*Test* (1993)
*Vol. 2, No.* 1–2, *pp.* 189–217



# Forecasting Point and Continuous Processes: Prequential Analysis

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SUMMARY

The problem considered in this paper is that of evaluating the performance of a forecaster who predicts the intensity of a point process or the drift and diffusion rates of a continuous process. It is shown that we can evaluate this performance in a "prequential" manner, without the usual assumption that the forecasts are generated in accordance with some probability distribution. Technically, the results in this paper are prequential counterparts of the Dambis-Dubins-Schwarz reduction of a continuous martingale, via a change of time, to a Wiener process, and the Papangelou-Meyer reduction of a counting process to a Poisson process.

*Keywords:* PROBABILITY FORECASTING; COMPENSATOR; MARTINGALE; PREQUENTIAL PRINCIPLE.

## 1. PROBLEM

Suppose we observe a point process (i.e., a sequence of point events of some kind). Let $N_t, t \geq 0$, be the number of the point events that have occurred in the time interval $]0, t]$. (In order words, $N_t$ is the counting process corresponding to our point process.) We are interested in the trend $A_t$ of $N_t$. Of course, $A_t$ must be nondecreasing. We are primarily interested in the case where

$$A_t = \int_0^t \lambda_s ds \tag{1}$$

Received July 93; Revised October 93.

for some *intensity function* $\lambda_t \geq 0$. We assume that $A_0 = 0$ and $A_t$ is continuous throughout the paper.

How, then, do we interpret the assertion that $A_t$ is the trend of $N_t$? Let us consider two examples.

**Example 1.** *Consider a chandelier with controllable brightness level* $\lambda \in [0, 1]$ *(when* $\lambda = 0$*, the chandelier is off, and* $\lambda = 1$ *corresponds to maximum brightness). Let* $\lambda_t$ *be the level of brightness at time* $t \geq 0$ *and* $N_t$ *be the total number of electric bulbs in the chandelier that burned out during the time period* $]0, t]$. *Consider the hypothesis that* $\lambda_t$ *is the intensity of* $N_t$*, i.e., that the process* $A_t$ *defined by (1) is the trend of* $N_t$. *What does this hypothesis lead us to believe about the world? (alternatively, what future observations will make us reject this hypothesis?)*

**Example 2.** *Let* $N_t$ *be the number of major earthquakes in the time interval* $]0, t]$. *For each* $t \geq 0$*, the forecaster specifies, prior to time* $t$*, some number* $A_t$ *and claims that the function* $A_t$ *(which is assumed to be continuous) is the trend of* $N_t$. *We cannot utilise these forecasts until we understand the full meaning of this claim.*

One possible interpretation of $A_t$ being the trend of $N_t$ is that both $A_t$ and $N_t$ are stochastic processes governed by some global probability distribution and that $A_t$ is the compensator of $N_t$ in the sense of the standard theory of martingales (see e.g., Dellacherie and Meyer, 1982; Elliott, 1982; Jacod and Shiryaev, 1987). This interpretation is not completely satisfactory since it presupposes that, in Example 1, an event such as $\lambda_{10} = 0$ (at time 10, the chandelier is off), which depends on deliberate actions taken by people, can be ascribed some probability.

The following interpretation of the claim that $A_t$ is the trend of $N_t$ seems more satisfactory: the difference $\mu_t := N_t - A_t$ is a *martingale* in the sense that, for any two time points $t < s$, $\mu_t$ is the fair price at time $t$ for the uncertain future value $\mu_s$. (Martingales are usually defined via probability, but we shall move in the opposite direction.)

*Remark 1.* Considering a situation which essentially includes Example 2 as a special case, Dawid (1984, 1985) (see also Seillier-Moiseiwitsch and Dawid, 1993) puts forward the *prequential principle*, which requires that our conclusions should depend only on the realized paths $N_t$ and $A_t$. The approach proposed in this paper is compatible with the prequential principle and the underlying ideas are due to Dawid (1985, Section 13.2), who suggests exploiting the fact that some processes are martingales (for details, see Vovk, 1993, Section 3).

## 2. IDEAL PICTURE, I

We consider here the foundations of the mathematical theory of martingales. Of course, no mathematical theory can adequately describe reality but, instead, allows analysis of an "ideal picture" (e.g., Euclidean geometry describes not real points but fictitious points without size). Moreover, there may be more than one way of relating the ideal picture of a mathematical theory to reality (or to the ideal pictures of other mathematical theories). Thus, as an introduction, we give an informal outline of the ideal picture of martingale theory and its possible relations to reality (exact definitions will be given in the later Sections).

Time is continuous and has an origin, which can be represented by the interval $[0, \infty[$. We consider a set of stochastic processes, called martingales, and an infinitely rich *guarantor* who, at each time $t$, for each real (possibly negative) constant $c$, and for each martingale $M$, allows us to *stake c units on $M$*, i.e., is willing to abide by the following agreement with us: at each future time point $s > t$, his debt to us is $c(M_s - M_t)$ provided the agreement is still valid at time $s$; we can always end the agreement by settling the debt. (The phrase "the guarantor owes us $c$ units" for $c < 0$ may be interpreted to mean that we owe him $|c|$ units; as usual, money is assumed to be infinitely divisible.) The guarantor is willing, as soon as we need it, to lend us any amount of money without charging interest. Strategies for staking on the martingales and borrowing money are called *gambling strategies*. No money is spent on consumption. Our success is measured by our terminal wealth (our wealth being the amount of money that we hold, less our debt to the guarantor).

The notion of the ideal picture allows us to define the meaning of the forecaster's claim that $A_t$ is the trend of $N_t$. First, the forecaster may mean that he is willing to play a role analogous to that of the guarantor in the ideal picture with the martingale $N_t - A_t$. This interpretation will be called the *E-interpretation*. Second, the forecaster can refuse to play such a role (e.g., because he is insufficiently rich) but can assert instead that it is practically impossible to make much money out of 1 using a predefined "honest" gambling strategy (i.e., ensuring that the debt to the guarantor, if any, is eventually paid off) against the (maybe, imaginary) guarantor for the martingale $N_t - A_t$. This is the *P-interpretation*. Under the $E$-interpretation, our aim is to win as much money as possible; under the $P$-interpretation, we try to falsify the forecaster's claim.

The $E$-interpretation is closely connected with such notions as value (or fair price) and mathematical expectation; the $P$-interpretation is connected with belief and probability. A discrete-time variant of the $P$-interpretation is treated in Vovk (1993).

