X_i \times \mathbb{C}\right)_g$$

as well as

$$g^{-1}(\mathbb{C}\backslash\{a\}) = \left(\bigcup_{\substack{i=1 \\ i \neq i_0}}^{r} X_i \times \mathbb{C}\right)_{g-a}. \qquad \square$$

Let us for a moment assume that in addition the $\mathbb{C}^+$-action is even free. In that case $\pi : W \to X$ is a $\mathbb{C}^+$-principal bundle, and $\mathbb{C}^+$-principal bundles over X are classified by elements of the cohomology group $H^1(X, \mathcal{O})$; so for an affine base $X \cong Z$ we find $W \cong X \times \mathbb{C}$. But for a nonseparated base space the condition given in 1.4 provides us with a lot of nontrivial $\mathbb{C}^+$-surfaces. Danielewski used surfaces of this type to construct his counterexample to the Zariski-cancellation problem, i.e. he found non-isomorphic varieties W, W', such that forming their cartesian product with the complex affine line $\mathbb{C}$ one gets isomorphic varieties, cf. [1]. The following remark is basic for his examples:

1.5. REMARK. Let $\pi : W \to X, \pi' : W' \to X$ be $\mathbb{C}^+$-principal bundles with affine total spaces W, W'. Then we have an isomorphism

$$W \times \mathbb{C} \cong W' \times \mathbb{C}.$$

Proof. In the cartesian square

$$\begin{array}{ccc} W' \times_X W & \xrightarrow{\rho'} & W \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \pi} \\ W' & \xrightarrow[\pi']{} & X \end{array}$$

all occuring maps are bundle projections of $\mathbb{C}^+$-principal bundles; since W, W' are affine, we have $H^1(W, \mathcal{O}) = 0 = H^1(W', \mathcal{O})$ and thus $W \times \mathbb{C} \cong W' \times_X W \cong W' \times \mathbb{C}$. $\square$

So it remains to find an invariant by means of which we can distinguish between surfaces W and W' of the above type. That invariant will be the first homology group at infinity

$$H_1^\infty(W) := \varprojlim_{K \subset\subset W} H_1(W \backslash K)$$

which we compute in the second section, cf. Th. 2.4 and 2.5.

Our next aim is to describe $\mathbb{C}$-invariant neighbourhoods of a connected fibre $\pi^{-1}(x_0)$ of multiplicity $m \geq 2$ and fixed point order $\mu \geq 0$ of the quotient map $\pi : V \to X$. Let us first introduce the local models:

1.6. EXAMPLE. Let X be a smooth connected affine curve – so in particular X is separated here – , $x_0 \in X$ and $\psi : Y \to X$ with Y smooth a finite cyclic Galois covering of order $m \geq 2$ which is unramified over $X^* := X \backslash \{x_0\}$ and has a ramification point y_0 of order m over x_0. After removing finitely many poins $\neq x_0$ from X we may assume that Y is of the form $Y = \{(x, z) \in X \times \mathbb{C};\, z^m = b(x)\}$ with a regular function $b \in \mathcal{O}(X)$ which generates the maximal ideal $\mathbf{m}_{x_0} \subset \mathcal{O}_{X,x_0}$ and has no other zeros than x_0, set $h := pr_{\mathbb{C}}|_Y \in \mathcal{O}(Y)$. The Galois group of Y over X is the group C_m of m-th roots of unity, which acts by multiplication on the second factor, and on $Q(\mathcal{O}(Y))$ by $\varepsilon f(y) := f(\varepsilon^{-1} y)$.

For $Y^* := \psi^{-1}(X^*)$ choose a regular function $f \in \mathcal{O}(Y^*)$ of trace $\mathrm{Tr}\,(f) = 0$ in the field extension $Q(\mathcal{O}(Y)) \supset Q(\mathcal{O}(X))$. Denote by n the order of the orbit $C_m \bar{f}$ of the residue class of f in $Q(\mathcal{O}_{Y,y_0})/h^{-\mu}\mathcal{O}_{Y,y_0}$. For $Y \subset X \times \mathbb{C}$ as described above, we see, using the isomorphism

$$Q(\mathcal{O}_{Y,y_0}) \cong \bigoplus_{\nu=0}^{m-1} Q(\mathcal{O}_{X,x_0}) h^\nu$$

that for $f = \Sigma_{\nu=0}^{m-1} f_\nu h^\nu$ this means nothing but: $f_0 = 0$ and $n = m/l$ for $l := l(f, \mu) :=$ gcd $(m, \nu : m\, \mathrm{ord}_{x_0}(f_\nu) + \nu < -\mu)$.

Let $\tilde{Y}$ be the smooth prevariety obtained from Y by replacing the point y_0 by n points $y_1, \ldots, y_n$, set $Y_i := Y^* \cup \{y_i\} \subset \tilde{Y}$ with $Y^* := Y \backslash \{y_0\}$ and $\varepsilon = e^{2\pi i/m}$.

We define $W_f^\mu(\psi)$ to be the $\mathbb{C}$-principal bundle over $\tilde{Y}$ defined by the transition functions

$$f_{ij}(y) := -h(\varepsilon^{-1} y)^\mu \sum_{\lambda=i}^{j-1} f(\varepsilon^{-\lambda} y)$$

for $1 \leq i < j \leq r$ and $y \in Y_i \cup Y_j = Y^*$, i.e.

$$W_f^\mu(\psi) = \bigcup_{i=1}^{n} Y_i \times \mathbb{C}/\sim$$

with $Y_i^* \times \mathbb{C} \ni (y, u) \sim (y, u + f_{ij}(y)) \in Y_j^* \times \mathbb{C}$. The fact that the C_m-orbit of the residue class of f in $Q(\mathcal{O}_{Y,y_0})/h^{-\mu}\mathcal{O}_{Y,y_0}$ has n elements implies that for $j > i$, the function f_{ij} has a pole at y_0. So, by 1.4, the variety $W_f^\mu(\psi)$ is affine.

We endow $W_f^\mu(\psi)$ with the $\mathbb{C}^+$-action $t * (y, u) := (y, u + th(y)^\mu) \in Y_i \times \mathbb{C}$ for $(y, u) \in Y_i \times \mathbb{C}$.

Denote by $\varrho : W_f^\mu(\psi) \to Y$ the separated, by $\tilde{\varrho} : W_f^\mu(\psi) \to \tilde{Y}$ the connected quotient morphism. Now consider the automorphism

$$\varphi : W_f^\mu(\psi) = \bigcup_{i=1}^{m} Y_i \times \mathbb{C} \to W_f^\mu(\psi)$$

such that for $(y, u) \in Y_i \times \mathbb{C}$, $1 \le i < n$ with the identification $Y_i = Y = Y_j$

$$\varphi(y, u) = (\varepsilon y, \varepsilon^\mu u) \in Y_{i+1} \times \mathbb{C},$$

while for $(y, u) \in Y_n \times \mathbb{C}$ we have

$$\varphi(y, u) := \left(\varepsilon y, \varepsilon^\mu u + h(y)^\mu \sum_{\lambda=0}^{n-1} f(\varepsilon^{-\lambda} y)\right) \in Y_1 \times \mathbb{C}.$$

We remark that $h^\mu \sum_{\lambda=0}^{n-1} \varepsilon^\lambda f \in \mathcal{O}(Y)$, since $0 = \overline{\mathrm{Tr}(f)} = l \sum_{\lambda=0}^{n-1} \varepsilon^\lambda \bar{f}$ in $Q(\mathcal{O}_{Y,y_0})/h^{-\mu}\mathcal{O}_{Y,y_0}$. Furthermore note that for $(y, u) \in Y^* \times \mathbb{C} \subset Y_1 \times \mathbb{C}$ one has

$$\varphi(y, u) = (\varepsilon y, \varepsilon^\mu u + h(y)^\mu f(y)) \in Y_1 \times \mathbb{C},$$

and

$$\varphi^k(y, u) = \left(\varepsilon^k y, \varepsilon^{k\mu} u + h(\varepsilon^{k-1} y)^\mu \sum_{\lambda=0}^{k-1} f(\varepsilon^\lambda y)\right).$$

Thus, since $\mathrm{Tr}(f) = 0$, φ has order m, such that we obtain an action of C_m on $W_f^\mu(\psi)$: let $\varepsilon \in C_m$ act via the automorphism φ. This action is $\mathbb{C}^+$-equivariant, hence

$$V_f^\mu(\psi) := W_f^\mu(\psi)/C_m$$

is an affine $\mathbb{C}^+$-surface as well, and its quotient morphism is

$$\pi : V_f^\mu(\psi) \to X$$
$$[y, u]_i \mapsto \psi(y),$$

where $[y, u]_i$ denotes the orbit $C_m(y, u)$ for $(y, u) \in Y_i \times \mathbb{C}$.

