The spherical derivative of integral and meromorphic functions

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1. Introduction

In a recent paper LEHTO and VIRTANEN [2] introduced the spherical derivative

$$\varrho(f(z)) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

as a measure of the growth of $f(z)$ near an isolated singularity. This point of view was further pursued by LEHTO [1]. If the singularity is taken to be at $z = \infty$ then LEHTO obtained the following results.

Theorem A. *Suppose that $f(z)$ is meromorphic for $R < |z| < \infty$, and has an essential singularity at $z = \infty$. Then*

$$\limsup_{z \to \infty} |z| \varrho(f(z)) \geq \tfrac{1}{2}. \tag{1.2}$$

Equality holds for functions of the form

$$f(z) = \prod_{1}^{\infty} \frac{a_\nu - z}{a_\nu + z}, \tag{1.3}$$

where a_ν is a sequence of complex numbers such that

$$\left|\frac{a_{\nu+1}}{a_\nu}\right| \to \infty \quad (\nu \to \infty). \tag{1.4}$$

Theorem B. *If $f(z)$ satisfies the hypotheses of Theorem A and in addition $f(z)$ is regular near $z = \infty$, then (1.2) can be replaced by*

$$\limsup_{z \to \infty} |z| \varrho(f(z)) = \infty. \tag{1.5}$$

Following LEHTO, we denote by $h(r)$ a positive function such that $h(r) = o(r)$ $(r \to \infty)$. The connection between $\varrho(f(z))$ and PICARD's Theorem is strikingly brought out by the following result of LEHTO [1].

Theorem C. *Let $f(z)$ be meromorphic for $R < |z| < \infty$. If for a sequence* $\{z_\nu\}$, $\lim_{\nu \to \infty} z_\nu = \infty$ *and*

$$\lim_{\nu \to \infty} h(|z_\nu|)\varrho(f(z_\nu)) = \infty \tag{1.6}$$

then Picard's *Theorem holds for $f(z)$ in the union of any infinite subsequence of the discs*

$$C_\nu = \{z : |z - z_\nu| < \epsilon h(|z_\nu|)\} \tag{1.7}$$

for each $\epsilon > 0$.

Conversely if there exist discs (1.7) such that Picard's *Theorem is true in every union $\bigcup_{k=1}^{\infty} C_{\nu_k}$ for every $\epsilon > 0$ then (1.6) is satisfied.* (*V. Gavrilov has pointed out to us that the converse must be modified here. (1.6) is satisfied for a sequence z'_ν instead of z_ν, where $|z'_\nu - z_\nu| = o\{h(|z_\nu|)\}$. This condition is also sufficient for the existence of the disks (1.7)*).

In particular it follows that if $f(z)$ has an essential singularity at $z = \infty$ then $f(z)$ possesses a Julia direction provided that

$$\limsup_{z \to \infty} |z|\varrho(f(z)) = \infty. \tag{1.8}$$

From Theorem B we see that every transcendental integral function possesses a Julia direction. If (1.8) is not satisfied there is not, in general, a Julia direction as the examples (1.3) show if $a_\nu > 0$.

2. Some further results for meromorphic functions

Our aim in this paper is to obtain some extensions of Theorems A and B. We may suppose without loss of generality that $f(z)$ is meromorphic in the whole plane. First we consider whether or not a restriction on the growth of $f(z)$ as defined by its order imposes any restriction on $\varrho(f(z))$, or conversely. For meromorphic functions no restriction on $\varrho(f(z))$ is implied by a restriction on the growth of the characteristic $T(r, f)$. Consider, for instance,

$$f(z) = \frac{\prod_1^{\infty}(1 - z/a_n)}{\prod_1^{\infty}(1 - z/b_n)}$$

where $\Sigma|a_n|^{-1}$, $\Sigma|b_n|^{-1}$ converge. Since $f(a_n) = 0$, $f(b_n) = \infty$ it follows that

$$\int \varrho(f(z))\,|dz| \geq \pi,$$

where the integral is taken along the segment Γ_n joining a_n to b_n. In particular

$$\varrho(f(z_n)) \geq \frac{\pi}{|b_n - a_n|}$$

for some point z_n on Γ_n. By choosing a_n, b_n close enough together we can make the right hand side bigger than any preassigned function of $|z_n|$.

On the other hand a result in the opposite direction is possible. It is convenient to set

$$\mu(r, f) = \sup_{|z| = r} \varrho(f(z)).$$

Suppose that for $r > r_0$ we have

$$\mu(r, f) < K r^{\sigma}. \tag{2.1}$$

By Theorem A this is only possible when $\sigma > -1$ or when $\sigma = -1$ and $K \geq \frac{1}{2}$. In the usual notation of NEVANLINNA Theory,

$$T_0(r, f) = \int_0^r \frac{S(t, f)}{t} dt$$

where

$$S(r, f) = \frac{1}{\pi} \int_0^r \int_0^{2\pi} \varrho^2(f(te^{i\varphi}))\, t dt d\varphi$$

$$\leq 2 \int_0^r \mu^2(t, f)\, t dt.$$

Thus if $\sigma = -1$ in (2.1),

$$S(r, f) = O(\log r),\ T_0(r, f) = O(\log^2 r). \tag{2.2}$$

The examples (1.3) with $a_\nu = A^\nu (A > 1)$ show that the order of magnitude in (2.2) cannot be sharpened.

If (2.1) is satisfied with $\sigma > -1$ we obtain

$$S(r, f) = O(r^{2\sigma+2}),\ T_0(r, f) = O(r^{2\sigma+2}). \tag{2.3}$$

Hence a meromorphic function of proper order $k > 0$ cannot satisfy (2.1) for any $\sigma < \frac{k}{2} - 1$. The implication from (2.1) to (2.3) is sharp as our first theorem shows.

Theorem 1. *Suppose that* $0 < \lambda < \infty$ *and that*

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n^{\lambda n} - z^n}. \tag{2.4}$$

Then $f(z)$ *has perfectly regular growth of order* $2/\lambda$ *and satisfies* (2.1) *with* $\sigma = \frac{1}{\lambda} - 1$.

The function $f(z)$ has poles at the points $z = n^\lambda e^{\frac{2\nu\pi i}{n}}$ ($\nu = 0, 1, \ldots, n-1$; $n \geq 1$). The number of poles in $|z| \leq r$ is $\frac{1}{2}p(p+1)$ where p is the largest integer such that $p^\lambda \leq r$, i.e. $p = [r^{1/\lambda}]$. Thus $n(r, f)$, the number of poles of $f(z)$ in $z \leq r$, satisfies

$$n(r, f) \sim \tfrac{1}{2}p^2 \sim \tfrac{1}{2} r^{2/\lambda} \, (r \to \infty),$$

and so

$$N(r, f) = \int_0^r \frac{n(t, f)}{t} \, dt \sim \frac{\lambda}{4} r^{2/\lambda} (r \to \infty). \tag{2.5}$$

We now estimate $|f(z)|$. Assume that

$$(p - \tfrac{3}{4})^\lambda \leq |z| \leq (p + \tfrac{3}{4})^\lambda, \tag{2.6}$$

where p is a positive integer. $A(\lambda)$ denotes a positive constant depending only on λ and is not necessarily the same at each occurrence. Let n be an integer satisfying $n > p$ and put $n = p + \nu$ so that $\nu \geq 1$. We have, in the range (2.6),

$$\left|\frac{z}{n^\lambda}\right|^n \leq \left(\frac{n - \nu + \frac{3}{4}}{n}\right)^{\lambda n} = \left\{1 - \frac{(\nu - \frac{3}{4})}{n}\right\}^{\lambda n}$$

$$\leq e^{-(\nu - \frac{3}{4})\lambda}.$$

Hence, when z lies in the range (2.6),

$$\left|\sum_{n=p+1}^{\infty} \frac{(-1)^n z^n}{n^{\lambda n} - z^n}\right| \leq \sum_{\nu=1}^{\infty} \frac{e^{-(\nu - \frac{3}{4})\lambda}}{1 - e^{-(\nu - \frac{3}{4})\lambda}} = A(\lambda). \tag{2.7}$$

When $1 \leq n < p$ and z lies in the range (2.6) then, if $n = p - \nu$ with $\nu \geq 1$,

$$\left|\frac{z}{n^\lambda}\right|^n \geq \left(\frac{n + \nu - \frac{3}{4}}{n}\right)^{\lambda n} \geq \left(1 + \frac{\nu - \frac{3}{4}}{n}\right)^{\lambda n}$$

$$\geq \left(1 + \frac{\nu - \frac{3}{4}}{k}\right)^{\lambda k} (n \geq k). \tag{2.8}$$

Now

$$\frac{(-1)^n z^n}{n^{\lambda n} - z^n} = (-1)^{n+1} + \frac{(-1)^n n^{\lambda n}}{n^{\lambda n} - z^n}$$

and so if we choose k in (2.8) to be $\left[\frac{2}{\lambda}\right] + 1$ so that $\lambda k > 2$, assuming that $p > \left[\frac{2}{\lambda}\right] + 1$, we find that in the range (2.6)

$$\left|\sum_{n=1}^{p-1} \frac{(-1)^n z^n}{n^{\lambda n} - z^n}\right| \leq 1 + \left|\sum_{n=1}^{p-1} \frac{(-1)^n n^{\lambda n}}{n^{\lambda n} - z^n}\right|$$

$$\leq 1 + \sum_{n=1}^{k-1} \frac{1}{\left(\frac{|z|}{n}\right)^{\lambda n} - 1} + \sum_{\nu=1}^{\infty} \frac{1}{\left(1 + \frac{\nu - \frac{3}{4}}{k}\right)^2 - 1} = A(\lambda).$$

From this and (2.7) we obtain

$$\left|f(z) - \frac{(-1)^p z^p}{p^{\lambda p} - z^p}\right| \leq A(\lambda) \tag{2.9}$$

in the range (2.6) for $p > \left[\frac{2}{\lambda}\right] + 1$. It is easy to see that consequently (2.9) holds in the range (2.6) for $p \geq 1$.