Consider a contract whose future value $\eta = \eta(\omega)$ is uncertain (i.e., depends on the unknown "state of the the world" $\omega$); we are interested in its current value. If, for some constant $c$, some gambling strategy enables us, starting with $c$, to win no less than $\eta(\omega)$, whatever the true value of $\omega$, we can say that now the contract is worth no more than $c$. The infimum of such numbers $c$ is called the *upper value* of the contract. Under the $E$-interpretation, this is virtually the minimal price for which, in the absence of additional information or opportunities, we can safely sell the contract. We may define the *lower value* of $\eta$ in an analogous way. In those cases where the upper value of $\eta$ coincides with its lower value, we call this common value the *fair price* for $\eta$.

Now consider a prespecified uncertain event $E$. Suppose that, for some very large constant $c$, there exists a gambling strategy which, when applied to the initial capital 1, never incurs a debt and wins no less than $c$ if $\omega \in E$. Then, under the $P$-interpretation, we can be sure that $E$ will not occur, provided that we believe the forecaster.

*Remark 2.* This idea of the ideal picture was borrowed from Shafer (1990b) (see also Shafer, 1992). Shafer's "ideal picture of probability" includes, besides belief and fair price, one further element: frequency.

## 3. MAIN RESULTS, I

Suppose we suspect that the forecaster from Example 2, whose claim is understood in the sense of the $P$-interpretation, systematically underestimates the true trend (or that the true intensity in Example 1 is greater than $\lambda_t$). In other words, we suspect that $N_t$ is typically much larger than $A_t$. Let $c > 0$ be a large constant and $\alpha_c$ be the stopping time defined as $\inf\{t : A_t = c\}$ ($\inf \emptyset$ is always taken to be $\infty$). For a well-calibrated forecaster, we would have $N_{\alpha_c} \approx c$ (assuming $\alpha_c < \infty$), but we suspect that $N_{\alpha_c} \gg c$. In Section 7 (Theorem 5) we shall see that, when $N_t - A_t$ is a martingale, the random variable $N_{\alpha_c}$ is distributed as $\mathcal{P}_c$ (denoting by $\mathcal{P}_c$ the Poisson distribution on the set $\mathrm{N} := \{0, 1, \ldots\}$ with mean $c$) in the sense that:

(a) for any bounded function $U$, the upper value of $U(N_{\alpha_c})$ (when $\alpha_c = \infty$, we define $U(N_{\alpha_c})$ to be $\inf U$) does not exceed $\int U d\mathcal{P}_c$;
(b) moreover, the fair price for $U(N_{\alpha_c})$ exists and equals $\int U d\mathcal{P}_c$ when it is known that $A_t$ tends to infinity.

Part (a) of this result enables us to judge whether the difference $N_{\alpha_c} - c$ is large enough to justify rejecting the forecaster's claim: we can fix in advance a very small value for $\delta > 0$ and hope that $N_{\alpha_c}$ will exceed the upper $\delta$-quantile of $\mathcal{P}_c$; in this case, we conclude that the forecaster has been proven wrong (or, in the situation of Example 1, we conclude that the hypothesis is false). Under the $E$-interpretation of the forecaster's claim, $U$ is interpreted as a loss function.

Let us now consider a similar problem for a continuous process $W_t$ with $W_0 = 0$. There now exist two associated continuous processes $B_t$ and $A_t$ such that $A_t$ is nondecreasing and $A_0 = B_0 = 0$. It is claimed that $B_t$ is the *drift compensator* of $W_t$ and $A_t$ is the *diffusion compensator* of $W_t$, in the sense that $W_t - B_t$ and $(W_t - B_t)^2 - A_t$ are martingales. It will be shown (Section 7, Theorem 6) that, for any constant $c > 0$, the random variable $W_{\alpha_c} - B_{\alpha_c}$ is distributed as $\mathcal{N}_{0,c}$ (which denotes the normal distribution on the real line $\Re$ with mean 0 and variance $c$), in an analogy of the case of a counting process. The interpretation of this result is also analogous.

## 4. GAMBLING STRATEGIES

We now begin the formal exposition. The purpose of this Section is to define the notion of a gambling strategy; in this paper, we need only consider gambling strategies which we call elementary.

Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F})$ be a filtered space, where $\Omega$ denotes a set, $\mathcal{F}$ denotes a $\sigma$-algebra on $\Omega$ and $(\mathcal{F}_t)$ denotes an increasing family ($\mathcal{F}_s \subseteq \mathcal{F}_t$ when $s \leq t$) of sub-$\sigma$-algebras of $\mathcal{F}$. Each $\mathcal{F}_t$ may be interpreted as the knowledge available at time $t$. We always assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_\infty = \mathcal{F}(\mathcal{F}_\infty$ is the $\sigma$-algebra generated by all $\mathcal{F}_t, t \geq 0)$. Set $F := \Re^d$ for a fixed positive integer $d$ (in Section 7, $d = 1$ or $d = 2$). We fix a *basic martingale* $\mu$, which is an $F$-valued adapted (meaning that each $\mu_t$ is $\mathcal{F}_t$-measurable) càdlàg (i.e., right continuous with limits existing on the left) process. (The word "basic" refers to the fact that we shall use $\mu$ as a basis from which to construct other martingales.) We shall always assume that $\mu_0 = 0$. Note that we do not fix any probability distribution in $(\Omega, \mathcal{F})$.

A stochastic process $H_t$ is an *elementary gambling strategy* if there exists a sequence

$$0 = \tau_m \leq \tau_{m+1} \leq \ldots \leq \tau_n \leq \tau_{n+1} = \infty$$

($m$ and $n$ being integers with $m \leq n$) of stopping times and a sequence $h_m, h_{m+1}, \ldots, h_n$ of *partial* $F$-valued random elements (the term "partial" implying that the domain of $h_i$ can be a measurable set other than the whole space $\Omega$) such that:

(a) each $h_i$ is defined on $\{\tau_i < \infty\}$ and is $\mathcal{F}_{\tau_i}$-measurable;
(b) $H_0 = 0$ and $H_t = h_i$ for $t \in ]\tau_i . \tau_{i+1}]$.

The definition and basic properties of the $\sigma$-algebras $\mathcal{F}_\tau$, $\tau$ ranging over the stopping times, may be found in Elliott (1982, Chapter 2); the inclusion $\{\tau_i < \infty\} \in \mathcal{F}_{\tau_i}$ follows from Theorem 2.12 in Elliott (1982). For $t \geq 0$, we define the *stochastic integral* $(H \cdot \mu)_t$, where $\mu$ is the basic martingale, by the equality

$$(H \cdot \mu)_t := \sum_{i=m}^{n} h_i \cdot \left(\mu_{\tau_{i+1}\wedge t} - \mu_{\tau_i \wedge t}\right).$$

On the right-hand side, $\cdot$ denotes the inner product of two vectors in $F$ and $\wedge$ denotes the minimum of two numbers. It is easy to see that $(H \cdot \mu)_t$

does not depend on the representation of $H$. Those stochastic processes which can be represented in the form $c + (H \cdot \mu)_t$, for some $c \in \Re$ and elementary gambling strategy $H$, are called *elementary martingales.* They are adapted and càdlàg.