Now suppose that l and μ are relatively prime. In that case C_m acts freely on $W_f^\mu(\psi)\backslash C_m(y_1, a(0)(1-\varepsilon^{n\mu})^{-1})$, where $a := \varepsilon^{1-n}h^\mu \Sigma_{\lambda=0}^{n-1}\varepsilon^{-\lambda}f \in \mathcal{O}(Y)$; hence the residue map $W_f^\mu(\psi) \to V_f^\mu(\psi)$ is étale outside a finite set, and we can conclude, that the fibre $\pi^{-1}(x_0)$ has fixed point order μ and multiplicity m: the former is obvious, since the fibres $\tilde{\varrho}^{-1}(y_i)$ have fixed point order μ, while for the multiplicity consider the coordinate function $b \in \mathbf{m}_{x_0}$ near $x_0 \in X$. Obviously the function $b \circ \psi \circ \varrho$ vanishes of order m along $\{y_i\} \times \mathbb{C}$ for $1 \le i \le n$; now, the residue map $W_f^\mu(\psi) \to V_f^\mu(\psi)$ being étale outside a finite set, it follows that $b \circ \pi$ has order m along $\pi^{-1}(x_0)$.

Note that under the assumption $(\mu, l) = 1$ the surface $V_f^\mu(\psi)$ is smooth iff $l = 1$; and this is in particular the case for $\mu \in m\mathbb{Z}$. On the other hand, for $l > 1$ there is exactly one singular pont in $V_f^\mu(\psi)$, cf. also [6].

1.7. THEOREM. *Let V be a connected normal affine $\mathbb{C}^+$-surface with connected quotient morphism $\pi : V \to X$, and $x_0 \in X$ a point, such that the fibre $\pi^{-1}(x_0)$ has multiplicity $m \ge 2$ and fixed point order $\mu \ge 0$. Fix a neighbourhood U of x_0, such that there is an equivariant isomorphism $\pi^{-1}(U^*) \cong U^* \times \mathbb{C}$ with $U^* := U\backslash\{x_0\}$, together with a finite cyclic Galois covering $\psi : Y \to U$ of order m as in* 1.6. *Then there is a regular function $f \in \mathcal{O}(Y^*)$, where $Y^* := \psi^{-1}(U^*)$, of trace* $\mathrm{Tr}(f) = 0$, *such that $\pi^{-1}(U) \cong V_f^\mu(\psi)$ and μ is coprime to the order $l(f, \mu)$ of the isotropy group of the residue class $\bar{f}$ of f in $Q(\mathcal{O}_{Y,y_0})/\mathbf{m}_{y_0}^{-\mu}$ (where $\mathbf{m}_{y_0}^{-\mu} := h^{-\mu}\mathcal{O}_{Y,y_0}$ for a generator h of the maximal ideal $\mathbf{m}_{y_0} \subset \mathcal{O}_{Y,y_0}$).*

Furthermore we have $V_f^\mu(\psi) \cong V_{f'}^\mu(\psi)$ if and only if the residue classes of f and f' in $Q(\mathcal{O}_{Y,y_0})/\mathbf{m}_{y_0}^{-\mu}$ are conjugate under the action of C_m.

Proof. We may assume $U = X$ and fix $b \in \mathbf{m}_{x_0}$ and $h \in \mathbf{m}_{y_0}$ with $h^m = b$ as in 1.6.

Consider the fibre product $Y \times_X V$. At a generic point of $\{y_0\} \times \pi^{-1}(x_0)$ it decomposes into m analytic branches. The projection pr_Y restricts to a submersion on each of these branches, and the group C_m acts transitively on them. Consequently the normalization W of the reduction of $Y \times_X V$ is a normal affine $\mathbb{C}^+$-surface with a connected quotient $\tilde{Y}$, which is obtained from Y by replacing y_0 by n points $y_1, \ldots, y_n$, and these points, with respect to the induced action of C_m on $\tilde{Y}$, form one orbit. Thus $n \mid m$, and we may assume $\varepsilon y_i = y_{i+1}$ for $1 \le i \le n$ with $y_{n+1} := y_1$. From the above considerations we can also conclude, that there are only finitely many non-principal orbits, i.e., orbits with less than m elements; so the quotient morphism $W \to W/C_m \cong V$ is outside a finite set étale; in particular, if $\tilde{\varrho} : W \to \tilde{Y}$ denotes the connected quotient morphism, the fibres $\tilde{\varrho}^{-1}(y_i)$ have fixed point order μ. On the other hand $\tilde{\varrho}$ has only reduced fibres; so, if we set $Y_i := \tilde{Y}^* \cup \{y_i\} \cong Y$, there is by Lemma 1.2, for $h \in \mathcal{O}(Y)$ as in 1.6, a trivialization

$\tau : Y_1 \times \mathbb{C} \cong \mathrm{Y} \times \mathbb{C} \to \tilde{\varrho}^{-1}(Y_1)$, which is $\mathbb{C}^+$-equivariant if we consider on $Y \times \mathbb{C}$ the action $t * (y, u) := (y, u + th(y)^\mu)$. Since $\tilde{\varrho}^{-1}(Y_1^*)$ is C_m-invariant, we obtain via the trivialization τ on $Y^* \times \mathbb{C}$ a C_m-action commuting with the $\mathbb{C}^+$-action, which thus is necessarily of the form $\varepsilon(y, u) = (\varepsilon y, \varepsilon^\mu u + h(y)^\mu f(y)) =: \varphi(y, u)$ with a regular function $f \in \mathcal{O}(Y^*)$.

Now consider the trivializations $\tau_i : Y_i \times \mathbb{C} \to \tilde{\varrho}^{-1}(Y_i)$ defined by $\tau_i(y, u) := \varepsilon^{i-1}\tau(\varepsilon^{1-i}y, \varepsilon^{\mu(1-i)}u)$. Using these trivializations we find

$$W = \bigcup_{i=1}^{n} Y_i \times \mathbb{C}/\sim,$$

where $Y_i^* \times \mathbb{C} \ni (y, u) \sim (y, u + f_{ij}(y)) \in Y_j^* \times \mathbb{C}$ with

$$f_{ij}(y) := -h(\varepsilon^{-1}y)^\mu \sum_{\lambda=i}^{j-1} f(\varepsilon^{-\lambda}y)$$

for $1 \leq i < j \leq n$. Now by a reasoning as in 1.6 we find that $\operatorname{Tr}(f) = 0$ and the C_m-orbit of the residue class of f in $Q(\mathcal{O}_{Y,y_0})/h^{-\mu}\mathcal{O}_{Y,y_0}$ has n elements. Note that the fact that the natural map $W \to V$ is étale outside a finite set implies $(\mu, l) = 1$. Thus we finally arrive at an isomorphism $V \cong V_f^\mu(\psi)$.