If $|z| = t$ and (2.6) is satisfied then using (2.9) we see, in the notation of NEVANLINNA Theory, that

$$\begin{aligned} m(t, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(te^{i\vartheta})|\, d\vartheta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|\frac{t^p}{p^{\lambda p} - t^p e^{ip\vartheta}}\right| d\vartheta + A(\lambda) \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|\frac{1}{\sin p\vartheta}\right| d\vartheta + A(\lambda) \\ &= A(\lambda). \end{aligned}$$

From this and (2.5) we deduce that

$$T(r, f) = m(r, f) + N(r, f) \sim \frac{\lambda}{4} r^{2/\lambda}, \ (r \to \infty)$$

so that $f(z)$ is of perfectly regular growth, order $\frac{2}{\lambda}$ and type $\frac{\lambda}{4}$.

It remains to be proved that $f(z)$ satisfies (2.1) with $\sigma = \frac{1}{\lambda} - 1$.

We have

$$\begin{aligned} f'(z) &= \sum_{n=1}^{\infty} (-1)^n \frac{n^{\lambda n + 1} z^{n-1}}{(n^{\lambda n} - z^n)^2} \\ &= (-1)^p \frac{p^{\lambda p + 1} z^{p-1}}{(p^{\lambda p} - z^p)^2} + f_p'(z), \quad \text{say,} \end{aligned}$$

where $f_p(z)$ is defined by the series for $f(z)$ with the pth term omitted. Now, by the above, $f_p(z)$ is regular and bounded by $A(\lambda)$ in $(p - 3/4)^\lambda \leq |z| \leq (p + 3/4)^\lambda$

and each point in $(p - 1/2)^\lambda \le |z| \le (p + 1/2)^\lambda$ is the centre of a disc which lies in the larger annulus with radius $\dfrac{p^{\lambda-1}}{A(\lambda)}$. Hence, from Cauchy's integral,

$$|f_p'(z)| \le A(\lambda) p^{1-\lambda} < A(\lambda) |z|^{1/\lambda - 1},$$

for

$$(p - 1/2)^\lambda \le |z| \le (p + 1/2)^\lambda \qquad (p \ge 1). \tag{2.10}$$

Therefore in the range (2.10),

$$\begin{aligned} |f'(z)| &\le \left| \frac{p^{\lambda p+1} z^{p-1}}{(p^{\lambda p} - z^p)^2} \right| + A(\lambda) |z|^{\frac{1}{\lambda} - 1} \\ &= \frac{p^{\lambda p+1}}{|z|^{p+1}} \left| \left(\frac{z^p}{p^{\lambda p} - z^p} \right)^2 \right| + A(\lambda) |z|^{\frac{1}{\lambda} - 1} \\ &\le A(\lambda) \frac{p^{\lambda p+1}}{|z|^{p+1}} (1 + |f(z)|^2) + A(\lambda) |z|^{\frac{1}{\lambda} - 1} \end{aligned}$$

by (2.9). Consequently, in the range (2.10),

$$\begin{aligned} \frac{|f'(z)|}{1 + |f(z)|^2} &\le A(\lambda) \frac{p}{|z|} + A(\lambda) |z|^{1/\lambda - 1} \\ &< A(\lambda) |z|^{1/\lambda - 1}. \end{aligned}$$

Since the ranges (2.10) cover all the plane apart from a disc, the proof of the theorem is complete.

3. Positive theorems for integral functions

The remainder of the paper will be devoted to obtaining improvements of Theorem B and to showing that these are best possible. We assume without loss of generality that $f(z)$ is an integral function. It will also be assumed that $f(z)$ is always transcendental. In this section we state our positive theorems.

Theorem 2. *If $f(z)$ is an integral function of proper order σ $(0 \le \sigma \le \infty)$, then*

$$\limsup_{r \to \infty} \frac{r \mu(r, f)}{\log M(r, f)} \ge A_0(\sigma + 1), \tag{3.1}$$

where A_0 is an absolute constant. In particular

$$\limsup_{r\to\infty} \frac{r\,\mu(r,f)}{\log r} = \infty. \tag{3.2}$$

Inequality (3.2) sharpens (1.5) which is equivalent to

$$\limsup_{r\to\infty} r\,\mu(r,f) = \infty.$$

Theorem 3. *If $f(z)$ is an integral function satisfying (2.1) for all large r with $-1 < \sigma < \infty$, then for large r*

$$\log M(r,f) < \frac{A_1 K}{\sigma+1}\, r^{\sigma+1}, \tag{3.3}$$

where $A_1 = 25e\ \log 2$.

It follows from (1.5) that the restriction $\sigma > -1$ is necessary in Theorem 3. The theorem shows that for integral functions (2.1) implies that

$$T(r,f) = O(r^{\sigma+1}).$$

This is significantly stronger than (2.3) which is the best possible result for meromorphic functions by Theorem 1. Note that if $f(z)$ is of perfectly regular growth then Theorem 3 is a consequence of Theorem 2.

As we shall see later, if $f(z)$ is an integral function such that the growth of $\log M(r,f)$ is properly of the order of $\log^2 r$ in the sense that

$$0 < \limsup_{r\to\infty} \frac{\log M(r,f)}{\log^2 r} < \infty,$$

then no improvement of (3.2) is possible. On the other hand our next theorems show that if $\log M(r,f) \neq O(\log^2 r)$ or $\log M(r,f) = o(\log^2 r)$ then we can improve (3.2), the improvement depending on how large or how small $\dfrac{\log M(r,f)}{\log^2 r}$ becomes respectively. However, there is no sharp difference in the behaviour of $\mu(r,f)$ as we pass from one of the above classes of functions to another. By this we mean that if $\varphi(r) \to \infty (r \to \infty)$, then there is an $f(z)$ from each of the above classes such that

$$\limsup_{r\to\infty} \frac{r\,\mu(r,f)}{\varphi(r)\log r} < \infty.$$

Before stating our next theorem we give an indication of how one arrives at an improvement of (3.2) if $\log M(r,f) \neq O(\log^K r)$ for K suitably large. If

$\mu(r, f) < K \dfrac{\log^2 r}{r}$ for large r then, from the inequality involving $T_0(r, f)$ and $\mu(r, f)$ in § 2, it follows that

$$T_0(r, f) = O(\log^6 r).$$

Hence if $\log M(r, f) \neq O(\log^6 r)$ we see that (3.2) can be improved to

$$\limsup_{r \to \infty} \frac{r\mu(r, f)}{\log^2 r} = \infty.$$

Our next result gives the improvement of (3.2) for functions $f(z)$ such that $\log M(r, f) \neq O(\log^2 r)$, but $\log M(r, f) = O(\log^6 r)$.

Theorem 4. *If $f(z)$ is an integral function and $\varphi(r) \nearrow \infty$ $(r \nearrow \infty)$ and*

$$\limsup_{r \to \infty} \frac{\log M(r, f)}{\varphi(r) \log^\alpha r} > 0, \ \log M(r, f) = O(\log^{\alpha+1} r), \tag{3.4}$$

where $2 \leq \alpha < \infty$, then

$$\limsup_{r \to \infty} \frac{r\mu(r, f)}{\varphi(r) \log^{\alpha-1} r} > 0. \tag{3.5}$$

When $\alpha = 2$ in (3.4) then (3.5) is the improved form of (3.2). For functions such that $\log M(r, f) \neq O(\log^3 r)$, $\log M(r, f) = O(\log^6 r)$ take $\varphi(r) = \{\log(r+1)\}^{1/2}$ and choose α so that both conditions (3.4) are satisfied and $\alpha \geq 2 \cdot 5$. The improved form of (3.2) is then

$$\limsup_{r \to \infty} \frac{r\mu(r, f)}{(\log r)^2} > 0.$$

To deal with functions such that $\log M(r, f) = o(\log^2 r)$ we have the following result.