Intuitively, an elementary gambling strategy is a rule determining our stakes at each point in time; the $\tau_i$'s represent those times when we change stakes (the word "elementary" indicates that we do this only finitely often). The stochastic integral $(H \cdot \mu)_t$ represents the profits gained from using the strategy $H$ (negative profit implying a loss).

## 5. UPPER EXPECTATION

Let $\tau$ be a stopping time. (For the exact formulation and proof of the results mentioned in Section 3, it would be sufficient to consider only the case $\tau = 0$.) For $\omega \in \Omega$, define the *information* about $\omega$ at time $\tau$ as

$$I_\tau(\omega) := \cap\{E \in \mathcal{F}_\tau : \omega \in E\},$$

so that $I_\tau(\omega)$ represents the set of $\psi \in \Omega$ which are indistinguishable from $\omega$ at time $\tau$ (note that $I_0(\omega) = \Omega$ does not depend on $\omega$). For a stochastic process $\xi_t$, let $\xi_\tau(\omega)$ denote $\xi_{\tau(\omega)}(\omega)$, which is undefined when $\tau(\omega) = \infty$. We define the *upper* and *lower expectation* of a (finite) random variable $\eta$ at time $\tau$ as

$$\begin{aligned} \bar{E}_\mu[\eta|\tau](\omega) := \inf \Big\{ & M_\tau(\omega) : \\ & \underline{\lim}_{t\to\infty} M_t(\psi) \geq \eta(\psi), \forall \psi \in I_\tau(\omega) \Big\}, \end{aligned} \tag{2}$$

$$\begin{aligned} \underline{E}_\mu[\eta|\tau](\omega) := \sup \Big\{ & M_\tau(\omega) : \\ & \overline{\lim}_{t\to\infty} M_t(\psi) \leq \eta(\psi), \forall \psi \in I_\tau(\omega) \Big\}, \end{aligned}$$

where $M$ ranges over the elementary martingales. The upper and lower expectations are undefined when $\tau(\omega) = \infty$. Note that

$$\underline{E}_\mu[\eta|\tau](\omega) = -\bar{E}_\mu[-\eta|\tau](\omega).$$

When $\tau = 0$, the upper and lower expectations no longer depend on $\omega$, and we write simply $\bar{E}_\mu[\eta]$ and $\underline{E}_\mu[\eta]$.

The number $\bar{E}_\mu[\eta|\tau](\omega)$ is interpreted as the minimal price for which the owner of $\eta$ (i.e., of the contract whose future value is $\eta(\omega)$, with $\eta(\omega) < 0$ interpreted as a debt to the guarantor) must be willing (in the ideal picture or under the $E$- interpretation of the forecaster's claim) to sell it at time $\tau(\omega)$. It is less obvious that the number $\underline{E}_\mu[\eta|\tau](\omega)$ can be interpreted as the maximal price for which we must be willing to buy $\eta$ at time $\tau(\omega)$. If $c < \underline{E}_\mu[\eta|\tau](\omega)$, then there exists an elementary martingale $M$ such that

$$M_\tau(\omega) = c, \qquad \overline{\lim}_{t\to\infty} M_t \le \eta \text{ on } I_\tau(\omega).$$

We now show that, at time $\tau(\omega)$, it is sensible to buy $\eta$ for $c$. Let our capital at time $\tau(\omega)$ be $C$. After the deal, our capital becomes $(C - c)$. Borrowing $c$ from the guarantor, we increase it to the initial $C$. Then we stake $-1$ unit on $M$ thus ensuring that, at each time $t \ge \tau(\omega)$, our debt to the guarantor will be $M_t(\omega)$. Since $\overline{\lim} M_t \le \eta$ on $I_\tau(\omega)$, buying $\eta$ and staking on $M$ will not decrease our terminal wealth.

The basic martingale $\mu$ is *coherent* if, for any elementary martingale $M$, any constant $t \ge 0$, and any $\omega \in \Omega$,

$$\inf_{\psi \in I_t(\omega)} \underline{\lim}_{s\to\infty} M_s(\psi) \le M_t(\omega).$$

Intuitively, it is required that no gambling strategy can ever ensure a guaranteed profit in the future. (This requirement is close to de Finetti's axiom of coherence (1964, 1975).)

We shall be dealing with expressions involving *partial* random variables $\xi(\omega)$, which can be undefined for some $\omega$ (like $\bar{E}_\mu[\eta|\tau]$). Such an expression is defined to be true if it holds for all $\omega$ for which all the partial random variables that enter into it are defined. We let $\bar{E}^*_\mu[\eta|\tau]$ denote the right-hand side of (2) with $M$ ranging over the elementary martingales such that $M_t(\psi) \ge \inf \eta$, for all $\psi \in I_\tau(\omega)$ and $t \ge \tau(\omega)$ (so $\bar{E}^*_\mu[\eta|\tau]$ differs from $\bar{E}_\mu[\eta|\tau]$ only when $\eta$ is bounded below).

**Theorem 1.** *The coherence of $\mu$ is equivalent to:*
*(a)* $\bar{E}_\mu[0|t] \ge 0$, *for all constant stopping times $t \ge 0$;*
*(b)* $\bar{E}_\mu[\eta|\tau] = \bar{E}^*_\mu[\eta|\tau]$, *for all $\eta$ and $\tau$.*

**Theorem 2.** *Let $\mu$ be coherent. For any stopping time $\tau$, random variables $\eta, \xi$, and constant c,*
*(a)* $\bar{E}_\mu[\eta + \xi|\tau] \leq \bar{E}_\mu[\eta|\tau] + \bar{E}_\mu[\xi|\tau]$;
*(b)* $c \geq 0 \Longrightarrow \bar{E}_\mu[c\eta|\tau] = c\bar{E}_\mu[\eta|\tau]$,
$c \leq 0 \Longrightarrow \bar{E}_\mu[c\eta|\tau] = c\underline{E}_\mu[\eta|\tau]$;
*(c)* $\bar{E}_\mu[1|\tau] = 1$;
*(d)* $\eta \geq 0 \Longrightarrow \bar{E}_\mu[\eta|\tau] \geq 0$.