Now assume that the residue classes of f and f' in $Q(\mathcal{O}_{Y,y_0})/h^{-\mu}\mathcal{O}_{Y,y_0}$ are conjugate under the action of C_m. Evidently it suffices to discuss the two cases $f' - f \in h^{-\mu}\mathcal{O}_{Y,y_0}$ and $f' = \varepsilon f$. In the first case, since $\operatorname{Tr}(f' - f) = 0$, we find a function $g \in h^{-\mu}\mathcal{O}(Y)$ with $f'(y) - f(y) = g(\varepsilon y) - g(y)$ for $y \in Y^*$. Then the map

$$W_f^\mu(\psi) = \bigcup_{i=1}^{n} Y_i \times \mathbb{C} \to W_{f'}^\mu(\psi) = \bigcup_{i=1}^{n} Y_i \times \mathbb{C},$$

$$W_f^\mu(\psi) \supset Y_i \times \mathbb{C} \ni (y, u) \mapsto (y, u + h(\varepsilon^{-1}y)^\mu g(\varepsilon^{1-i}y)) \in Y_i \times \mathbb{C} \subset W_{f'}^\mu(\psi)$$

induces an isomorphism $V_f^\mu(\psi) \cong V_{f'}^\mu(\psi)$.

Secondly, if $f' = \varepsilon f$, then we can apply the map with

$$W_f^\mu(\psi) \supset Y_i \times \mathbb{C} \ni (y, u) \mapsto (\varepsilon y, \varepsilon^\mu u) \in Y_i \times \mathbb{C} \subset W_{f'}^\mu(\psi),$$

for $1 \leq i \leq n$, which again provides an isomorphism $V_f^\mu(\psi) \cong V_{f'}^\mu(\psi)$.

On the other side every isomorphism $V_f^\mu(\psi) \cong V_{f'}^\mu(\psi)$ lifts to an isomorphism

$$\vartheta : W_f^\mu(\psi) \xrightarrow{\cong} W_{f'}^\mu(\psi)$$

by the naturality of the normalization of the reduction of the fibre product with Y. Now it is not difficult to see that ϑ is a composition of a morphims of the above type and the action of a suitable power of ε either on $W_f^\mu(\psi)$ or on $W_{f'}^\mu(\psi)$; and this yields easily the reverse direction of the equivalence. □

The next step in order to achieve a global classification of normal affine $\mathbb{C}^+$-surfaces is to describe the germs of invariant neighbourhoods of a fibre $q^{-1}(z_j)$ of the separated quotient morphism $q := p \circ \pi : V \to Z$. For this it is enough to consider the case where the separation morphism $p : X \to Z$ has only one fibre $p^{-1}(z_0) = \{x_1, \ldots, x_r\}$ of order $r > 1$. Again we use the notation $Z^* = Z \setminus \{z_0\}$, $X^* = p^{-1}(Z^*)$ and $X_i = X^* \cup \{x_i\}$. A generalization of 1.4 is the following

1.8. THEOREM. *Let V_i be affine $\mathbb{C}^+$-surfaces with connected quotient morphisms $\pi_i : V_i \to X_i \cong Z$, such that there are equivariant isomorphisms $\Psi_i : V_i^* := \pi_i^{-1}(X^*) \xrightarrow{\cong} X^* \times \mathbb{C}$ (with $\mathbb{C}^+$ acting by translation on the second factor) and V be the result of gluing the V_i, $1 \le i \le r$, over X^* via the maps $\Psi_j^{-1} \circ \Psi_i : V_i^* \to V_j^*$. Then V is affine if and only if for no two different indices i, j the transition isomorphism $\Psi_j^{-1} \circ \Psi_i : V_i^* \to V_j^*$ extends to a morphism $V_i \to V_j$.*

REMARK. Note that, if the fixed point orders μ_i and μ_j of the central fibres $\pi_i^{-1}(z_0)$ resp. $\pi_j^{-1}(z_0)$ coincide, then every extension of $\Psi_j^{-1} \circ \Psi_i$ is necessarily an isomorphism. Hence our condition is satisfied if the V_i are pairwise non-isomorphic with the same fixed point orders μ_i of $\pi_i^{-1}(z_0)$.

Proof. As in 1.4 the nontrivial part is to show that the condition is sufficient: Denote by m_i the multiplicity of the fibre $\pi_i^{-1}(z_0)$ and by μ_i its fixed point order. As above choose a function $b \in \mathcal{O}(Z)$ which generates the maximal ideal in the local ring $\mathcal{O}_{Z,z_0}$ and has no other zeros than z_0; let $Y := \{(z, \zeta) \in Z \times \mathbb{C}; \zeta^m = b(z)\}$ with $m := \operatorname{lcm}(m_1, \ldots, m_r)$ and $Y^i := \{(z, \zeta) \in Z \times \mathbb{C}; \zeta^{m_i} = b(z)\}$, denote by $\psi : Y \to Z$ and $\psi_i : Y^i \to Z$ the morphisms $\psi(z, \zeta) = z$ resp. $\psi_i(z, \zeta) = z$. Then according to Th. 1.7 there is a representation $V_i \cong W_i / C_{m_i}$ where W_i is an affine $\mathbb{C}^+$-surface lying submersively over its separated quotient Y^i. We shall construct a global representation $V \cong W / C_m$, where W is an affine $\mathbb{C}^+$-surface with separated quotient Y. We have

$$W_i \cong W_{f_i}^{\mu_i}(\psi_i) = \bigcup_{k=1}^{n_i} Y_k^i \times \mathbb{C},$$

where the Y_k^i, $1 \le k \le n_i$, are copies of Y^i, n_i is the order of the orbit $C_{m_i} \bar{f}_i \subset Q(\mathcal{O}_{Y^i,y_0}) / h_i^{-\mu_i} \mathcal{O}_{Y^i,y_0}$ for the regular function f_i on $(Y^i)^*$ and $h_i := pr_{\mathbb{C}}|_{Y^i}$. Denote by $\vartheta_i : Y \to Y^i$ the covering $\vartheta_i(z, \zeta) = (z, \zeta^{\lambda_i})$ with $\lambda_i := m/m_i$.

Consider now $\tilde{W}_i := \vartheta_i^*(W_i) := Y \times_{Y^i} W_i = \bigcup_{k=1}^{n_i} Y_k \times \mathbb{C}$ with copies Y_k, $1 \le k \le n_i$, of Y. In this situation the group C_m acts on $\tilde{W}_i$ such that $\varepsilon := e^{2\pi i/m}$ induces on $\tilde{W}_i$ the fibre product of the maps $Y \to Y$, $y \mapsto \varepsilon y$ and $\varphi_i\colon W_i \to W_i$, which both cover the transformation $Y^i \to Y^i$, $y \mapsto \varepsilon^{\lambda_i} y$. In local coordinates it is given by

$$Y_k \times \mathbb{C} \ni (y, u) \mapsto (\varepsilon y, \varepsilon^{\mu_i \lambda_i} u) \in Y_{k+1} \times \mathbb{C},$$

for $1 \le k < n_i$ and

$$Y_{n_i} \times \mathbb{C} \ni (y, u) \mapsto \left(\varepsilon y, \varepsilon^{\mu_i \lambda_i} u + h_i(\vartheta_i(y))^{\mu_i} \sum_{\lambda=0}^{n_i - 1} f_i(\vartheta_i(\varepsilon^{-\lambda} y)) \right) \in Y_1 \times \mathbb{C},$$

while on $\tilde{W}_i^* = Y_1^* \times \mathbb{C} \cong Y^* \times \mathbb{C}$ this action is nothing but

$$Y_i^* \times \mathbb{C} \ni (y, u) \mapsto (\varepsilon y, \varepsilon^{\mu_i \lambda_i} u + h_i(\vartheta_i(y))^{\mu_i} f_i(\vartheta_i(y))) \in Y_1^* \times \mathbb{C}.$$