Theorem 5. *If $\varphi(r)$ is increasing and $f(z)$ is an integral function such that*

$$\log M(r, f) = O\left\{\frac{\log^2 r}{\varphi(r)}\right\} \ (r \to \infty) \tag{3.6}$$

then

$$\limsup_{r \to \infty} \frac{r\mu(r, f)}{\varphi(r) \log r} = \infty. \tag{3.7}$$

4. Proofs of the positive theorems

4.1. We require a number of preliminary lemmas.

Lemma 1. *Let* $f(z) = a_0 + a_1(z - z_0) + \dots$ *be regular in* $|z - z_0| \le \delta$ *and satisfy* $|f(z)| \ge 1$ *there. Then*

$$|a_1| \le \frac{2|a_0| \log |a_0|}{\delta}\,, \tag{4.1}$$

and for $|z - z_0| \le r < \delta$

$$|a_0|^{\frac{\delta-r}{\delta+r}} \le |f(z)| \le a_0^{\frac{\delta+r}{\delta-r}}. \tag{4.2}$$

If further $|f(z_1)| = 1$ *for some* z_1 *with* $|z_1 - z_0| = \delta$ *then for some* z *on the segment joining* z_0 *to* z_1

$$\varrho(f(z)) \ge \frac{\log |a_0|}{10\delta \log 2} \ge \frac{|a_1|}{20\,|a_0| \log 2}\,. \tag{4.3}$$

(4.1) and (4.2) are classical.

Suppose that

$$|f(z_0 + \delta e^{i\varphi})| = 1 \quad (z_1 = z_0 + \delta e^{i\varphi}).$$

If

$$|f(z_0 + \varrho e^{i\varphi})| \le 2 \quad (0 \le \varrho \le \delta) \tag{4.4}$$

then $|a_0| \le 2$ and

$$|a_0| - 1 \le |f(z_0 + \delta e^{i\varphi}) - f(z_0)| \le \int_0^\delta |f'(z_0 + te^{i\varphi})|\, dt$$

$$\le \delta \max_{0 \le t \le \delta} |f'(z_0 + te^{i\varphi})|.$$

If $\zeta = z_0 + t_0 e^{i\varphi}$ is a point where the maximum on the right is attained then,

$$|f'(\zeta)| \ge \frac{|a_0| - 1}{\delta} \ge \frac{\log |a_0|}{\delta}$$

and so

$$\varrho(f(\zeta)) = \frac{|f'(\zeta)|}{1 + |f(\zeta)|^2} \ge \frac{|f'(\zeta)|}{5} \ge \frac{\log |a_0|}{5\,\delta}\,.$$

Hence the first inequality of (4.3) is true in this case.

If (4.4) is false let ϱ be the largest number with $0 \le \varrho < \delta$ such that $|f(z_0 + \varrho e^{i\varphi})| = 2$. Take $\zeta = z_0 + t_1 e^{i\varphi}$ to be a point for which $|f'(z)|$ is greatest when $z = z_0 + te^{i\varphi} (\varrho \le t \le \delta)$. Then $|f(\zeta)| \le 2$ and so

$$\frac{|f'(\zeta)|}{1 + |f(\zeta)|^2} \ge \frac{|f'(\zeta)|}{5}\,.$$

Also

$$1 \le |f(z_0 + \delta e^{i\varphi}) - f(z_0 + \varrho e^{i\varphi})| \le \int_{\varrho}^{\delta} |f'(z_0 + te^{i\varphi})|\, dt$$
$$\le (\delta - \varrho)\, |f'(\zeta)|.$$

Further, by (4.2) and the fact that $|f(z_0 + \varrho e^{i\varphi})| = 2$, we have

$$|a_0|^{\frac{\delta-\varrho}{\delta+\varrho}} \le 2,$$

and hence

$$\delta - \varrho \le \frac{(\delta + \varrho) \log 2}{\log |a_0|} \le \frac{2\delta \log 2}{\log |a_0|}.$$

From the above it follows that

$$\varrho(f(\zeta)) = \frac{|f'(\zeta)|}{1 + |f(\zeta)|^2} \ge \frac{|f'(\zeta)|}{5} \ge \frac{1}{5(\delta - \varrho)} \ge \frac{\log |a_0|}{10\delta \log 2}.$$

This completes the proof of the first inequality of (4.3). The second follows immediately from (4.1).

Lemma 2. *Suppose that $f(z)$ is an integral function such that for some $r_1 > 0$*

$$\min_{|z| = r_1} |f(z)| = 1, \tag{4.5}$$

and that

$$|f(z)| > 1 \ (r_1 < |z| < 3r_1). \tag{4.6}$$

Then for some r satisfying $r_1 < r < 2r_1$ we have

$$\mu(r, f) > \frac{e^{-4\pi} \log M(r, f)}{10 r \log 2}. \tag{4.7}$$

In particular if the conditions are satisfied for arbitrarily large r_1 then,

$$\limsup_{r \to \infty} \frac{r \mu(r, f)}{\log M(r, f)} \ge \frac{e^{-4\pi}}{10 \log 2}. \tag{4.8}$$

Let $r_0 = 2r_1$ and let $z_0 = r_0 e^{i\vartheta_0}$ be such that

$$|f(z_0)| = M(r_0, f).$$

There is a ϑ_1 with $|\vartheta_1 - \vartheta_0| \le \pi$ such that

$$|f(r_1 e^{i\vartheta_1})| = 1.$$

For each ζ, with $|\zeta| = r_0$, $|f(z)| > 1$ for $|z - \zeta| < r_1 = \frac{r_0}{2}$ and so (4.1) gives

$$\frac{|f'(\zeta)|}{|f(\zeta)| \log |f(\zeta)|} \leq \frac{4}{r_0}.$$

Thus

$$\left| \frac{\partial}{\partial \vartheta} \log \log |f(r_0 e^{i\vartheta})| \right| \leq 4$$

and so

$$\left| \log \frac{\log |f(r_0 e^{i\vartheta_1})|}{\log |f(r_0 e^{i\vartheta_0})|} \right| \leq 4\pi,$$

from which it follows that

$$\log |f(r_0 e^{i\vartheta_1})| \geq e^{-4\pi} \log |f(r_0 e^{i\vartheta_0})| = e^{-4\pi} \log M(r_0, f).$$

In the closed disc $|z - r_0 e^{i\vartheta_0}| \leq \frac{r_0}{2}$ we have $|f(z)| \geq 1$ and, at the point $z_1 = r_1 e^{i\vartheta_1}$ on the boundary, $|f(z_1)| = 1$. Consequently, by (4.3) with $\delta = \frac{r_0}{2}$, there is a point ξ on the segment joining $r_0 e^{i\vartheta_1}$ to z_1 for which

$$\varrho(f(\xi)) \geq \frac{\log |f(r_0 e^{i\vartheta_1})|}{5 r_0 \log 2} \geq \frac{e^{-4\pi} \log M(r_0, f)}{5 r_0 \log 2}.$$

If $|\xi| = r$, then $\frac{r_0}{2} \leq r \leq r_0$ and hence we deduce that

$$\mu(r, f) \geq \frac{e^{-4\pi} \log M(r, f)}{10 r \log 2}.$$

This proves Lemma 2.

The next lemma is required to cope with possible irregularities in the growth of $\log M(r, f)$.

Lemma 3. *Suppose that $\varphi(r)(r_0 \leq r < \infty)$ is continuous, positive and strictly increasing with a sectionally continuous locally bounded derivative $\varphi'(r)$. [At points of discontinuity we define $\varphi'(r)$ as the limit from the left.] Suppose that for positive α, β*

$$\limsup_{r \to \infty} \frac{\varphi(r)}{r^{\alpha}} > \beta. \tag{4.9}$$

Then given α' $(0 < \alpha' < \alpha)$ there exist arbitrarily large r for which the following are satisfied:

$$\frac{\varphi(r)}{r^{\alpha}} \geq \beta e^{-5}; \tag{4.10}$$

$$\frac{\varphi'(r)}{\varphi(r)} \geq \frac{\alpha'}{r}\ ; \tag{4.11}$$

$$\varphi\left\{r + 2\,\frac{\varphi(r)}{\varphi'(r)}\right\} < e^{4}\varphi(r). \tag{4.12}$$

We assume that $\varphi'(r)$ is never zero. This really involves no loss of generality. First of all we show that there are arbitrarily large values of r such that (4.11) and

$$\frac{\varphi(r)}{r^{\alpha}} \geq \beta \qquad (4.10)'$$

are satisfied. Now $\frac{\varphi(r)}{r^{\alpha'}}$ is unbounded as $r \to \infty$ and so for arbitrarily large r it must be locally nondecreasing. For such r,

$$\frac{d}{dr}\left\{\frac{\varphi(r)}{r^{\alpha'}}\right\} = \frac{\varphi(r)}{r^{\alpha'}}\left\{\frac{\varphi'(r)}{\varphi(r)} - \frac{\alpha'}{r}\right\} \geq 0$$

and so (4.11) is satisfied. If for all large r, $\varphi(r) \geq \beta r^{\alpha}$ then we obtain the desired result. Otherwise there are arbitrarily large values of r such that $\varphi(r) < \beta r^{\alpha}$. From (4.9) there is a smallest $R > r$ such that $\varphi(R) = \beta R^{\alpha}$. But then $\frac{\varphi(r)}{r^{\alpha}}$ is nondecreasing at R and so $\frac{\varphi'(R)}{\varphi(R)} \geq \frac{\alpha}{R}$, as in the previous argument, and $\frac{\varphi(R)}{R^{\alpha}} = \beta$. Hence the result.