**Corollary 1.** *If $\mu$ is coherent, then*

$$\underline{E}_\mu[\eta|\tau] \leq \bar{E}_\mu[\eta|\tau], \forall \eta, \tau.$$

Henceforth, we assume that the basic martingale $\mu$ is coherent.
The *(upper) probability* of an event $E \in \mathcal{F}$ at time $\tau$ is defined by

$$\Pr_\mu[E|\tau] := \bar{E}_\mu[\chi_E|\tau],$$

where $\chi_E$ is the indicator of $E$, and the *lower probability* (or belief function) is defined to be

$$\underline{P}_\mu[E|\tau] := \underline{E}_\mu[\chi_E|\tau].$$

The belief functions of the Dempster-Shafer theory (Dempster, 1967; Shafer, 1976, 1990a) are closely related to lower probabilities — see the discussion in Vovk (1993). Theorem 2 implies that

$$\Pr_\mu[E \cup F|\tau] \leq \Pr_\mu[E|\tau] + \Pr_\mu[F|\tau],$$

$$\underline{P}_\mu[E|\tau] = 1 - \Pr_\mu[E^c|\tau].$$

*Remark 3.* In the framework of conventional probability theory, the notions of probability and mathematical expectation are equivalent to each other and so we have probability theory but do not have a distinguishable "expectation theory", as such. Either of these two notions can be taken as basic (Lebesgue's scheme vs. Daniel's scheme). In the prequential framework, probability reduces to upper expectation, but not vice versa. So, from a technical point of view, upper expectation yields a richer theory. Besides, probability, as a special case of upper expectation, can be interpreted as the upper value of some contract. However, the importance of probability stems from the fact that it can also be used under the $P$-interpretation of the forecaster's claim, and, in this case, lower probability is an explication of warranted belief. (This understanding of probability is described in detail in Vovk, 1993.) So, the principal intuitive meaning of probability does not reduce to fair price.

## 6. EXPECTATION

Recall that the basic martingale $\mu$ is assumed to be coherent. Henceforth, we shall usually remove the subscript $\mu$ from the relevant notation. A random variable $\eta$ is *$\tau$-integrable*, where $\tau$ is a finite stopping time, if $\underline{E}[\eta|\tau] = \bar{E}[\eta|\tau]$; this common value is denoted by $E[\eta|\tau]$.

**Theorem 3.** *For any finite stopping time $\tau$, random variables $\eta$ and $\xi$, and constant $c \in \Re$,*
*(a) if $\eta$ and $\xi$ are $\tau$-integrable, then $\eta + \xi$ is also $\tau$-integrable and*

$$E[\eta + \xi|\tau] = E[\eta|\tau] + E[\xi|\tau];$$

*(b) if $\eta$ is $\tau$-integrable, then $c\eta$ is $\tau$-integrable and*

$$E[c\eta|\tau] = cE[\eta|\tau];$$

*(c) the identical 1 is $\tau$-integrable and $E[1|\tau] = 1$;*
*(d) if $\eta \geq 0$ is $\tau$-integrable, then $E[\eta|\tau] \geq 0$.*

The next simple theorem shows that, in some (rather weak) sense, the theory of this paper is in agreement with conventional martingale theory.

**Theorem 4.** *Let $P$ be a probability distribution in $(\Omega, \mathcal{F})$. If the basic martingale $\mu$ is a local martingale with respect to $(\mathcal{F}_t)$ and $P$ then, for each bounded* 0*-integrable random variable $\eta$,*

$$\int_\Omega \eta dP = E_\mu[\eta].$$

## 7. MAIN RESULTS, II

A function $n : [0, \infty[\to \mathrm{N}$ is *counting* if it is càdlàg, nondecreasing, has unit jumps, and satisfies $n(0) = 0$. An adapted process $N_t$ is *counting* if all its sample paths are counting. An adapted process $W_t$ is *continuous* if all its sample paths are continuous. A family $\alpha_t, t \geq 0$, of stopping times is a *change of time* if $\alpha_t \leq \alpha_s$ for $t \leq s$. The change of time $\alpha_t$ is *proper* if $\alpha_t(\omega) < \infty$, for all $t$ and $\omega$.

*Remark 4.* It is usually required (e.g., Dellacherie and Meyer, 1982, Chapter VI, n. 56) that a change of time be a right continuous process. We have modified the definition in order to avoid imposing the unnatural requirement that the filtration $(\mathcal{F}_t)$ be right continuous.

We say that a counting process $\xi_t$ is a *Poisson process* with respect to a proper change of time $\alpha_t$ if, for any $t > s \geq 0$ and any bounded function $U : \mathrm{N} \to \Re$, the random variable $U(\xi_{\alpha_t} - \xi_{\alpha_s})$ is $\alpha_s$-integrable and

$$E\left[U(\xi_{\alpha_t} - \xi_{\alpha_s})|\alpha_s\right] = \int_{\mathrm{N}} U(u)\mathcal{P}_{t-s}(du).$$

A continuous process $\xi_t$ is a *Wiener process*, with respect to the proper change of time $\alpha_t$, if $\xi_0 = 0$ and, for any $t > s \geq 0$ and any bounded continuous function $U : \Re \to \Re$, the random variable $U(\xi_{\alpha_t} - \xi_{\alpha_s})$ is $\alpha_s$-integrable and

$$E\left[U(\xi_{\alpha_t} - \xi_{\alpha_s})|\alpha_s\right] = \int_{\Re} U(u)\mathcal{N}_{0,t-s}(du).$$

We must also deal with changes of time $\alpha_t$ which are not proper. A counting process $\xi_t$ is a *Poisson process* with respect to a change of time $\alpha_t$ if, for any $t > s \geq 0$ and any bounded function $U : \mathrm{N} \to \Re$,

$$\bar{E}\left[U(\xi_{\alpha_t} - \xi_{\alpha_s})|\alpha_s\right] \leq \int_{\mathrm{N}} U(u)\mathcal{P}_{t-s}(du);$$

in the case where $\alpha_t = \infty$, we take

$$U(\xi_{\alpha_t} - \xi_{\alpha_s}) := \inf_u U(u).$$

Analogously, a continuous process $\xi_t$ with $\xi_0 = 0$ is a *Wiener process* with respect to $\alpha_t$ if, for any $t > s \geq 0$ and any bounded continuous function $U : \Re \longrightarrow \Re$,

$$\bar{E}\left[U(\xi_{\alpha_t} - \xi_{\alpha_s})|\alpha_s\right] \leq \int_{\Re} U(u)\mathcal{N}_{0,t-s}(du),$$

with the same convention for $\alpha_t = \infty$.