Then we have $V_i \cong \tilde{W}_i / C_m$. Now let us try to cover the trivializations $\Psi_i : V_i^* \to X^* \times \mathbb{C} \cong Z^* \times \mathbb{C}$ by C_m-equivariant trivializations $\Phi_i : \tilde{W}_i^* \to Y^* \times \mathbb{C}$ where C_m acts only on Y^*. To that end choose $g_i \in \mathcal{O}(Y^{i*})$ such that $f_i(y) = g_i(\varepsilon^{\lambda_i} y) - g_i(y)$ for $y \in Y^{i*}$ – this is possible (at least after shrinking Z a little bit) with an argument analogous to that in the proof of Theorem 1.7. Then the map

$$\Phi_i : Y_1^* \times \mathbb{C} \to Y^* \times \mathbb{C}$$
$$(y, u) \mapsto (y, h_i(\vartheta_i(y))^{-\mu_i} u - g_i(\vartheta_i(y)))$$

intertwines the two actions of C_m.

Now since $g_i \in \mathcal{O}(Y^{i*})$ is determined only up to a pull back of a regular function on Z^*, we can choose the Φ_i in order to cover the given trivializations Ψ_i.

Then patch together the W_i, $1 \le i \le r$, to W via the respective Φ_i and Φ_j:

$$\tilde{W}_i \supset \tilde{W}_i^* = Y_1^* \times \mathbb{C} \xrightarrow{\Phi_i} Y^* \times \mathbb{C} \xleftarrow{\Phi_j} Y_1^* \times \mathbb{C} = \tilde{W}_j^* \subset \tilde{W}_j.$$

We want to show that W is affine. As a consequence of 1.4 and the remark thereafter it is enough to show that no map

$$\Phi_j^{-1} \circ \Phi_i : \tilde{W}_i^* \to \tilde{W}_j^*$$

extends to a morphism

$$\tilde{W}_i \supset Y_k \times \mathbb{C} \to Y_l \times \mathbb{C} \subset \tilde{W}_j$$

for some k, l with $1 \leq k \leq n_i$ and $1 \leq l \leq n_j$.

Using the action of C_m and the fact that $C_m \cdot Y_k \times \mathbb{C} = \tilde{W}_i$ we may extend it once more to a morphism $\tilde{W}_i \hookrightarrow \tilde{W}_j$. Obviously this extension respects both the action of $\mathbb{C}$ and C_m and thus induces a morphism $V_i \to V_j$ contrary to our hypothesis. □

As a consequence of the vanishing of the cohomology group $H^1(Z, \mathcal{O})$ we have eventually:

1.9. THEOREM. *Let V be a normal affine $\mathbb{C}^+$-surface, denote by $q : V \to Z$ the separated quotient morphism, let $z_1, \ldots, z_s$ be the points in Z, near which q is not equivariantly locally trivial. Then V is determined up to equivariant isomorphism over Z by the germs $V_1, \ldots, V_s$ of invariant neighbourhoods of the exceptional fibres $\pi^{-1}(z_j)$, $1 \leq j \leq s$.*

On the other hand for every finite set of points $z_1, \ldots, z_s \in Z$ and prescribed germs V_j, $1 \leq j \leq s$, of invariant neighbourhoods, there is an affine $\mathbb{C}^+$-surface realizing these data and being locally trivial elsewhere. □

2. Equivariant compactification and homology for smooth surfaces

Let $\bar{Z}$ denote the smooth projective closure of the smooth affine curve Z, fix a line bundle L on $\bar{Z}$ together with a nontrivial section $\sigma : \bar{Z} \to L$.

We use σ in order to define a $\mathbb{C}^+$-action on L:

$$\mathbb{C}^+ \times L \ni (t, x) \mapsto t * x := x + t\sigma(pr_L(x)) \in L;$$

and this action extends to the projectivization $M := \mathbb{P}(L \times \mathbb{C})$ of the line bundle L, which is obtained from $L \cong \mathbb{P}(L \times \mathbb{C}^*)$ by adding the section at infinity $\mathbb{P}(L^* \times \{0\})$, where L^* is L with the zero section removed. The fixed point set $M^{\mathbb{C}^+}$ is the union of the section at infinity and $f^{-1}(N_\sigma)$, where $f : M \to \bar{Z}$ is the projection of the $\mathbb{P}_1$-bundle M over $\bar{Z}$ and N_σ denotes the zero set of σ. Note that the invariant curve $f^{-1}(z)$ has fixed point order $\text{ord}_z(\sigma)$.

Now an algebraic $\mathbb{C}^+$-action on a complex algebraic surface carries over to its blow up in a fixed point. Let $\tilde{M}$ be the result of successively applying this procedure to M, with the restriction, that the "modified points" (i.e. which have a positive dimensional fibre with respect to the modification map $\tilde{M} \to M$) are contained in

$L^{\mathbb{C}^+} = L \cap f^{-1}(N_\sigma)$. In order to control the above process consider a $\mathbb{C}^+$-equivariant modification $\varphi : M_0 \to M$ of the above type. To each irreducible component D_i of $(f \circ \varphi)^{-1}(N_\sigma)$ we can associate three numbers: its self intersection number $a_i := D_i^2 \in \mathbb{Z}_{\leq 0}$, the multiplicity $m_i \in \mathbb{N}_{\geq 1}$ of D_i as irreducible component of a fibre of the morphism $f \circ \varphi : M_0 \to \bar{Z}$, and the fixed point order $\mu_i \in \mathbb{N}$.

Suppose that M_0 contains no isolated fixed points and consider the blow up $\varphi_0 : M_1 \to M_0$ of M_0 in a fixed point x_0. It is contained in an irreducible component D_1 of $(f \circ \varphi)^{-1}(N_\sigma)$ with fixed point order $\mu_1 > 0$. If x_0 is not a crossing point of irreducible components of $(f \circ \varphi)^{-1}(N_\sigma)$, then for $D_2 := \varphi_0^{-1}(x_0)$ we have $a_2 = -1$, $m_2 = m_1$ and $\mu_2 = \mu_1 - 1$. Note that for $\mu_1 = 1$ and the strict transform $\tilde{D}_1$ of D_1 the difference $D_2 \backslash \tilde{D}_1$ is one orbit. As data for $\tilde{D}_1$ we find $\tilde{a}_1 = a_1 - 1$, $\tilde{m}_1 = m_1$ as well as $\tilde{\mu}_1 = \mu_1$.

If x_0 is contained in two irreducible components, say D_1 and D_2, of $(f \circ \varphi)^{-1}(N_\sigma)$ and $D_3 = \varphi_0^{-1}(x_0)$, then $a_3 = -1$, $m_3 = m_1 + m_2$ and $\mu_3 = \mu_1 + \mu_2$, while for the strict transforms $\tilde{D}_i$ of the D_i, $i = 1, 2$, we have again $\tilde{a}_i = a_i - 1$, $\tilde{m}_i = m_i$ and $\tilde{\mu}_i = \mu_i$. Note that no isolated fixed points have been created in M_1, so we may go on with M_1 instead of M_0.