Now set $h = h(r) = 2\,\frac{\varphi(r)}{\varphi'(r)}$ and note that

$$\log\varphi(r+h) - \log\varphi(r) = \int_{r}^{r+h} \frac{\varphi'(t)}{\varphi(t)}\,dt \leq h \max_{r \leq t \leq r+h} \frac{\varphi'(t)}{\varphi(t)}\,.$$

Consequently if (4.12) is false for $r = r_0$ there is an r_1 such that $r_0 < r_1 \leq r_0 + h(r_0)$ and

$$\frac{\varphi'(r_1)}{\varphi(r_1)} \geq \frac{4}{h(r_0)} = 2\,\frac{\varphi'(r_0)}{\varphi(r_0)}\,.$$

Suppose that $r_0, r_1, \ldots, r_n$ have been defined in this way so that (4.12) is false for $r = r_\nu (0 \leq \nu \leq n)$ and

$$r_\nu < r_{\nu+1} \leq r_\nu + 2\,\frac{\varphi(r_\nu)}{\varphi'(r_\nu)}\ (0 \leq \nu \leq n-1),$$

$$\frac{\varphi'(r_{\nu+1})}{\varphi(r_{\nu+1})} \geq 2\,\frac{\varphi'(r_\nu)}{\varphi(r_\nu)}\ (0 \leq \nu \leq n-1).$$

Then we can define r_{n+1} so that

$$\frac{\varphi'(r_{n+1})}{\varphi(r_{n+1})} \geq 2\,\frac{\varphi'(r_n)}{\varphi(r_n)}\,,\quad r_n < r_{n+1} \leq r_n + 2\,\frac{\varphi(r_n)}{\varphi'(r_n)}\,.$$

If this process continued indefinitely then we should have

$$\frac{\varphi'(r_n)}{\varphi(r_n)} \to \infty \quad (r \to \infty)$$

and

$$\sum_{n=0}^{\infty}(r_{n+1}-r_n) \leq 2\sum_{n=0}^{\infty}\frac{\varphi(r_n)}{\varphi'(r_n)}$$

$$\leq 2\,\frac{\varphi(r_0)}{\varphi'(r_0)}\sum_{0}^{\infty}2^{-n}$$

$$= 4\,\frac{\varphi(r_0)}{\varphi'(r_0)}\,.$$

Thus r_n would tend to a finite limit and so $\dfrac{\varphi'(r_n)}{\varphi(r_n)} \to \infty$. This contradiction shows that the construction of the r_n must terminate after a finite number of steps.

Take now as r_0 a value such that (4.10)′ and (4.11) are satisfied for $r = r_0$. If (4.12) is not satisfied for $r = r_0$ then there is a sequence $r_0, r_1, \ldots, r_N$ as above such that it is not satisfied for $r = r_n$ $(0 \leq n \leq N-1)$ but it is satisfied for $r = r_N$. Then for $0 \leq n < N$,

$$\frac{\varphi'(r_{n+1})}{\varphi(r_{n+1})} \geq 2\,\frac{\varphi'(r_n)}{\varphi(r_n)} \geq 2^{n+1}\,\frac{\varphi'(r_0)}{\varphi(r_0)}$$

and so

$$r_N - r_0 = \sum_{0}^{N-1}(r_{n+1}-r_n) \leq 2\,\frac{\varphi(r_0)}{\varphi'(r_0)}\sum_{n=0}^{N-1}\frac{1}{2^n}$$

$$< 4\,\frac{\varphi(r_0)}{\varphi'(r_0)}$$

$$< 4\,\frac{r_0}{\alpha'}$$

by (4.11). Hence if α' is near enough to α,

$$r_N < r_0\left(1 + \frac{4}{\alpha'}\right) \leq r_0(1 + 5/\alpha).$$

Since (4.10)′ holds for $r = r_0$,

$$\varphi(r_N) \geq \varphi(r_0) \geq \beta r_0^\alpha \geq \beta r_N^\alpha (1 + 5/\alpha)^{-\alpha} > \beta e^{-5} r_N^\alpha.$$

Also

$$\frac{\varphi'(r_N)}{\varphi(r_N)} \geq \frac{\varphi'(r_0)}{\varphi(r_0)} \geq \frac{\alpha'}{r_0} \geq \frac{\alpha'}{r_N}.$$

Hence the proof of Lemma 3 is complete.

4.2. Proofs of Theorems 2 and 3 for $\sigma \geq 6$.

Suppose now that $f(z)$ is an integral function of order $\sigma \geq 6$. We apply Lemma 3 with $\sigma > \alpha' > 5$ to $\varphi(r) = \log M(r, f)$ so that for some arbitrarily large r, (4.10), (4.11) and (4.12) hold simultaneously. For such an r there is a point $z_0 = re^{i\vartheta}$ so that [see e.g. 3, Lemma 2, p. 136.]

$$|f(z_0)| = M(r, f),$$

$$\left|\frac{f'(z_0)}{f(z_0)}\right| = \varphi'(r).$$

It now follows from Lemma 1 that if $\delta = \delta(r)$ is the radius of the largest disc with centre z_0 in which $|f(z)| > 1$ then, by (4.1),

$$\delta(r) \leq 2\,\frac{|f(z_0)| \log |f(z_0)|}{|f'(z_0)|} = 2\,\frac{\varphi(r)}{\varphi'(r)} \leq \frac{2r}{\alpha'} < \frac{2}{5}r.$$

By (4.3) there is a point z with $|z - z_0| < \delta(r)$ and

$$\begin{aligned} \varrho(f(z)) &\geq \frac{\log |f(z_0)|}{10\,\delta(r) \log 2} \\ &= \frac{\varphi(r)}{10\,\delta(r) \log 2} \\ &\geq \frac{\alpha' \varphi(r)}{20 r \log 2}. \end{aligned} \tag{4.13}$$

If $|z| = R$, then $R < r + \delta(r)$ and so, by (4.12),

$$\varphi(R) \leq \varphi(r + \delta(r)) \leq \varphi\left(r + 2\,\frac{\varphi(r)}{\varphi'(r)}\right) \leq e^4 \varphi(r).$$

Hence, since also $R > r - \delta(r) > 3/5\, r$,

$$\mu(R, f) \geq \varrho(f(z)) \geq \frac{\alpha' e^{-4} \varphi(R)}{20\,(2R) \log 2}$$

$$= \frac{\alpha' e^{-4} \log M(R, f)}{40 R \log 2} .$$

From $R > \frac{3}{5} r$ it follows that as $r \to \infty$ then $R \to \infty$ and so we arrive at

$$\limsup_{R \to \infty} \frac{R \mu(R, f)}{\log M(R, f)} \geq \frac{\sigma e^{-4}}{40 \log 2} ,$$

since α' can be taken as near to σ as we please. This proves (3.1) and so Theorem 2.

We next prove Theorem 3 for $\sigma \geq 5$. Suppose in fact that (3.3) is false for some arbitrarily large r where A_1 is some positive constant. We may apply Lemma 3 as before with $\alpha = \sigma + 1$, $\alpha' = \sigma$ and any quantity β such that

$$0 < \beta < \frac{A_1 K}{\sigma + 1} . \tag{4.14}$$

Then (4.13) yields for some z with $|z| = R$

$$\varrho(f(z)) \geq \frac{\sigma \varphi(r)}{20 r \log 2} \geq \frac{\sigma \beta e^{-5} r^{\sigma}}{20 \log 2} . \tag{4.15}$$

Also

$$|z| = R < r + \delta(r) \leq r + 2\, \frac{\varphi(r)}{\varphi'(r)} \leq r \left(1 + \frac{2}{\sigma}\right)$$

by (4.11). Therefore

$$R^{\sigma} \leq r^{\sigma} \left(1 + \frac{2}{\sigma}\right)^{\sigma} \leq e^{2} r^{\sigma}.$$

Then (4.15) shows that

$$\mu(R, f) \geq \frac{\sigma \beta e^{-7}}{20 \log 2} R^{\sigma}$$

for arbitrarily large values of R. From (4.14) we see that

$$\frac{\sigma A_1 K}{\sigma + 1} \; \frac{e^{-7}}{20 \log 2} \leq K,$$

and so

$$A_1 \leq \frac{\sigma + 1}{\sigma}\, 20\, e^{7} \log 2 < 25\, e^{7} \log 2 .$$

Consequently it is only for such A_1 that the result of the theorem is false. Hence it must be true with $A_1 = 25\, e^{7} \log 2$. This proves (3.3) for $\sigma \geq 5$.

4.3. Completion of proof of Theorem 3

Suppose that the hypotheses of Theorem 3 hold with $-1 < \sigma < 5$. Let n be a positive integer such that

$$n(\sigma + 1) \geq 6 \tag{4.16}$$

and consider $F(z) = f(z^n)$. Then for all large r we have

$$\varrho(F(z)) = \frac{|F'(z)|}{1 + |F(z)|^2} = \frac{n r^{n-1} |f'(z^n)|}{1 + |f(z^n)|^2} < K n r^{n-1} r^{n\sigma} \quad (|z| = r)$$

by (2.1). Hence $F(z)$ satisfies (2.1) with Kn in place of K and $n(\sigma + 1) - 1$ in place of σ. In view of (4.16) we can apply the previous result to $F(z)$ and obtain

$$\log M(r, F) \leq \frac{A_1 K n r^{n(\sigma+1)}}{n(\sigma + 1)} = \frac{A_1 K}{\sigma + 1} r^{n(\sigma+1)}.$$

As $M(r, F) = M(r^n, f)$ this completes the proof of Theorem 3.