*Remark 5.* It is easy to see that the two definitions of a Poisson process (as well as those for a Wiener process) agree with each other. Indeed, when $\xi_t$ is a Poisson process, in the sense of the second definition, with respect to a proper change of time $\alpha_t$, we have

$$\underline{E}\left[U(\xi_{\alpha_t} - \xi_{\alpha_s})|\alpha_s\right] = -\bar{E}\left[-U(\xi_{\alpha_t} - \xi_{\alpha_s})|\alpha_s\right]$$
$$\geq -\int_{\mathrm{N}} -U(u)\mathcal{P}_{t-s}(du) = \int_{\mathrm{N}} U(u)\mathcal{P}_{t-s}(du).$$

**Theorem 5.** *Suppose that $N_t$ is a counting process, $A_t$ is a nondecreasing continuous process with $A_0 = 0$ and the basic martingale $\mu_t$ is defined by $\mu_t := N_t - A_t$. Then $N_t$ is a Poisson process with respect to the change of time $\alpha_t$ defined by*

$$\alpha_c := \inf\{t : A_t = c\}. \tag{3}$$

This result is closely connected with the result of Meyer (1971) and Papangelou (1972), although we cannot say that either of these results implies the other. In Examples 1 and 2 (Section 1), we have the "canonical" situation where:

(a) $\Omega$ is the set of pairs $(n, a)$, where $n : [0, \infty[\rightarrow \mathrm{N}$ is a counting function and $a : [0, \infty[\rightarrow \Re$ is a nondecreasing continuous function with $a(0) = 0$;
(b) $N_t(\omega)$ and $A_t(\omega)$, where $\omega = (n, a)$, are defined as $n(t)$ and $a(t)$, respectively;

(c) each $\sigma$-algebra $\mathcal{F}_t$ is generated by the random variables $N_s$ and $A_s, s \leq t$.

It is easy to see that the basic martingale $N_t - A_t$ is coherent in this case.

Applying Theorem 5 to the case $A_t = t$ (which corresponds to a proper change of time $\alpha_t$), we obtain the prequential counterpart of Watanabe's (1964) characterization of a Poisson process.

**Theorem 6.** *Suppose $W_t, B_t, A_t$ are continuous processes with $W_0 = B_0 = A_0 = 0$, $A_t$ is nondecreasing and the basic martingale $\mu_t$ is defined as the $\Re^2$-valued stochastic process*

$$\left(W_t - B_t, (W_t - B_t)^2 - A_t\right).$$

*Then $W_t - B_t$ is a Wiener process with respect to the change of time $\alpha_t$ defined by (3).*

This result is an analogue of the result due to Dambis (1965) and Dubins and Schwarz (1965). The "canonical" situation, where the coherence of $\mu$ is obvious, is constructed in the same way as for a counting process. Theorem 6, in the special case where $B_t = 0, A_t = t$, parallels Lévy's (1948) (see also Doob (1953), Theorem 11.9) characterization of a Wiener process.

*Remark 6.* In this paper we consider only the case of "left quasicontinuous" processes, whose compensators are continuous. For such processes it is impossible to predict jumps. Vovk (1993), Section 6, considers the case of discrete time. By the usual embedding of the discrete-time processes into the continuous-time processes (Jacod and Shiryaev, 1987, Chapter I, Section 1f), this case corresponds to the situation where jumps are allowed only at times $1, 2, \ldots$

## 8. IDEAL PICTURE, II

In this Section, we focus our attention on some questionable aspects of the ideal picture. First, it is an unrealistic feature of the ideal picture that we are interested only in our fortune in the infinitely remote future, and yet this feature is essentially used in our interpretation of the definition of upper and lower expectation. (Such situations are common in applications of mathematics: e.g., one never observes the sizeless points studied by Euclidean geometry.) This deficiency must be remedied at the stage of application of the ideal picture to reality: e.g., this definition can be safely applied when $\eta(\omega)$ depends on $\omega$ only through $I_t(\omega)$, where $t$ is a moderately large number. Another unrealistic feature of the ideal picture is the stability of money which manifests itself as the willingness of the guarantor to make interest-free loans: in reality, we experience inflation and hyperinflation. We must be flexible when applying the ideal picture to the real world; e.g., it may be useful to interpret capital $c$ as the market price for $c$ units of a riskless security or even as a value of $c$ for some utility function.

## 9. APPENDIX

*Proof of Theorem 1.* Since (b) obviously implies (a), we are reduced to proving that the coherence of $\mu$ implies (b) and that (a) implies the coherence of $\mu$. Assume that $\mu$ is coherent but

$$\bar{E}_\mu[\eta|\tau](\omega) \neq \bar{E}^*_\mu[\eta|\tau](\omega),$$

for some $\eta$ (which must be bounded below) and some $\tau$, $\omega$ with $\tau(\omega) < \infty$. Then there exists an elementary martingale $M$ such that $\underline{\lim} M_t \geq \eta$ on $I_\tau(\omega)$, and $M_t(\psi) < \inf \eta$ for some $\psi \in I_\tau(\omega)$ and $t \geq \tau(\omega)$. This contradicts the coherence of $\mu$ since

$$\left.\begin{matrix} \psi \in I_\tau(\omega) \\ t \geq \tau(\omega) \end{matrix}\right\} \Longrightarrow I_t(\psi) \subseteq I_\tau(\omega).$$

The last implication is intuitively obvious, but we give a formal proof here.

*Proof.* Let $\psi \in I_\tau(\omega), t \geq \tau(\omega), \varphi \in I_t(\psi)$, and $\omega \in E \in \mathcal{F}_\tau$; we are required to prove that $\varphi \in E$. From $\psi \in I_\tau(\omega)$ and $\tau(\omega) \leq t$ we deduce $\tau(\psi) \leq t$ (since $\{\tau \leq t\} \in \mathcal{F}_\tau$); $\psi \in I_\tau(\omega)$ and $\omega \in E \in \mathcal{F}_\tau$ implies $\psi \in E$. Thus, $\psi \in E \cap \{\tau \leq t\}$. By Theorem 2.10 (ii) in Elliot (1982), $E \cap \{\tau \leq t\} \in \mathcal{F}_t$. The last two inclusions and $\varphi \in I_t(\psi)$ imply $\varphi \in E \cap \{\tau \leq t\}$.