2.1. THEOREM. *Let V be a smooth affine $\mathbb{C}^+$-surface and $Z := \mathrm{Sp}\,(\mathcal{O}(V)^{\mathbb{C}^+})$ its separated quotient. Then V admits a smooth $\mathbb{C}^+$-equivariant compactification $\bar{V} \cong \tilde{M}$, where $\tilde{M}$ is an equivariant modification of a $\mathbb{C}^+$-surface $M = \mathbb{P}(L \times \mathbb{C})$ of the above type, and the divisor at infinity $D := \bar{V} \backslash V$ contains all irreducible components of $\varphi^{-1}(\mathbb{P}(L \times \{0\}) \cup f^{-1}(N_\sigma))$, which are not terminal in the dual graph of that system of curves.*

Proof. Choose a function $f \in \mathrm{Ker}\,(D^2) \backslash \mathrm{Ker}\,(D)$ as in the proof of Lemma 1.1, and define $a \in \mathcal{O}(Z)$ to be the regular function on Z with $a \circ q = Df$. Then the zero set N_a includes the points $z_1, \ldots, z_s \in Z$ with exceptional fibre $q^{-1}(z_j)$, i.e. $q^{-1}(z_j)$ is either unconnected or has multiplicity > 1 or consists entirely of fixed points.

Now consider the line bundle $L := \mathcal{O}_D$ over $\bar{Z}$ for the divisor $D := \Sigma_{j=1}^{s} \mathrm{ord}_{z_j}(a) z_j$ together with the section $\sigma \in \mathcal{O}_D(\bar{Z}) \subset \mathcal{M}(\bar{Z})$ corresponding to the rational function $\equiv 1$. Then the equivariant map $V_{Df} \to L \cap f^{-1}(Z), v \mapsto f(v)(\sigma(q(v))/a(q(v)))$ extends to $V_0 := q^{-1}(Z_0)$, where $Z_0 := Z_a \cup \{z_1, \ldots, z_s\}$; and as a consequence of the vanishing of $H^1(Z, \mathcal{O})$ there is a function $b \in \mathcal{O}(Z_0)$ such that $v \mapsto (b(q(v)) + f(v))(\sigma(q(v))/a(q(v)))$ extends even to an equivariant morphism $g : V \to L \subset M$, which restricts to an isomorphism $g^{-1}(Z^*) \xrightarrow{\cong} L \cap f^{-1}(Z^*)$ for $Z^* := Z \backslash \{z_1, \ldots, z_s\}$.

Let us now turn to the construction of an equivariant modification $\varphi : \tilde{M} \to M$, such that g factors through $\tilde{M}$ via an open embedding $\tilde{g} : V \to \tilde{M}$. The centres of the sequence of blow ups φ is composed of will lie over $L \cap f^{-1}(N_\sigma)$; so we may replace M with $f^{-1}(Z) \cap L$ and since the problem then is local with respect to the separated quotient Z, we may assume that there is only one exceptional fibre $q^{-1}(z_0)$ and V has a representation $V \cong W/C_m$ as in the proof of Th. 1.8, where W is a smooth affine $\mathbb{C}^+$-surface without multiple fibres and with a separated quotient Y which can be realized as an m-sheeted cyclic Galois cover $\psi : Y \to Z$ having only one ramification point y_0, situated above z_0 and of order m.

Let us first consider the case $m = 1$, i.e. $V = W$. We have $W = \bigcup_{i=1}^r X_i \times \mathbb{C}/\sim$ as in the discussion preceding Prop. 1.4 and may assume $f^{-1}(Z) \cap L \cong Z \times \mathbb{C}$ where $t \in \mathbb{C}^+$ acts by $t * (z, u) := (z, u + th(z)^n)$. Then we have $g(x, u) = (p(x), h(p(x))^{n-\mu_i}u + b_i(p(x)))$ for $(x, u) \in X_i \times \mathbb{C}$ and functions $b_i \in \mathcal{O}(Z)$ satisfying the relations

$$b_i = h^{n-\mu_j} f_{ij} + b_j,$$

where of course $\mu_j \leq n$ for $1 \leq j \leq r$. We have the following two possibilities: Either $n = \mu_j$ for some j, in which case $f_{ij} = b_i - b_j \in \mathcal{O}(Z)$ implies $r = 1$, cf. Prop. 1.4, and g already is an isomorphism, or $n > \mu_j$ for every j: Then $g(q^{-1}(z_0))$ is finite and we have $g(\{x_i\} \times \mathbb{C}) = g(\{x_j\} \times \mathbb{C})$ if and only if $n - \mu_j - n_{ij} > 0$ with $n_{ij} := -\mathrm{ord}_{z_0}(f_{ij})$. Denote by $\beta_1 : B_1 \to B_0 := Z \times \mathbb{C}$ the blow up of B_0 in all the points of $g(q^{-1}(z_0))$, let F be the strict transform of $\{z_0\} \times \mathbb{C}$ in B_1 and $E_i := \beta_1^{-1}(g(\{x_i\} \times \mathbb{C}))$, the exceptional fibre over $g(\{x_i\} \times \mathbb{C})$, set $B_1^i := B_1 \backslash F \cup E_1 \cup \cdots \cup \hat{E}_i \cup \cdots \cup E_r$. Since the pull back of the reduced ideal sheaf of $g(q^{-1}(z_0))$ is an invertible sheaf, the morphism $g_0 := g : W \to B_0 = Z \times \mathbb{C}$ lifts to a morphism $g_1 : W \to B_1$ with $g_1(X_i \times \mathbb{C}) \subset B_1^i$ for $1 \leq i \leq r$.

Now an easy computation shows $B_1^i \cong Z \times \mathbb{C}$ equivariantly with the action $t * (z, u) = (z, u + th(z)^{n-1})$ on $Z \times \mathbb{C}$, so we come across the same alternative: either $g_1|_{g_1^{-1}(B_1^i)} : g_1^{-1}(B_1^i) \to B_1^i$ is an isomorphism or we may apply the same procedure as previously. Doing this where ever it is possible we obtain a second blow up $B_2 \to B_1$ such that g_1 factors through a morphism $g_2 : W \to B_2$. After at least n steps this process becomes stationary, and $g_n : W \to B_n$ is an open embedding.

Let us mention some details we will need later:

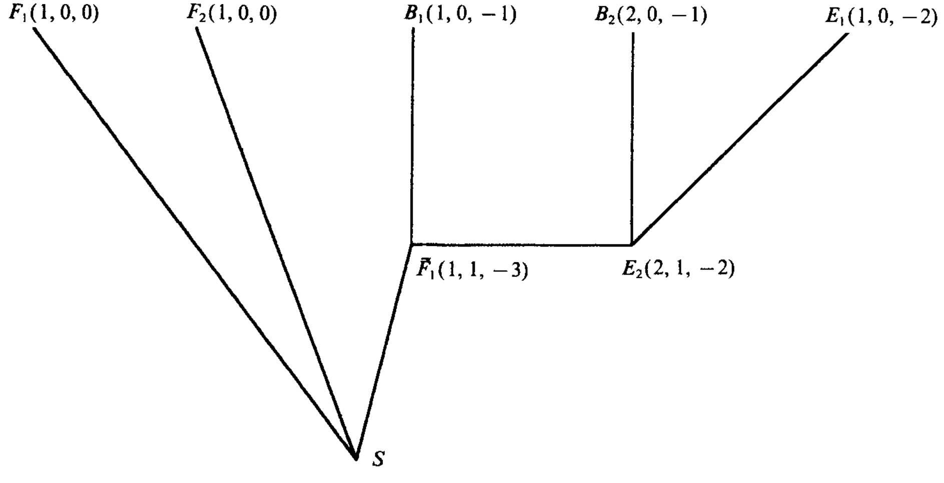
The images $g_k(\{x_i\} \times \mathbb{C})$ and $g_k(\{x_j\} \times \mathbb{C})$ are either equal (iff $k < n - \mu_j - n_{ij}$) or disjoint (iff $k \geq n - \mu_j - n_{ij}$); and $g_k(\{x_i\} \times \mathbb{C})$ is a curve iff $k \geq n - \mu_i$ iff $g_k|_{X_i \times \mathbb{C}}$ is an open embedding. Consequently the dual graph of the system of irreducible curves lying over $\{z_0\} \times \mathbb{C}$ is a tree emanating from e_0, the vertex corresponding to F, the strict transform of $\{z_0\} \times \mathbb{C}$, and having the vertices e_i, $1 \leq i \leq r$, corresponding to the closures $\overline{g_n(\{x_i\} \times \mathbb{C})}$ as terminal points. Note that the path from e_i to e_0 consists of $n - \mu_i$ edges and after exactly $n_{ji} = n_{ij} - \mu_i + \mu_j$ edges it joins the path from e_j to e_0.