4.4. Completion of proof of Theorem 2

We assume that $f(z)$ is of order $\sigma < 6$ and consider $F(z) = f(z^{12})$. Since, as above,

$$\varrho(F(z)) = 12 |z|^{11} \varrho(f(z^{11}))$$

and $F(z)$ is of order 12σ it follows that if (3.1) holds for $F(z)$ then

$$\limsup_{r \to \infty} \frac{r \mu(r, f)}{\log M(r, f)} \geq \frac{1}{12} A_0 (12\sigma + 1)$$

and this is the result for $f(z)$ if A_0 is adjusted. Consequently it is sufficient for $\sigma < 6$ to prove the theorem for $F(z)$.

Now for some constant A_2 we have

$$\log M(4r, F) \leq A_2 \log M(r, F) \tag{4.17}$$

for arbitrarily large values of r. Otherwise for some r_0 we find that

$$\log M(4^n r_0, F) \geq A_2^n \log M(r_0, F) \quad (n \geq 1)$$

so that the order of $F(z)$ is at least $\frac{\log A_2}{\log 4}$. This is impossible if $A_2 \geq 4^{72}$ as $F(z)$ is of order less than 72.

We consider arbitrarily large r for which (4.17) is true. If for an infinite sequence of such r, $|f(z)| \geq 1$ $(r \leq |z| \leq 3r)$ then the result follows from Lemma 2. Hence we assume always that for some R in $r \leq R \leq 3r$ there is a z on $|z| = \mathrm{R}$ where $|f(z)| < 1$. From the periodic nature of $F(z)$ we see that there is a disc $S(R)$ centred on ζ where $|\zeta| = R$, $|F(\zeta)| = M(R, F)$ such that $|F(z)| \geq 1$ in $S(R)$, $|F(z)| = 1$ at some boundary point and the radius of $S(R)$ does not exceed $\frac{\pi R}{12}$. By Lemma 1 it follows that

$$\mu(t, F) \geq \frac{12 \log M(R, F)}{10\pi R \log 2},$$

for some t satisfying $R - \frac{\pi R}{12} < t < R + \frac{\pi R}{12}$, so that $\frac{2}{3} R < t < \frac{4}{3} R$. If $t \leq R$ then we get

$$\mu(t, R) \geq \frac{12 \log M(t, F)}{10\pi \cdot \frac{3}{2} t \log 2} = \frac{4 \log M(t, F)}{5\pi t \log 2}.$$

If $t > R$ then, since $R \leq 3r$, $t < 4r$ and so, using (4.17) we have

$$\mu(t, F) \geq \frac{12 \log M(t, F)}{A_2 10\pi t \log 2} = \frac{6 \log M(t, F)}{5 A_2 \pi t \log 2}.$$

As $t > \frac{2}{3} R \geq \frac{2}{3} r$ it follows that one of the above inequalities must hold for arbitrarily large t. Hence the proof of Theorem 2 is complete.

4.5. Proof of Theorem 4

For any function $f(z)$ of order less than 1 with $f(0) \neq 0$ we have the well known inequalities [see e.g. 4, p. 28]

$$\int_0^r \frac{n(t)}{t}\, dt \leq \log\left(\frac{M(r, f)}{|f(0)|}\right) \leq \int_0^r \frac{n(t)}{t}\, dt + r \int_r^\infty \frac{n(t)}{t^2}\, dt, \qquad (4.18)$$

where $n(t)$ is the number of zeros of $f(z)$ in $|z| \leq t$. The restriction $f(0) \neq 0$

clearly involves no loss of generality. From the second condition of (3.4) and the left hand inequality of (4.18) it follows that

$$n(r) = O(\log^{\alpha} r). \tag{4.19}$$

From (4.19) we find that

$$r\int_{r}^{\infty} \frac{n(t)}{t^2}\,dt = O(\log^{\alpha} r). \tag{4.20}$$

Hence for r such that $\log M(r, f) > \eta\varphi(r)\log^{\alpha} r$, where η is some positive constant implied in the first condition of (3.4), we obtain, from (4.18) and (4.20),

$$\log M(r, f) = \{1 + o(1)\}\int_{0}^{r} \frac{n(t)}{t}\,dt. \tag{4.21}$$

Assume now that we are dealing with values r of the above kind. By a known result we have for some R in $\left(\frac{r}{4}, \frac{r}{2}\right)$, $\log|f(z)| > H\log M(R, f)$ $(|z| = R)$ where, here and elsewhere, H depends only on $f(z)$ [5, pp. 64–65]. For sufficiently large r let R' be the smallest number such that $|f(z)| > 1$ $(R' < |z| < R)$. We deal with two cases: a) $R' > \frac{r}{12}$; b) $R' \leq \frac{r}{12}$ for arbitrarily large values of R'. It is clear that in fact R' does take arbitrarily large values.

Case a). If $|f(\zeta)| = 1$ $(\zeta = R'e^{i\varphi})$ we consider the largest disc D centred on $Re^{i\varphi}$ in which $|f(z)| > 1$. The radius of D is at most $\frac{r}{2} - \frac{r}{12} = \frac{5}{12}r$ and so D lies in $|z| < \frac{r}{2} + \frac{5}{12}r < r$. By Lemma 1, (4.3), for some t in $\frac{r}{12} < t < r$ we have

$$\mu(t, f) > \frac{H\log M(R, f)}{r}.$$

From (4.18), (4.19) and (4.21) it follows that

$$\begin{aligned}\log M\left(\frac{r}{12}, f\right) &> H\log M(r, f) - \int_{r^{1/12}}^{r} \frac{n(t)}{t}\,dr + O(\log^{\alpha} r)\\ &> H\log M(r, f) + O(\log^{\alpha} r)\\ &= H(1 + o(1))\log M(r, f).\end{aligned}$$

Hence we see that

$$\mu(t, f) > H\,\frac{\varphi(r)\log^\alpha r}{r} > H\,\frac{\varphi(t)\log^\alpha t}{t},$$

for arbitrarily large values of t. This proves the theorem in this case.

Case b). In this case $|f(z)| > 1\,(R' < |z| < 3R')$ and $|f(\zeta)| = 1\,(\zeta = R'e^{i\varphi})$. We see from the proof of Lemma 2 that

$$\mu(t, f) > H\,\frac{\log M(2R', f)}{R'} \tag{4.22}$$

for some t satisfying $R' < t < 2R'$. Now from (4.19) and (4.21)

$$n\left(\frac{r}{4}\right)\log r > H\int_0^{r/4}\frac{n(t)}{t}\,dt = H\left(\int_0^{r}\frac{n(t)}{t}\,dt - \int_{r/4}^{r}\frac{n(t)}{t}\,dt\right) > H\varphi(r)\log^\alpha r - H\log^\alpha r$$

and so

$$n\left(\frac{r}{4}\right) > H\varphi(r)\log^{\alpha-1} r.$$

But $\left(R', \frac{r}{4}\right)$ is free from zeros and so

$$n(R') > H\varphi(r)\log^{\alpha-1} r.$$

Hence, by (4.18),

$$\frac{\log M(2R', f)}{|f(0)|} \geq \int_{R'}^{2R'}\frac{n(t)}{t}\,dt = n(R')\log 2 > H\varphi(r)\log^{\alpha-1} r.$$

Therefore we find that in (4.22),

$$\mu(t, f) > \frac{H\varphi(t)\log^{\alpha-1} t}{t}.$$

Since this holds for arbitrarily large values of t the theorem is proved in this case.

4.6. Proof of Theorem 5

From the left hand inequality of (4.18) we get

$$n(r)\log r \leq \int_r^{r^2}\frac{n(t)}{t}\,dt \leq \log M(r^2, f) = O\left\{\frac{\log^2 r}{\varphi(r^2)}\right\}$$

and so, since $\varphi(r)$ is increasing,

$$n(r) = O\left\{\frac{\log r}{\varphi(r)}\right\}. \tag{4.23}$$

Using (4.23) we obtain

$$r\int_r^\infty \frac{n(t)}{t^2}\,dt = O\left\{\frac{1}{\varphi(r)}\cdot r\int_r^\infty \frac{\log t}{t^2}\,dt\right\}$$

$$= O\left\{\frac{\log r}{\varphi(r)}\right\}. \tag{4.24}$$

Hence if we put $\beta(r) = \eta\sqrt{\dfrac{\log r}{\varphi(r)\log M(r)}}$, where $\eta > 0$ and depends on $f(z)$, then, by a known result [5, pp. 64–65], in $r(1-\beta(r)) < |z| < r(1+\beta(r))$

$$\log|f(z)| > H\log M(|z|, f)$$

outside a set of circles the sum of whose radii is at most $Hr\beta^2(r)$.