Now assume that (a) holds but $\mu$ is not coherent, i.e., there are an elementary martingale $M$, a constant $c \geq 0$, and $\omega \in \Omega$ such that

$$\inf_{\psi \in I_c(\omega)} \underline{\lim}_t M_t(\psi) \quad > \quad M_c(\omega).$$

We insert a real number $C$ between the two sides of this inequality. Then the elementary martingale $X_t := M_t - C$ satisfies $X_c(\omega) < 0$ and $\underline{\lim} X_t(\psi) > 0$ for all $\psi \in I_c(\omega)$. Hence,

$$\bar{E}_\mu[0|c](\omega) < 0.$$

*Proof of Theorem 2.*

(a) It suffices to note that the sum of two elementary gambling strategies is again an elementary gambling strategy and

$$\left.\begin{array}{l}\underline{\lim}_t M_t \geq \eta \\ \underline{\lim}_t X_t \geq \xi\end{array}\right\} \Longrightarrow \underline{\lim}_t (M_t + X_t) \geq \eta + \xi.$$

(b) Let $M$ range over the elementary martingales. For $c > 0$,

$$\bar{E}_\mu[c\eta|\tau](\omega) = \inf\{M_\tau(\omega) : \underline{\lim} M_t \geq c\eta \text{ on } I_\tau(\omega)\}$$

$$= \inf\{cX_\tau(\omega) : \underline{\lim} X_t \geq \eta \text{ on } I_\tau(\omega)\} = c\bar{E}_\mu[\eta|\tau](\omega),$$

putting $X := M/c$; for $c < 0$,

$$\bar{E}_\mu[c\eta|\tau] = \bar{E}_\mu[(-c)(-\eta)|\tau] = -c\bar{E}_\mu[-\eta|\tau] = c\underline{E}_\mu[\eta|\tau].$$

When $c = 0$, it suffices to note that the identical 0 is an elementary martingale and make use of item (d).

(c) Since the identical 1 is an elementary martingale, we have $\bar{E}_\mu[1|\tau] \leq 1$; the opposite inequality follows from Theorem 1(b).

(d) By Theorem 1(b),

$$\eta \geq 0 \quad \Longrightarrow \quad \bar{E}_\mu[\eta|\tau] = \bar{E}^*_\mu[\eta|\tau] \geq 0.$$

*Proof of Corollary 1.* We are required to prove that

$$\bar{E}_\mu[\eta|\tau] + \bar{E}_\mu[-\eta|\tau] \geq 0.$$

This immediately follows from parts (a) and (d) of Theorem 2:

$$\bar{E}_\mu[\eta|\tau] + \bar{E}_\mu[-\eta|\tau] \geq \bar{E}_\mu[\eta - \eta|\tau] = \bar{E}_\mu[0|\tau] \geq 0.$$

*Proof of Theorem 3.*
(a) By Theorem 2(a),

$$\bar{E}[\eta+\xi|\tau] \le \bar{E}[\eta|\tau] + \bar{E}[\xi|\tau];$$
$$\underline{E}[\eta+\xi|\tau]$$
$$= -\bar{E}[-\eta-\xi|\tau] \ge -\bar{E}[-\eta|\tau] - \bar{E}[-\xi|\tau] = \underline{E}[\eta|\tau] + \underline{E}[\xi|\tau].$$

(b) It suffices to apply Theorem 2(b) and note that

$$c \ge 0 \quad \Longrightarrow \quad \underline{E}[c\eta|\tau] = -\bar{E}[-c\eta|\tau] = -c\,\bar{E}[-\eta|\tau] = c\,\underline{E}[\eta|\tau],$$
$$c \le 0 \quad \Longrightarrow \quad \underline{E}[c\eta|\tau] = -\bar{E}[-c\eta|\tau] = c\,\bar{E}[\eta|\tau].$$

(c) Since the identical $-1$ is an elementary martingale, we have

$$\underline{E}[1|\tau] = -\bar{E}[-1|\tau] \ge -(-1) = 1.$$

The result $\bar{E}[1|\tau] = 1$ was proved earlier in Theorem 2(c).
(d) Immediate from Theorem 2(d).

*Proof of Theorem 4.* Let $|\eta| \le c$. We are required to prove that

$$\bar{E}[\eta] \ge \int \eta\, dP; \tag{4}$$

indeed, in this case, we have

$$\underline{E}[\eta] = -\bar{E}[-\eta] \le -\int (-\eta)\, dP = \int \eta\, dP,$$

so $\underline{E}[\eta] = \bar{E}[\eta]$ implies $E[\eta] = \int \eta\, dP$. Furthermore, we may assume that $\eta \ge 0$; the general case reduces to this as follows:

$$\bar{E}[\eta] = \bar{E}[\eta + c] - c \ge \int (\eta + c) dP - c = \int \eta\, dP.$$

Thus, we need only prove the result for $\eta \ge 0$. By Theorem 1(b), we may replace $\bar{E}$ by $\bar{E}^*$ in (4). It suffices to prove that there exists no non-negative elementary martingale $M$ such that

$$M_0 < \int \eta\, dP, \quad \underline{\lim}_{t\to\infty} M_t(\omega) \ge \eta(\omega), \forall \omega. \tag{5}$$

Let us assume that such an $M$ exists. Then $M$ is a non-negative local martingale, with respect to $(\mathcal{F}_t)$ and $P$, (Dellacherie and Meyer, 1982, Ch. VIII, Nos. 3 and 9) and, hence, a non-negative supermartingale (Dellacherie and Meyer, 1982, Ch. VI, N. 29). Now, from (5), we can deduce that $M_0 < \int M_\infty dP$ (where $M_\infty := \underline{\lim} M_t$), which contradicts the fact that $M_n, n = 0, \ldots, \infty$, is a supermartingale (Dellacherie and Meyer, 1982, Ch. V, Nos. 28 and 29).

*Proof of Theorem 5.* We are required to prove that, for any two non-negative numbers $a < b$ and any bounded $U : \mathrm{N} \to \Re$,

$$\bar{E}\left[U\left(N_{\alpha_b} - N_{\alpha_a}\right)|\alpha_a\right] \leq \int_{\mathrm{N}} U(u)\mathcal{P}_{b-a}(du). \tag{6}$$

We fix $a, b$, and $U$. Specify a very large integer $L > b$ and put

$$c_i := a + i\frac{b-a}{L}, i = 0, \ldots, L,$$

so that $c_0 = a$ and $c_L = b$, and

$$\tau_i := \inf\{t : A_t = c_i\}, i = 0, \ldots, L,$$

$$\tau_{-1} := 0, \tau_{L+1} := \infty.$$

We define inductively, for $n \in \mathrm{N}$, the quantities

$$U_L(n) := U(n),$$

$$U_{i-1}(n) := \frac{b-a}{L}U_i(n+1) + \left(1 - \frac{b-a}{L}\right)U_i(n), i = L, \ldots, 1,$$

in the hope that this "backward averaging" will transform $U(N_{\tau_L} - N_{\tau_0})$ into $U_0(0) \approx \int U d\mathcal{P}_{b-a}$ in terms of $U(N_{\tau_L} - N_{\tau_0})$. Let $\sigma$ be the stopping time

$$\sigma := \inf\left\{t : t \leq \tau_L, \exists i : N_{\tau_{i+1}\wedge t} - N_{\tau_i \wedge t} \geq 2\right\},$$