Let us now deal with the general case: The morphism $g : V \to Z \times \mathbb{C}$ induces a C_m-equivariant morphism $\hat{g} : W \to Y \times \mathbb{C}$, which is the composition of $\psi^*(g) : \hat{V} := Y \times_Z V \to Y \times \mathbb{C} \cong Y \times_Z (Z \times \mathbb{C})$ and the reduction-normalization morphism $W \to \hat{V}$. Now let us carry out the above construction for $\hat{g} : W \to Y \times \mathbb{C}$. We obtain modifications $\hat{\beta}_k : \hat{B}_k \to Y \times \mathbb{C}$ and liftings $\hat{g}_k : W \to \hat{B}_k$. The blow ups $\hat{B}_k$ inherit a natural C_m-action, and the morphisms $\hat{g}_k$ are necessarily C_m-equivariant. Now take the quotient mod C_m of the final step $\hat{g}_n : W \to \hat{B}_n$ and obtain thus an open embedding $g_n : V \cong W/C_m \to B_n := \hat{B}_n/C_m$, where B_n in every case is a normal analytic or rather algebraic space. Since V is smooth, the singular points of B_n lie outside $g_n(V)$ and being isolated they are fixed point of the $\mathbb{C}^+$-action. Now choose $\tilde{B}$ as the minimal $\mathbb{C}^+$-equivariant resolution of B_n, and take $\tilde{g} : V \to \tilde{B}$ to be the lifting of g_n. The composed morphism $\tilde{B} \to B_n \to Z \times \mathbb{C}$ is as a modification of smooth surfaces a sequence of blow ups, and from the construction it is clear that $\tilde{B} \backslash \tilde{g}(V)$ consists of all non-terminal irreducible curves lying over $\{z_0\} \times \mathbb{C}$ together with the terminal curves in the linear subchains of the dual graph of that system of curves which result from resolving the singularities of B_n. □

Let us return to the general situation we started with in the beginning of this section. Denote by $\bar{V} = \tilde{M}$ an equivariant compactification of the above type and let B_{ij}, $1 \le i \le r_j$, $1 \le j \le s$ denote the closures in $\bar{V}$ of the irreducible curves in V above $z_j \in Z$, $1 \le j \le s$. Let S be the strict transform with respect to $\varphi : \bar{V} = \tilde{M} \to \mathbb{P}(L \times \mathbb{C})$ of the section at infinity $\mathbb{P}(L \times \{0\})$, $F_1, \ldots, F_l$ the fibres of points in $\bar{Z} \backslash Z$ with respect to the map $f \circ \varphi : \bar{V} \to \bar{Z}$, while $\tilde{F}_j$ is the strict transform of $f^{-1}(z_j)$ in $\bar{V}$.

Let us denote by E_{kj}, $k \in I_j$, the irreducible components of $(f \circ \varphi)^{-1}(z_j)$ different from $\tilde{F}_j$ and the B_{ij}, $1 \le i \le r_j$.

The following diagram shows the weighted dual graph of $(\bar{V} \backslash V) \cup \bigcup_{i,j} B_{ij}$ for a surface V with one exceptional fibre $q^{-1}(z_1) = (B_1 \cap V) \cup (B_2 \cap V)$, $B_i := B_{i1}$ and separated quotient $Z = \mathbb{C}^*$. The triples (m, μ, a) indicate the multiplicity, fixed point order and self intersection number of the corresponding curve.



Let us now as a first application compute the singular homology of V: Denote by m_{ij} resp. n_{kj} the multiplicity of B_{ij} resp. E_{kj} in the fibre $(f \circ \varphi)^{-1}(z_j)$, set $m_j := \gcd(m_{1j}, \ldots, m_{r_j j})$.

2.2. THEOREM. *With the above notation the integral singular homology groups of a connected smooth affine $\mathbb{C}^+$-surface V are given by*

$$H_q(V) \cong \begin{cases} \mathbb{Z}, & q = 0; \\ H_1(Z) \oplus \bigoplus_{j=1}^{s} \mathbb{Z}_{m_j}, & q = 1; \\ \mathbb{Z}^r \text{ with } r = \Sigma_{j=1}^{s} (r_j - 1), & q = 2; \\ 0, & q > 2. \end{cases}$$

The following corollary is a generalization of a result of Rentschler, which describes algebraic $\mathbb{C}^+$-actions on the affine plane, cf. [5]:

2.3. COROLLARY. *Every acyclic affine $\mathbb{C}^+$-surface V is equivariantly isomorphic to $\mathbb{C}^2$ endowed with an action $t * (z, w) = (z, w + p(z)t)$ for some nonzero polynomial $p(z) \in \mathbb{C}[z]$.*

Proof. The vanishing of $H_2(V)$ yields that $q : V \to Z$ has connected fibres, while $H_1(V) = 0$ means that they all have multiplicity 1. On the other hand, $H_1(Z) = 0$ implies $Z \cong \mathbb{C}$. Thus, by 1.2 and 1.9, V is determined up to equivariant isomorphism by the fixed point orders μ_j of the fibres $q^{-1}(z_j)$, $1 \le j \le s$. Hence V is as given with $p(z) := \Pi_{j=1}^{s} (z - z_j)^{\mu_j}$. □

Proof of 2.3. Since V is Stein, we have $H_q(V) = \{0\}$ for $q > 2$. Denote by $D := \bar{V} \setminus V$ the divisor at infinity.

Relative Poincaré duality applied to the pair $(\bar{V}, D)$ yields $H_q(V) \cong H^{4-q}(\bar{V}, D)$; so we have to consider the following part of the long exact cohomology sequence of $(\bar{V}, D)$:

$$H^2(\bar{V}, D) \hookrightarrow H^2(\bar{V}) \to H^2(D) \to H^3(\bar{V}, D) \to H^3(\bar{V}) \to 0,$$

where we have used $H^1(\bar{V}) \cong H^1(D)$ and $H^3(D) = \{0\}$.

Furthermore $H^2(\bar{V}) \cong H_2(\bar{V})^*$ as well as $H^2(D) \cong H_2(D)^*$; and $H_2(\bar{V})$ is freely generated by the homology classes of the curves S, F_1, E_{kj}, $k \in I_j$, B_{ij}, $1 \le i \le r_j$, $1 \le j \le s$, while $H_2(D)$ has as a base the homology classes of $S, F_1, \ldots, F_l$, $\tilde{F}_1, \ldots, \tilde{F}_s$, E_{kj}, $k \in I_j$, $1 \le j \le s$.

Now replace $[\tilde{F}_j] \in H_2(D)$ with $\xi_j := [\tilde{F}_j] - [F_1] + \Sigma_{k \in I_j} n_{kj}[E_{kj}]$ in order to obtain a new base of $H_2(D)$. Then for the image $\alpha^* \in H_2(D)^*$ of a linear form $\alpha \in H_2(\bar{V})^*$

we find

$$\alpha^*([S]) = \alpha([S]), \qquad \alpha^*([F_k]) = \alpha([F_1]),$$

$$\alpha^*(\xi_j) = -\sum_{i=1}^{r_j} m_{ij}\alpha([B_{ij}]), \qquad \alpha^*([E_{kj}]) = \alpha([E_{kj}]),$$

where the third row is a consequence of the fact that the fibre F_1 is homologous in $\bar{V}$ to $\tilde{F}_j + \Sigma_{k \in I_j} n_{kj} E_{kj} + \Sigma_{i=1}^{r_j} m_{ij} B_{ij}$.