Consider now values of r such that $f(z)$ has a zero on $|z| = r$. Let $z_0 = re^{i\vartheta_0}$ be such a zero. Then from the above, if r is large enough, for some R satisfying $r - Hr\beta^2(r) < R < r$ we have

$$\log|f(Re^{i\vartheta_0})| > H\log M(R, f).$$

Let D be the disc with centre $Re^{i\vartheta_0}$ in which $|f(z)| > 1$, assuming r is sufficiently large, with $|f(z)| = 1$ somewhere on the boundary. Then, by Lemma 1 and the above for some z in this disc

$$\varrho(f(z)) > \frac{H\log M(R, f)}{r\beta^2(r)}. \tag{4.25}$$

Now as $\beta(r)\to 0$ as $r\to\infty$ it follows that for large r, $\frac{r}{2} < R < r$ and so

$$\log M(R, f) = \{1 + o(1)\}\int_0^R \frac{n(t)}{t}\,dt$$

$$> \{1 + o(1)\}\left\{\log M(r, f) - \int_R^r \frac{n(t)}{t}\,dt\right\}$$

$$= \{1 + o(1)\}\{\log M(r, f) + O(\log r)\}$$

$$= \{1 + o(1)\}\log M(r, f),$$

where we have used (4.23), (4.24), (4.18) and the obvious result that $\log r = o(\log M(r, f))$. Hence, from (4.25),

$$\varrho(f(z)) > \frac{H \log M(r, f)}{r\beta^2(r)}$$
$$= \frac{H\varphi(r)\log r}{\eta^2 r}\left\{\frac{\log M(r, f)}{\log r}\right\}^2.$$

Now in (4.25), $\frac{r}{2} < |z| < r$ for large r and so if $|z| = t$ then for large r we find that

$$\mu(t, f) > H\,\frac{\varphi(t)\log t}{\eta^2 t}\left(\frac{\log M(r, f)}{\log r}\right)^2$$

since $\varphi(t)$ is increasing. As the final factor above tends to ∞ with r and the inequality holds for some arbitrarily large t this proves Theorem 5.

5. Counter examples

The first theorem shows that (3.2) is best possible and that the properties of $f(z)$ referred to in §3 preceding Theorem 4 do in fact hold.

Theorem 6. *Given $\varphi(r) \nearrow \infty$ $(r \nearrow \infty)$ there is a sequence of increasing integers k_n such that if*

$$f(z) = \prod_1^\infty \left(1 - \frac{z}{2^{k_n}}\right)^{k_n}, \quad f_1(z) = \prod_1^\infty \left(1 - \frac{z}{2^{nk_n}}\right)^{k_n},$$

$$f_2(z) = \prod_1^\infty \left(1 - \frac{z}{2^{k_n/n}}\right)^{k_n}$$

then for $g(z) = f(z), f_1(z)$ or $f_2(z)$

$$\limsup_{r\to\infty} \frac{r\mu(r, g)}{\varphi(r)\log r} < \infty.$$

The sequence $\{k_n\}$ will be seen later to satisfy $\frac{k_{n+1}}{k_n} \geq 4$ and in this case it is easy to verify that

$$0 < \limsup_{r\to\infty} \frac{\log M(r, f)}{\log^2 r} < \infty, \ \log M(r, f_1) = o(\log^2 r), \ \log M(r, f_2) \neq O(\log^2 r).$$

The next theorem shows that Theorem 2 is best possible

Theorem 7. *Given $\sigma(0 \leq \sigma < \infty)$ there is an integral function of proper order σ and very regular growth when $\sigma > 0$ such that*

$$\limsup_{r \to \infty} \frac{r\mu(r, f)}{\log M(r, f)} < C(\sigma + 1)$$

for some absolute constant C.

5.1. Proof of Theorem 6

The proof of the theorem requires a number of lemmas. We assume that besides any other conditions that the integers k_n will be required to satisfy, that they will always satisfy

$$\frac{k_{n+1}}{k_n} \geq 4\,(n > 1),\; k_1 \geq 2. \tag{5.1}$$

We confine our attention to $f(z)$. The proofs for $f_1(z)$ and $f_2(z)$ are similar.

Lemma 4. *On $|z| = 2^{k_n+1}$ and on $|z| = 2^{k_n-1}$,*

$$|f(z)| > H|z|.$$

On $|z| = 2^{k_n+1}$ we have

$$|f(z)| \geq \prod_{m=1}^{n} \left(\frac{2^{k_n+1}}{2^{k_m}} - 1\right)^{k_m} \cdot \prod_{m=n+1}^{\infty} \left(1 - \frac{2^{k_n+1}}{2^{k_m}}\right)^{k_m}.$$

From (5.1) each factor in the first product is at least 1 and so

$$\prod_{m=1}^{n} \left(\frac{2^{k_n+1}}{2^{k_m}} - 1\right)^{k_m} \geq \left(\frac{2^{k_n+1}}{2^{k_1}} - 1\right)^{k_1}$$

$$> H \cdot 2^{k_n+1} = H|z|. \tag{5.2}$$

Also, from (5.1),

$$\prod_{m=n+1}^{\infty} \left(1 - \frac{2^{k_n+1}}{2^{k_m}}\right)^{k_m} > \prod_{m=n+1}^{\infty} \left(1 - 2^{\frac{-k_m}{2}}\right)^{k_m}$$

$$> H. \tag{5.3}$$

From (5.2) and (5.3) the lemma follows for $|z| = 2^{k_n+1}$.

In dealing with $|z| = 2^{k_n-1}$ we assume for convenience that $n > 1$. This clearly involves no loss of generality. On $|z| = 2^{k_n-1}$ we have

$$|f(z)| \geq \prod_{m=1}^{n-1} \left(\frac{2^{k_n}}{2^{k_m+1}} - 1\right)^{k_m} \cdot 2^{-k_n} \prod_{m=m+1}^{\infty} \left(1 - \frac{2^{k_n}}{2^{k_m+1}}\right)^{k_m}.$$

By (5.1) each factor in the first product is at least 1 and so

$$\prod_{m=1}^{n-1}\left(\frac{2^{k_n}}{2^{k_m+1}}-1\right)^{k_m} > \left(\frac{2^{k_n}}{2^{k_1+1}}-1\right)^{k_1}$$

$$> H\cdot 2^{2k_n-1} \tag{5.4}$$

since $k_1 \geq 2$. As before

$$\prod_{n=m+1}^{\infty}\left(1-\frac{2^{k_n}}{2^{k_m+1}}\right)^{k_m} > H. \tag{5.5}$$

Hence on $|z| = 2^{k_n-1}$, by (5.4) and (5.5),

$$|f(z)| > H\cdot 2^{2k_n-1}\cdot 2^{-k_n}$$

$$= H\,2^{k_n-1} = H\,|z|.$$

Hence the lemma follows for $|z| = 2^{k_n-1}$.

We see from Lemma 4 that when z is large the regions in which $|f(z)| < 1$ are disjoint, with one in each annulus $2^{k_n-1} < |z| < 2^{k_n+1}$. Denote these by D_n. Clearly D_n contains the zero at $z = 2^{k_n}$.

Lemma 5. *If the k_n increase sufficiently rapidly then on the boundary of D_n when n is large*

$$H\,2^{k_n-k_1-k_2-\ldots k_{n-1}} < |z-2^{k_n}| < H\,2^{k_n-k_1-\ldots k_{n-1}}.$$

We have

$$|f(z)| = \prod_{m=1}^{n-1}\left|1-\frac{z}{2^{k_m}}\right|^{k_m}\left(\frac{|z-2^{k_n}|}{2^{k_n}}\right)^{k_n}\cdot\prod_{m=n+1}^{\infty}\left|1-\frac{z}{2^{k_m}}\right|^{k_m}.$$

Now on the boundary of D_n

$$\prod_{m=1}^{n-1}\left|1-\frac{z}{2^{k_m}}\right|^{k_m} = \frac{|z|^{k_1+\ldots+k_{n-1}}}{2^{k_1^2+k_2^2+\ldots+k_{n-1}^2}}\prod_{m=1}^{n-1}\left|1-\frac{2^{k_m}}{z}\right|^{k_m}. \tag{5.6}$$

When n is large then $2^{k_n-1} < |z| < 2^{k_n+1}$ by Lemma 4 and so, if the k_n increase sufficiently rapidly to ensure that the final product in (5.6) lies between $\frac{1}{H}$ and H, we obtain on the boundary of D_n,

$$H\cdot\frac{2^{(k_n-1)(k_1+\ldots+k_{n-1})}}{2^{k_1^2+\ldots+k_{n-1}^2}} < \prod_{m=1}^{n-1}\left|1-\frac{z}{2^{k_m}}\right|^{k_m} < H\cdot\frac{2^{(k_n+1)(k_1+\ldots+k_{n-1})}}{2^{k_1^2+\ldots+k_{n-1}^2}}. \tag{5.7}$$

Again, from Lemma 4, it follows that on boundary of D_n when n is large,

$$H < \prod_{n=m+1}^{\infty} \left|1 - \frac{z}{2^{k_m}}\right|^{k_m} < H . \tag{5.8}$$

From (5.6), (5.7) and (5.8) we find that on the boundary of D_n when n is large

$$H \cdot 2^{k_n} \left\{ \frac{2^{\frac{1}{k_n}(k_1^2 + \dots + k_{n-1}^2)}}{2^{\left(1+\frac{1}{k_n}\right)(k_1+\dots+k_{n-1})}} \right\} < |z - 2^{k_n}| < H\, 2^{k_n} \left\{ \frac{2^{\frac{1}{k_n}(k_1^2 + \dots + k_{n-1}^2)}}{2^{\left(1-\frac{1}{k_n}\right)(k_1+\dots+k_{n-1})}} \right\} .$$