where it is desired that $\sigma = \infty$ in most cases. We let

$$\bar{N}_i := N_{\tau_i \wedge \sigma} - N_{\tau_0}, i = 0, \ldots, L,$$

where $\bar{N}_i$ is allowed to take the value $\infty$ and is defined as soon as $\tau_0 < \infty$; it is not required that $\tau_i \wedge \sigma < \infty$. Define $H$ to be the elementary gambling strategy associated with the sequence $(\tau_i \wedge \sigma)$ of stopping times and the sequence

$$h_i := U_{i+1}(\bar{N}_i + 1) - U_{i+1}(\bar{N}_i),\ i = 0, \ldots, L-1, h_{-1} := h_L := 0$$

of partial random variables (the domain of $h_i$ is $\{\tau_i \wedge \sigma < \infty\}$). Also, define $M$ to be the elementary martingale

$$M_t := U_0(0) + (H \cdot \mu)_t.$$

We shall prove that, for all $k = 0, \ldots, L$ and all $\omega$ with $\tau_k(\omega) < \sigma(\omega)$,

$$M_{\tau_i}(\omega) = U_i(\bar{N}_i(\omega)),\ i = 0, \ldots, k. \tag{7}$$

When $i = 0$, we have equality, so it suffices to prove that

$$M_{\tau_{i+1}} - M_{\tau_i} = U_{i+1}(\bar{N}_{i+1}) - U_i(\bar{N}_i), \quad i = 0, \ldots, k-1,$$

where, to simplify notation, we drop the argument $\omega$, i.e.,

$$h_i \cdot \left(\mu_{\tau_{i+1}} - \mu_{\tau_i}\right) = U_{i+1}(\bar{N}_{i+1}) - U_i(\bar{N}_i),$$

which is equivalent to

$$\begin{aligned}\left(U_{i+1}(\bar{N}_i + 1) - U_{i+1}(\bar{N}_i)\right) \left((\bar{N}_{i+1} - c_{i+1}) - (\bar{N}_i - c_i)\right) \\ = U_{i+1}(\bar{N}_{i+1}) - U_i(\bar{N}_i),\end{aligned} \tag{8}$$

i.e.,

$$\left(U_{i+1}(\bar{N}_i + 1) - U_{i+1}(\bar{N}_i)\right) \left(\bar{N}_{i+1} - \bar{N}_i - \frac{b-a}{L}\right)$$

$$= U_{i+1}(\bar{N}_{i+1}) - \frac{b-a}{L} U_{i+1}(\bar{N}_i + 1) - \left(1 - \frac{b-a}{L}\right) U_{i+1}(\bar{N}_i),$$

i.e.,

$$\left(U_{i+1}(\bar{N}_i + 1) - U_{i+1}(\bar{N}_i)\right) \left(\bar{N}_{i+1} - \bar{N}_i\right) = U_{i+1}(\bar{N}_{i+1}) - U_{i+1}(\bar{N}_i).$$

It is easy to see that this equality holds both when $\bar{N}_{i+1} = \bar{N}_i$ and when $\bar{N}_{i+1} = \bar{N}_i + 1$ (other possibilities are excluded by the requirement that $\tau_k(\omega) < \sigma(\omega)$). Thus, (7) is proved.

Note that $M_{\tau_0} = U_0(0)$ may be expressed as

$$\int_{\{0,1\}^L} U(u_1 + \cdots + u_L)\mathcal{B}(du_1)\ldots\mathcal{B}(du_L),$$

where $\mathcal{B}$ is the probability distribution on the set $\{0,1\}$ such that

$$\mathcal{B}\{1\} = \frac{b-a}{L}.$$

So, for large $L$, Poisson's theorem implies that $M_{\tau_0}$ is close to $\int U d\mathcal{P}_{b-a}$. This and (7) imply that $M$ is a suitable martingale for use in the proof of (6).

One essential requirement which $M$ may violate is $M \geq \inf U$. Let us show that $M_t$, where $t < \sigma$, cannot be much smaller than $\inf U$. Let $t \in ]\tau_i, \tau_{i+1} \wedge \sigma[$ (for $t = \tau_j$, the inequality $M_t \geq \inf U$ follows from (7)) and $|U| \leq c$. It is easy to see that (8) will hold when $\bar{N}_{i+1}$ is replaced with $N_t - N_{\tau_0}$. Therefore,

$$\left(U_{i+1}(\bar{N}_i + 1) - U_{i+1}(\bar{N}_i)\right)\left((N_t - N_{\tau_0} - A_t) - (\bar{N}_i - c_i)\right)$$
$$\geq U_{i+1}(N_t - N_{\tau_0}) - U_i(\bar{N}_i) - \frac{2c(b-a)}{L},$$

i.e.,

$$M_t - M_{\tau_i} \geq U_{i+1}(N_t - N_{\tau_0}) - U_i(\bar{N}_i) - \frac{2c(b-a)}{L}.$$

Adding this inequality to $M_{\tau_i} = U_i(\bar{N}_i)$ gives

$$M_t \geq U_{i+1}(N_t - N_{\tau_0}) - \frac{2c(b-a)}{L} \geq \inf U - \frac{2c(b-a)}{L}.$$

So, for large $L$, the elementary martingale

$$X_t := M_t + \frac{2c(b-a)}{L}$$

satisfies $X_t \geq \inf U$ for $t < \sigma, X_{\tau_0}$ is close to $\int U \, d\mathcal{P}_{b-a}$ and $X_{\tau_L} \geq U(\bar{N}_L)$ when $\tau_L < \sigma$.

Thus, proving (6) reduces to proving that $\Pr[\sigma < \infty|\tau_0]$ is small when $L$ is large (indeed, since $U$ and all the $h_i$ are bounded, a small value for this probability implies the existence of an elementary martingale $Y$ such that $Y \geq \inf U$ everywhere, $Y_{\alpha_a}$ is close to $\int U d\mathcal{P}_{b-a}$ and $Y_{\alpha_b} \geq U(N_{\alpha_b} - N_{\alpha_a})$ when $\alpha_b < \infty$). Fix $\varepsilon > 0$. It is evident that $\Pr[\bar{N}_L > r|\tau_0] \leq \varepsilon$ for some constant $r = r(\varepsilon)$, so we are reduced to proving that

$$\Pr\left[\bar{N}_L \leq r, \sigma < \infty|\tau_0\right] \leq \varepsilon.$$

The idea behind our construction is straightforward: at the time of a jump of $N$, we stake one unit on the basic martingale until reaching the next stopping time $\tau_i$.