Since $H^3(\bar{V}) \cong H_1(\bar{V}) \cong H_1(\bar{Z})$ is free, we thus obtain

$$H^3(\bar{V}, D) \cong H_1(\bar{Z}) \oplus \mathbb{Z}^{l-1} \oplus \bigoplus_{j=1}^{s} \mathbb{Z}_{m_j} \cong H_1(Z) \oplus \bigoplus_{j=1}^{s} \mathbb{Z}_{m_j},$$

and $H^2(\bar{V}, D)$ is free of rank $\operatorname{rk} H^2(\bar{V}) - (\operatorname{rk} H^2(D) - l + 1) = \Sigma_{j=1}^{s} r_j - s$. □

For the computation of the first homology group at infinity we recall some general facts: For a "good" neighbourhood (with respect to the complex topology) U of the divisor at infinity $D = \bigcup_{k=1}^{n} D_k$ we have $H_1^\infty(V) = H_1(U \backslash D)$, and D is a strong deformation retract of U.

Thus the exact sequence

$$\begin{array}{ccccccccc}
H_2(U) & \longrightarrow & H_2(U, U\backslash D) & \longrightarrow & H_1(U\backslash D) & \longrightarrow & H_1(U) & \longrightarrow & H_1(U, U\backslash D) \\
\cong & & \cong & & & & \cong & & \cong \\
H_2(D) & & H_2(\bar{V}, \bar{V}\backslash D) & & & & H_1(D) & & H_1(\bar{V}, \bar{V}\backslash D) \\
 & & \cong & & & & \cong & & \cong \\
 & & H^2(D) & & & & H_1(\bar{Z}) & & H^3(D) = \{0\} \\
 & & \cong & & & & & & \\
 & & H_2(D)^* & & & & & &
\end{array}$$

together with the fact, that the first homomorphism identifies elements of $H_2(D)$ with linear forms on it using the intersection product on $\bar{V}$, leads to the isomorphism

$$H_1^\infty(V) \cong H_1(\bar{Z}) \oplus \mathbb{Z}^n / \langle (D_k \cdot D_l) \rangle,$$

where for a matrix $A \in \mathbb{Z}^{(n,n)}$ we denote by $\langle A \rangle$ the submodule of $\mathbb{Z}^n$ generated by the row vectors of the matrix A. Let A_j denote the intersection matrix of $[\tilde{F}_j]; [E_{kj}], k \in I_j$ and arrange a base of $H_2(D)$ in the form $[S], [F_1], \ldots, [F_l]$, $[\tilde{F}_1], [E_{k1}], k \in I_1, \ldots, [\tilde{F}_s], [E_{ks}], k \in I_s$.

With respect to that base the intersection matrix $\langle (D_k \cdot D_l) \rangle$ takes the form

$$\left(\begin{array}{cccc|cccc|cccc|c}
a & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \\
1 & & & & 0 & & & & 0 & & & & \cdots \\
\vdots & & 0 & & \vdots & & 0 & & \vdots & & 0 & & \\
1 & & & & 0 & & & & 0 & & & & \\
\hline
1 & 0 & \cdots & 0 & & & & & & & & & \\
0 & & & & & & A_1 & & & & 0 & & \cdots \\
\vdots & & 0 & & & & & & & & & & \\
0 & & & & & & & & & & & & \\
\hline
1 & 0 & \cdots & 0 & & & & & & & & & \\
0 & & & & & & 0 & & & & A_2 & & \cdots \\
\vdots & & 0 & & & & & & & & & & \\
0 & & & & & & & & & & & & \\
\hline
 & & \vdots & & & & \vdots & & & & 0 & & \ddots
\end{array}\right),$$

where $a = S \cdot S = -\Sigma_{j=1}^{s} \operatorname{ord}_{z_j}(\sigma)$ is the self-intersection number of S.

Now we can easily prove

2.4. THEOREM. *Every connected affine $\mathbb{C}^+$surface V is connected at infinity and its first homology group at infinity is of the form*

$$H_1^{\infty}(V) \cong H_1(Z) \oplus \bigoplus_{j=1}^{s} T_j$$

with a torsion module T_j associated to every fibre $q^{-1}(z_j)$, $1 \leq j \leq s$, near which the separated quotient $q : V \to Z$ is not an equivariant product.

Proof. V is connected at infinity, since the divisor at infinity $D = \bar{V} \backslash V$ for an equivariant compactification as above is connected.

An easy exercise in linear algebra using the shape of the intersection matrix given above shows that

$$\mathbb{Z}^n / \langle (D_k \cdot D_l) \rangle \cong \mathbb{Z}^{l-1} \oplus \bigoplus_{j=1}^{s} \mathbb{Z}^{\operatorname{rk} A_j} / \langle A_j \rangle;$$

since $H_1(Z) \cong H_1(\bar{Z}) \oplus \mathbb{Z}^{l-1}$, it remains to show that $T_j := \mathbb{Z}^{\operatorname{rk} A_j} / \langle A_j \rangle$ is a torsion module for $1 \leq j \leq s$.

Consider for fixed j the submodule $M \subset H_2(\bar{V})$ generated by the (linearly independent) homology classes of $\tilde{F}_j$; E_{kj}, $k \in I_j$; B_{ij}, $1 \le i \le r_j$. We have $M = \mathbb{Z}\xi \oplus M_0$, where $\xi = [\tilde{F}_j + \Sigma_{k \in I_j} n_{kj} E_{kj} + \Sigma_{i=1}^{r_j} m_{ij} B_{ij}]$ and M_0 is generated by the $[E_{kj}]$, $k \in I_j$ and $[B_{ij}]$, $1 \le i \le r_j$.

The intersection form is negative definite on M_0 and $\xi \cdot M = 0$. So the intersection form is negative definite on every submodule M_1 of M, which does not contain a non-zero multiple of ξ. That applies in particular to the submodule M_1 generated by $[\tilde{F}_j]$; $[E_{kj}]$, $k \in I_j$. Since A_j is the associated intersection matrix, the claim follows immediately. □

Finally we want to compute the torsion module $T = T_j$ more explicitly in case that all the components $B_{ij} \cap V$, $1 \le i \le r := r_j$, of the fibre $q^{-1}(z_j)$ have multiplicity $m_{ij} = 1$. Let $B_i := B_{ij}$ and $p^{-1}(z_j) = \{x_i := x_{ij};\ 1 \le i \le r\}$.

For a first discussion of T we deal with the general case of arbitrary multiplicities and consider the tree in the dual graph of $D \cup B_1 \cup \cdots \cup B_r$ emanating from $\tilde{F} = \tilde{F}_j$.

Let $(f \circ \varphi)^{-1}(z_j) = \tilde{F} \cup B_1 \cup \cdots \cup B_r \cup E_{r+1} \cup \cdots \cup E_q$ and denote by $e_0, e_1, \ldots, e_r, e_{r+1}, \ldots, e_q$ the corresponding vertices in that tree.