From these inequalities the lemma follows provided the k_n increase sufficiently rapidly to ensure that

$$k_1^2 + \dots + k_{n-1}^2 = O(k_n) \quad (n \to \infty) . \tag{5.9}$$

Lemma 6. *For large n we have in $2^{k_n - 1} \le |z| \le 2^{k_n + 1}$, but outside D_n, provided that k_n increases quickly enough,*

$$\left|\frac{f'(z)}{f(z)}\right| < H \, \frac{k_n 2^{k_1 + \dots + k_{n-1}}}{|z|} .$$

We have

$$\frac{f'(z)}{f(z)} = \sum_{m=1}^{\infty} \frac{k_m}{z - 2^{k_m}} .$$

If the k_n increase sufficiently rapidly then, for $2^{k_n - 1} \le |z| \le 2^{k_n+1}$

$$\begin{aligned} \sum_{m=1}^{n-1} \frac{k_m}{z - 2^{k_m}} &\le \sum_{m=1}^{n-1} \frac{k_m}{2^{k_n - 1} - 2^{k_m}} \\ &< \frac{2}{2^{k_n - 1}} \sum_{m=1}^{n-1} k_m \\ &< H \, \frac{k_n}{2^{k_n}} . \end{aligned} \tag{5.10}$$

Also,

$$\begin{aligned} \sum_{m=n+1}^{\infty} \frac{k_m}{|z - 2^{k_m}|} &\le \sum_{m=n+1}^{\infty} \frac{k_m}{2^{k_m} - 2^{k_n+1}} \\ &< H \sum_{m=n+1}^{\infty} \frac{k_m}{2^{k_m}} \\ &< \frac{H}{2^{k_n}} \sum_{m=n+1}^{\infty} \frac{k_m}{2^{\frac{k_m}{2}}} \\ &< \frac{H}{2^{k_n}} . \end{aligned} \tag{5.11}$$

From Lemma 5 it follows that if the k_n increase rapidly enough then

$$\frac{k_n}{|z-2^{k_n}|} < H\,\frac{k_n 2^{k_1+\dots+k_{n-1}}}{2^{k_n}}\,. \tag{5.12}$$

From (5.10), (5.11) and (5.12) the lemma follows.

Lemma 7. *If the k_n increase sufficiently rapidly then for $2^{k_n+1} \le |z| \le 2^{k_{n+1}-1}$ we have*

$$\varrho(f(z)) = O\left(\frac{1}{|z|}\right).$$

If the k_n increase quickly enough then on $|z| = 2^{k_n+1}$ we obtain

$$\frac{|f'|}{|f|^2} < H\,\frac{k_n 2^{k_1+\dots+k_{n-1}}}{|z|^2}$$
$$< \frac{H}{|z|}$$

by Lemmas 4 and 6. The same inequality is also true for $|z| = 2^{k_{n+1}-1}$. Now $\left|\frac{zf'(z)}{f^2(z)}\right|$ is subharmonic in $2^{k_n+1} \le |z| \le 2^{k_{n+1}-1}$ and since it is bounded by H on the boundary it is bounded by H inside the annulus. Therefore in $2^{k_n+1} \le |z| \le 2^{k_{n+1}-1}$,

$$\varrho(f(z)) < \frac{|f'(z)|}{|f^2(z)|} = O\left(\frac{1}{|z|}\right).$$

Lemma 8. *In $2^{k_n-1} \le |z| \le 2^{k_n+1}$ we have*

$$\varrho(f(z)) \le H\,\frac{k_n 2^{k_1+\dots+k_{n-1}}}{|z|}$$

provided the k_n increase quickly enough.

In $2^{k_n-1} \le |z| \le 2^{k_n+1}$ but outside D_n it follows, if the k_n increase quickly enough, that

$$\left|\frac{zf'(z)}{f^2(z)}\right| < H k_n 2^{k_1+\dots+k_{n-1}} \tag{5.13}$$

by Lemmas 4 and 6 and the use of subharmonicity as before. Hence the lemma is true in this region.

On the boundary of D_n we get

$$|zf'(z)| < H k_n 2^{k_1+\dots+k_{n-1}} \tag{5.14}$$

and so, by the maximum modulus principle, this also holds inside D_n. From (5.13) and (5.14) the lemma follows.

Given $\varphi(r)$ as in the theorem choose an increasing sequence of integers k_n so that the above results hold and also

$$2^{k_1+\dots+k_{n-1}} < \varphi(2^{k_n-1}).$$

Then from Lemmas 7 and 8 we see that

$$\limsup_{r\to\infty} \frac{r\mu(r,f)}{\varphi(r)\log r} < \infty,$$

since $\varphi(r)$ is increasing.

This completes the proof of the theorem. In should perhaps be pointed out that given $\varphi(r)$ where $\varphi(r)\to\infty (r\to\infty)$ it is not difficult to find a $\psi(r)$ such that $\psi(r)\to\infty(r\to\infty)$, $\varphi(r)\geq\psi(r)$ and $\psi(r)$ is increasing. Consequently $\varphi(r)$ was assumed to be increasing in the theorem only for convenience.

5.2. Proof of Theorem 7

A number of lemmas are required.

Lemma 9. *If* $A>1$ *and* $f(z)=\prod_1^\infty\left(1+\frac{z}{e^{nA}}\right)^{[A^n]}$ *then* $f(z)$ *is a function of very regular growth and order* $\frac{\log A}{A}$.

For $e^{nA}\leq |z| \leq e^{(n+1)A}$ we have

$$\log M(r,f) \geq \log|f(e^{nA})| \geq (A^n-1)\log 2. \tag{5.15}$$

Also, in this range,

$$\begin{aligned}\log M(r,f) &\leq \log M(e^{(n+1)A},f)\\ &\leq \sum_{m=1}^{n+1} A^m\log\{1+e^{(n+1-m)A}\} + \sum_{m=n+2}^{\infty} A^m \log\{1+e^{(n+1-m)A}\}\\ &\leq \sum_{m=1}^{n+1} A^m\{\log 2+(n+1-m)A\} + \sum_{m=n+2}^{\infty} A^m e^{-(m-n-1)A}\\ &\leq \frac{A^{n+2}\log 2}{A-1} + A^{n+1}\sum_{\nu=1}^{n}\frac{\nu}{A^\nu} + A^{n+1}\sum_{\nu=1}^{\infty} A^\nu e^{-\nu A}\\ &< K(A)A^n.\end{aligned} \tag{5.16}$$

From (5.15) and (5.16) it follows that for $e^{nA}\leq|z|\leq e^{(n+1)A}$

$$\frac{(A^n - 1)\log 2}{A^{n+1}} < \frac{\log M(r, f)}{r^{(\log A)/A}} < \frac{K(A) \cdot A^n}{A^n},$$

and so the result follows.

Lemma 10. *If* $\varphi_n(z) = (\sum_1^{n-2} + \sum_{n+1}^{\infty})[A^m] \log \left| 1 + \frac{z}{e^{mA}} \right|$ *then for* $\frac{e^{nA}}{2} \leq |z| \leq 2e^{nA}$,

$$-\eta A^n \leq \varphi_n(z) \leq \eta A^n$$

where $\eta = \eta(A) > 0$ *and* $\eta \to 0 (A \to \infty)$; η *is not necessarily the same at each occurrence.*

We have, in the range of the lemma,

$$\begin{aligned}\sum_1^{n-2} [A^m] \log \left| 1 + \frac{z}{e^{mA}} \right| &\leq \sum_1^{n-2} A^m \log \left(1 + \frac{2e^{nA}}{e^{mA}} \right) \\ &\leq \sum_1^{n-2} A^m \{\log 4 + (n - m)A\} \\ &\leq \frac{A^{n-1} \log 4}{A - 1} + A^{n-1} \sum_{\nu=0}^{n-3} \frac{\nu + 2}{A^\nu} \\ &\leq \eta(A) \cdot A^n. \end{aligned} \tag{5.17}$$

Also, in the above range,

$$\begin{aligned}\sum_{n+1}^{\infty} [A^m] \log \left| 1 + \frac{z}{e^{mA}} \right| &\leq \sum_{n+1}^{\infty} A^m \log \left(1 + \frac{2e^{nA}}{e^{mA}} \right) \\ &\leq 2 \sum_{n+1}^{\infty} A^m e^{(n-m)A} \\ &= 2A^n \sum_{\nu=1}^{\infty} (Ae^{-A})^\nu \\ &\leq \eta(A) A^n. \end{aligned} \tag{5.18}$$

From (5.17) and (5.18) the right hand inequality of the lemma follows.

In the range of the lemma we also have, if $e^{2A} \geq 4$,

$$\begin{aligned}\sum_1^{n-2} [A^m] \log \left| 1 + \frac{z}{e^{mA}} \right| &\geq \sum_1^{n-2} [A^m] \log \left(\frac{e^{nA}}{2e^{mA}} - 1 \right) \\ &\geq 0, \end{aligned} \tag{5.19}$$

and, if $e^A > 4$,

$$\sum_{n+1}^{\infty} [A^m] \log \left| 1 + \frac{z}{e^{mA}} \right| \geq \sum_{n+1}^{\infty} A^m \log \left(1 - \frac{2e^{nA}}{e^{mA}} \right)$$

$$> - 4 \sum_{n+1}^{\infty} A^m e^{(n-m)A}$$

$$= - 4 A^n \sum_{\nu=1}^{\infty} (A e^{-A})^{\nu}$$

$$\geq - \eta(A) A^n, \tag{5.20}$$

From (5.19) and (5.20) the left hand inequality of the lemma follows.

Lemma 11. *For* $|z| = \frac{e^{nA}}{2}$ *and* $|z| = 2e^{nA}$,

$$(\tfrac{1}{4} - \eta) A^n \leq \log |f(z)| \leq (3 + \eta) A^n.$$

If $|z| = \frac{e^{nA}}{2}$ we have

$$[A^{n-1}] \log \left| 1 + \frac{z}{e^{(n-1)A}} \right| \leq A^{n-1} \log \left(1 + \frac{e^A}{2} \right)$$

$$\leq A^{n-1} (\log 2 + A)$$

$$\leq (1 + \eta) A^n. \tag{5.21}$$

Also for $|z| = \frac{e^{nA}}{2}$,

$$[A^n] \log \left| 1 + \frac{z}{e^{nA}} \right| \leq A^n \log 3/2$$

$$\leq A^n. \tag{5.22}$$

From (5.21) and (5.22) and Lemma 10, the right hand inequality of Lemma 11 follows for $|z| = \frac{e^{nA}}{2}$.

We have for $|z| = \frac{e^{nA}}{2}$, if $e^A > 4$,

$$[A^{n-1}] \log \left| 1 + \frac{z}{e^{(n-1)A}} \right| \geq [A^{n-1}] \log \left(\frac{e^A}{2} - 1 \right)$$

$$\geq (A^{n-1} - 1)(A - \log 4)$$

$$\geq (1 - \eta) A^n; \tag{5.23}$$

and

$$[A^n] \log \left| 1 + \frac{z}{e^{nA}} \right| > - A^n \log 2$$

$$> - \tfrac{3}{4} A^n. \tag{5.24}$$

From (5.23) and (5.24) and Lemma 10, the left hand inequality of the Lemma 11 follows for $|z| = \dfrac{e^{nA}}{2}$.

The result for $|z| = 2e^{nA}$ follows in a similar manner to the above.

Lemma 12. *If z satisfies $|z + e^{nA}| \geq \dfrac{e^{nA}}{4}$ and $\dfrac{e^{nA}}{2} \leq |z| \leq 2e^{nA}$ then*

$$\left| \frac{f'(z)}{f(z)} \right| \leq (4 + \eta) \frac{A^n}{e^{nA}}.$$

We have

$$\frac{f'(z)}{f(z)} = \sum_1^\infty \frac{[A^m]}{z + e^{mA}}.$$

For $|z| \geq \dfrac{e^{nA}}{2}$, if $e^A \geq 4$,

$$\left| \sum_1^{n-1} \frac{[A^m]}{z + e^{mA}} \right| \leq \sum_1^{n-1} \frac{A^m}{\dfrac{e^{nA}}{2} - e^{mA}}$$

$$\leq \frac{4}{e^{nA}} \sum_1^{n-1} A^m$$

$$< \frac{4A^n}{(A-1)e^{nA}}$$

$$\leq \eta \frac{A^n}{e^{nA}}; \tag{5.25}$$

and for $|z| \leq 2e^{nA}$, if $e^A \geq 4$,

$$\left| \sum_{n+1}^\infty \frac{[A^m]}{z + e^{mA}} \right| \leq \sum_{n+1}^\infty \frac{A^m}{e^{mA} - 2e^{nA}}$$

$$\leq 2 \sum_{n+1}^\infty \frac{A^m}{e^{mA}}$$

$$= 2 \frac{A^n}{e^{nA}} \sum_{\nu=1}^\infty (Ae^{-A})^\nu$$

$$\leq \eta \frac{A^n}{e^{nA}}. \tag{5.26}$$

Finally, if $|z + e^{nA}| \geq \dfrac{e^{nA}}{4}$, then

$$\frac{[A^n]}{|z + e^{nA}|} \leq \frac{4A^n}{e^{nA}} . \tag{5.27}$$

From (5.25), (5.26) and (5.27) the lemma follows.

Lemma 13. For $\dfrac{e^{nA}}{2} \leq |z| \leq 2\, e^{nA}$,

$$\mu(r, f) \leq K(A)\, \frac{A^n}{r} ,$$

provided A is sufficiently large.

When A is large enough we see from Lemma 11 that the set $|f(z)| < 1$ splits into a number of components. Each zero e^{nA} is contained in a component D_n, say, and D_n lies in $\dfrac{e^{nA}}{2} \leq |z| \leq 2\, e^{nA}$.

First of all we show that when A is large the disc $|z + e^{nA}| \leq \dfrac{e^{nA}}{4}$ is contained in D_n. From Lemma 10 it follows that in this disc,

$$\begin{aligned} \log |f(z)| &\leq [A^{n-1}] \log \left| 1 + \frac{z}{e^{(n-1)A}} \right| + [A^n] \log \left| 1 + \frac{z}{e^{nA}} \right| + \eta A^n \\ &\leq A^{n-1} \log \left(1 + \frac{5}{4}\, e^{A} \right) - (A^n - 1) \log 4 + \eta A^n \\ &\leq A^{n-1} \left(\log \frac{5}{2} + A \right) - (A^n - 1) \log 4 + \eta A^n \\ &< 0, \end{aligned}$$

provided A is large enough, independently of n. Hence we arrive at the desired conclusion.

From Lemma 12 and the above it follows that when A is large then on the boundary of D_n,

$$|f'(z)| = \left| \frac{f'(z)}{f(z)} \right| \leq (4 + \eta)\, \frac{A^n}{e^{nA}} .$$

Therefore in D_n and on its boundary,

$$\varrho(f(z)) \leq |f'(z)| \leq (4 + \eta)\, \frac{A^n}{e^{nA}} . \tag{5.28}$$

In the annulus $\dfrac{e^{nA}}{2} \leq |z| \leq 2e^{nA}$ outside D_n it follows that when A is large

$$\varrho\,(f(z)) \le \left|\frac{f'(z)}{f(z)}\right| \le (4+\eta)\,\frac{A^n}{e^{nA}}\,, \tag{5.29}$$

by Lemma 12.

Since $\dfrac{1}{e^{nA}} \le \dfrac{2}{r}$ for $\dfrac{e^{nA}}{2} \le r \le 2\,e^{nA}$ the lemma follows from (5.28) and (5.29).

Lemma 14. *For large* A, *if* $2\,e^{nA} \le r \le \dfrac{e^{(n+1)A}}{2}$ *then*

$$\mu\,(r, f) < \frac{K(A)}{r}\,.$$

From Lemmas 11 and 12 it follows that

$$\left|\frac{z f'(z)}{f(z)^2}\right| < K(A)$$

on the boundary of $2e^{nA} \le |z| \le \dfrac{e^{(n+1)A}}{2}$. Since the function on the left above is subharmonic in the annulus it follows that the inequality holds throughout the annulus. Hence the lemma follows because $\varrho\,(f(z)) \le \dfrac{|f'(z)|}{|f(z)|^2}$.

5.3. Before completing the proof of Theorem 7, we observe that the constants $K(A)$ appearing in Lemmas 13 and 14 remain bounded as $A \to \infty$. From Lemmas 11, 13 and 14 it follows that

$$\limsup_{r\to\infty} \frac{r\,\mu\,(r, f)}{\log M\,(r, f)} < B\,,$$

where B is an absolute constant for all $f(z)$ for which $A \ge A_0$, A_0 being some fixed value.

We proceed to prove Theorem 7.

If $0 < \sigma < \dfrac{\log A_0}{A_0}$ in Theorem 7 we take $f(z)$ as above with A given by $\sigma = \dfrac{\log A}{A}$. If $\sigma > \dfrac{\log A_0}{A_0}$ we proceed as follows. Let $A_1 > A_0$ be defined by $2\,\dfrac{\log A_1}{A_1} = \dfrac{\log A_0}{A_0}$. Let n be the smallest positive integer such that $\dfrac{\sigma}{n} \le \dfrac{\log A_0}{A_0}$. Then, since $n \ge 2$, $\dfrac{\sigma}{n-1} \ge \dfrac{\log A_0}{A_0}$ and so $\dfrac{\sigma}{n} \ge \dfrac{n-1}{n}\,\dfrac{\log A_0}{A_0} \ge \tfrac{1}{2}\,\dfrac{\log A_0}{A_0}$. Therefore $\dfrac{\sigma}{n} = \dfrac{\log A}{A}$ where $A_1 \le A \le A_0$.

We now take, as a function for Theorem 7, $F(z) = f(z^n)$ where $f(z)$ is constructed as in Lemma 9 with this value of A. Then

$$\begin{aligned}\limsup_{r\to\infty} \frac{r\,\mu(r, F)}{\log M(r, F)} &= \limsup_{r\to\infty} \frac{n r^n \mu(r^n, f)}{\log M(r^n, f)} \\ &\le n B \\ &= \frac{\log A}{A} \cdot n \cdot \frac{B \cdot A}{\log A} \\ &\le \frac{2 A_0 B}{\log A_0} \cdot \sigma .\end{aligned}$$

Thus the theorem is proved for $0 < \sigma < \infty$.

It can be shown by the same methods as above that if K is large enough then

$$F(z) = \prod_1^\infty (1 + z\, e^{-K n^2})^{n^n}$$

is a function of order 0 satisfying the conclusion of the theorem.

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