As in the introduction denote by a_i the self-intersection number of the curve represented by e_i (so a_i is the weight of the vertex e_i in the weighted dual graph of the fibre $(f \circ \varphi)^{-1}(z_j) \hookrightarrow \bar{V}$), by m_i its multiplicity as irreducible component of the fibre $(f \circ \varphi)^{-1}(z_j) \hookrightarrow \bar{V}$ and by μ_i its fixed point order. For $k \in I := \{0, \ldots, q\}$ let $I_k := \{i \in I \backslash \{k\};\ e_i$ and e_k are the common end points of an edge$\}$, and set

$$L := \bigoplus_{i=0}^{q} \mathbb{Z} e_i,$$

$$v_k := a_k e_k + \sum_{i \in I_k} e_i \in L$$

for $k \in I$. Then $T \cong L/L_1$ with the submodule

$$L_1 := \bigoplus_{i=1}^{r} \mathbb{Z} e_i \oplus \mathbb{Z} v_0 \oplus \bigoplus_{i=r+1}^{q} \mathbb{Z} v_i.$$

Furthermore, the fact that $[F_1] \cdot [C] = 0$ for every irreducible component C of the fibre $(f \circ \varphi)^{-1}(z_j)$ together with the homology $F_1 \sim \tilde{F} + \Sigma_{i=1}^{r} m_i B_i + \Sigma_{i=r+1}^{q} m_i E_i$ yields the following relation for the self intersection numbers a_i:

$$m_k a_k + \sum_{i \in I_k} m_i = 0.$$

We turn now to the special situation that the fibres $\pi^{-1}(x_i) = B_i \cap V$, $1 \le i \le r$, are reduced. Then by the construction of $\tilde{M}$ according to the proof of Th. 2.1 all multiplicities m_i equal 1, so the weight a_k of e_k is up to sign the valency of the vertex e_k in the dual graph.

Let us write $i \succeq j$ for $i, j \in I$, iff e_j lies on the (unique) path from e_i to e_0, and for $i \in I \setminus \{0\}$ denote by $i(1)$ the unique index such that $e_{i(1)}$ is the immediate successor of e_i on that path; define $i(v)$ by induction: $i(v+1) = i(v)(1)$ so far as it makes sense. Denote by $a_{ij} \in \mathbb{N}$ the number of edges in the path from e_i to e_0 one has to pass before reaching the junction point with the path from e_j to e_0.

If $V = W$ is of the form

$$W = \dot{\bigcup_{i=1}^{r}} X_i \times \mathbb{C}/\sim$$

with the identification

$$X_i \times \mathbb{C} \ni (x, u) \sim (x', u') \in X_j \times \mathbb{C} \Leftrightarrow x = x' \quad \text{and} \quad u' = h(p(x))^{\mu_j - \mu_i} u + f_{ij}(p(x))$$

as in Prop. 1.4, then, according to the proof of Th. 2.1, we have

$$a_{ij} = n_{ji} = n_{ij} - \mu_i + \mu_j$$

for $i \neq j$. Using that notation we arrive eventually at

2.5. THEOREM. *If z_j is a regular value of q, then the torsion module $T = T_j$ is of the form $T \cong \mathbb{Z}^{r+1}/\langle A \rangle$ with the matrix*

$$A = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & a_{kl} & \\ 1 & & & \end{pmatrix};$$

in particular, if $a_{kl} = a \in \mathbb{N}_{\geq 1}$ for every pair (k, l) with $k \neq l$, there is an isomorphism

$$T \cong \mathbb{Z}_{ra} \oplus \mathbb{Z}_a^{r-2}.$$

Proof. We consider the homomorphism

$$\psi : \mathbb{Z}^{r+1} \to L/L_1,$$

$$(c_0, \ldots, c_r) \mapsto c_0 e_0 + \sum_{i=1}^{r} c_i v_i + L_1;$$

since ψ obviously is onto, it suffices to prove $\operatorname{Ker}(\psi) = \langle A \rangle$.

For $L_0 := \{\Sigma_{i \in I} c_i e_i; \Sigma_{i \in I} c_i = 0\}$ we have $L = \mathbb{Z}e_0 \oplus L_0$ and $L_0 = \Sigma_{k \in I} \mathbb{Z}v_k$, where the only relation for the generators v_k is $\Sigma_{k \in I} v_k = 0$. So we find $(0, 1, \ldots, 1) \in \operatorname{Ker}(\psi)$. Furthermore for $i \in I \backslash \{0\}$ we have

$$e_{i(1)} - e_i = \sum_{k \succeq i} v_k,$$

such that for $i \in \{1, \ldots, r\}$ one finds with n as in the proof of Th. 2.1:

$$\begin{aligned} e_0 - e_i &= \sum_{\nu=0}^{n-\mu_i-1} e_{i(\nu+1)} - e_{i(\nu)} \\ &= \sum_{\nu=0}^{n-\mu_i-1} \left(\sum_{k \succeq i(\nu)} v_k \right) \\ &\equiv \sum_{k=1}^{r} (n - \mu_i - a_{ik}) v_k \mod L_1 \\ &\equiv -\sum_{k=1}^{r} a_{ik} v_k \mod L_1; \end{aligned}$$

since $e_i \in L_1$ for $1 \le i \le r$, this gives $(1, a_{i1}, \ldots, a_{ir}) \in \operatorname{Ker}(\psi)$.

Now let us turn to the other inclusion $\operatorname{Ker}(\psi) \subset \langle A \rangle$: For $(c_0, \ldots, c_r) \in \operatorname{Ker}(\psi)$ there exist $\alpha_0, \ldots, \alpha_q \in \mathbb{Z}$ with

$$c_0 e_0 + \sum_{k=1}^{r} c_k v_k = \alpha_0 v_0 + \sum_{i=1}^{r} \alpha_i e_i + \sum_{k=r+1}^{q} \alpha_k v_k$$

or equivalently

$$c_0 e_0 - \sum_{i=1}^{r} \alpha_i e_i = \alpha_0 v_0 - \sum_{k=1}^{r} c_k v_k + \sum_{k=r+1}^{q} \alpha_k v_k,$$

where the coefficients $\alpha_0, -c_1, \ldots, -c_r, \alpha_{r+1}, \ldots, \alpha_q$ are determined by the left hand side up to a common summand. On the other hand, since the right hand side is in L_0, we have $c_0 - \Sigma_{i=1}^{r} \alpha_i = 0$; so we can write

$$\begin{aligned} c_0 e_0 - \sum_{i=1}^{r} \alpha_i e_i &= \sum_{i=1}^{r} \alpha_i (e_0 - e_i) \\ &= \sum_{i=1}^{r} \alpha_i \left(\sum_{k=1}^{r} (n - \mu_i - a_{ik}) v_k \right) + \tilde{\alpha}_0 v_0 + \sum_{i=r+1}^{q} \tilde{\alpha}_i v_i \\ &= \sum_{k=1}^{r} \left(\sum_{i=1}^{r} \alpha_i (n - \mu_i - a_{ik}) v_k \right) + \tilde{\alpha}_0 v_0 + \sum_{i=r+1}^{q} \tilde{\alpha}_i v_i, \end{aligned}$$

so by comparing coefficients one finds

$$c_k = \sum_{i=1}^{r} \alpha_i(a_{ik} + \mu_i - n) + \alpha$$

for some $\alpha \in \mathbb{Z}$ and $1 \leq k \leq r$. In the whole

$$(c_0, \ldots, c_r) = \sum_{i=1}^{r} \alpha_i(1, a_{i1}, \ldots, a_{ir}) + \lambda(0, 1, \ldots, 1) \in \langle A \rangle$$

with $\lambda = \alpha + \Sigma_{i=1}^{r} \alpha_i(\mu_i - n)$. The explicit formula for the case $a_{kl} = a$ for every $k, l \in \{1, \ldots, r\}, k \neq l$ now follows easily with elementary methods of linear algebra. □

Addendum: While proofreading I learnt about the papers of J. Bertin, which are closely related to our subject; they are listed in the references without numbering.

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