Put $\sigma_{-1} := 0, \sigma_0 := \tau_0, \sigma_{2r+1} := \infty$; the remaining $\sigma_k, k = 1, \ldots, 2r$, will be defined inductively. Assume $\sigma_{2j}, j < r$, to be already specified. Put

$$\sigma_{2j+1} := \inf\{t : \sigma_{2j} < t \leq \tau_L, \Delta N_t = 1\},$$

with $\Delta N_t := N_t - N_{t-}$,

$$\sigma_{2j+2} := \inf\{t : t \geq \sigma_{2j+1}, \exists i : t = \tau_i\}. \tag{9}$$

Define

$$h_{-1} := h_{2r} := 0, h_{2j} := 0, h_{2j+1} := 1, j = 0, \ldots, r-1.$$

Let $H_t$ be the elementary gambling strategy corresponding to $(\sigma_k)$ and $(h_k)$ and $Z$ be the elementary martingale

$$Z_t := \varepsilon + (H \cdot \mu)_t.$$

Then $Z_{\tau_0} = \varepsilon$ and, provided $L$ is sufficiently large,

$$Z_t \geq \varepsilon - r\frac{b-a}{L} \geq 0, \forall t,$$

$$\bar{N}_L(\omega) \leq r \& \sigma(\omega) < \infty \Longrightarrow Z_\infty(\omega) \geq \varepsilon - r\frac{b-a}{L} + 1 \geq 1.$$

*Remark 7.* In the proof of Theorem 5, we have not used the coherence of the basic martingale. When the basic martingale is coherent, this proof implies that

$$\bar{E}\left[U(N_{\alpha_b}-N_{\alpha_a})|\alpha_a\right]=\underline{E}\left[U(N_{\alpha_b}-N_{\alpha_a})|\alpha_a\right]=\int U d\mathcal{P}_{b-a}$$

if, for $\alpha_b=\infty$,

$$U(N_{\alpha_b}-N_{\alpha_a}):=\begin{cases}\int U(N_\infty-N_{\alpha_a}+v)\mathcal{P}_{b-A_\infty}(dv), \text{ if } N_\infty<\infty,\\ 0, \text{ otherwise.}\end{cases}$$

*Proof of Theorem 6.* Without loss of generality, we assume that $B_t=0$, $\forall t$, (otherwise, we may redefine $W_t$ to be $W_t-B_t$). Fix $0\le a<b$ and bounded continuous $U:\Re\longrightarrow\Re$. We are required to prove that

$$\bar{E}\left[U\left(W_{\alpha_b}-W_{\alpha_a}\right)|\alpha_a\right]\le\int_{\Re}U(u)\mathcal{N}_{0,b-a}(du). \tag{10}$$

We assume that $U$ is a smooth function vanishing outside a finite interval (we can do this since $W_t^2-A_t$ is a component of the basic martingale).

We consider a very large integer $L>b$ and define $c_i,\tau_i$ as in the proof of Theorem 5. Define inductively, for $u\in\Re$, the quantities

$$U_L(u):=U(u),$$

$$U_{i-1}(u):=\int_{\Re}U_i(u+w)\mathcal{N}_{0,(b-a)/L}(dw), i=L,\dots,1.$$

Note that $U_0(0)=\int U d\mathcal{N}_{0,b-a}$. Define

$$\sigma:=\inf\left\{t:\sum_{i=0}^{L-1}|W_{\tau_{i+1}\wedge t}-W_{\tau_i\wedge t}|^3\ge 9\varepsilon\right\},$$

where $\varepsilon$ is an arbitrarily small positive constant, and let

$$\bar{W}_i:=W_{\tau_i\wedge\sigma}-W_{\tau_0},\tilde{W}_i:=W_{\tau_i\wedge\sigma}, i=0,\dots,L,$$

$$h_i:=\left(U'_{i+1}(\bar{W}_i)-U''_{i+1}(\bar{W}_i)\tilde{W}_i;\frac{1}{2}U''_{i+1}(\bar{W}_i)\right), i=0,\dots,L-1,$$

$$h_{-1} := h_L := (0; 0).$$

$\bar{W}_i, \tilde{W}_i$, and $h_i$ are undefined when $\tau_i \wedge \sigma = \infty$; intuitively, the first component of $h_i$ is our stake on $W_t$ and its second component is our stake on $W_t^2 - A_t$. Let $H$ be the elementary gambling strategy associated with $(\tau_i \wedge \sigma)$ and $(h_i)$ and define

$$M_t := U_0(0) + 9K\varepsilon + 3K\frac{(b-a)^{3/2}}{L^{1/2}} + (H \cdot \mu)_t, \tag{11}$$

where $K$ is a positive constant specified below. We wish to prove that

$$M_{\tau_i}(\omega) \geq U_i(\bar{W}_i(\omega)),\ i = 0, \ldots, k,$$

provided $\tau_k(\omega) < \sigma(\omega)$. It suffices to prove that

$$M_{\tau_{i+1}} - M_{\tau_i} + K|\bar{W}_{i+1} - \bar{W}_i|^3$$

$$\geq U_{i+1}(\bar{W}_{i+1}) - U_i(\bar{W}_i) - 3K\left(\frac{b-a}{L}\right)^{3/2},\ i = 0, \ldots, k-1,$$

again, dropping the argument $\omega$, i.e.,

$$h_i \cdot \left(\mu_{\tau_{i+1}} - \mu_{\tau_i}\right) + K|\bar{W}_{i+1} - \bar{W}_i|^3$$

$$\geq U_{i+1}(\bar{W}_{i+1}) - U_i(\bar{W}_i) - 3K\left(\frac{b-a}{L}\right)^{3/2},$$

which is equivalent to

$$\left(U'_{i+1}(\bar{W}_i) - U''_{i+1}(\bar{W}_i)\tilde{W}_i\right)(\bar{W}_{i+1} - \bar{W}_i)$$

$$+\frac{1}{2}U''_{i+1}(\bar{W}_i)\left((\tilde{W}_{i+1}^2 - c_{i+1}) - (\tilde{W}_i^2 - c_i)\right) + K|\bar{W}_{i+1} - \bar{W}_i|^3$$

$$\geq U_{i+1}(\bar{W}_{i+1}) - U_i(\bar{W}_i) - 3K\left(\frac{b-a}{L}\right)^{3/2}, \tag{12}$$

i.e.,

$$\left(U'_{i+1}(\bar{W}_i) - U''_{i+1}(\bar{W}_i)\tilde{W}_i\right)(\bar{W}_{i+1} - \bar{W}_i)$$

$$+\frac{1}{2}U''_{i+1}(\bar{W}_i)\left(\tilde{W}_{i+1}^2-\tilde{W}_i^2-\frac{b-a}{L}\right)+K|\bar{W}_{i+1}-\bar{W}_i|^3$$

$$\geq U_{i+1}(\bar{W}_{i+1})-\int U_{i+1}(\bar{W}_i+v)\,\mathcal{N}_{0,(b-a)/L}(dv)$$

$$-3K\left(\frac{b-a}{L}\right)^{3/2}. \qquad (13)$$

There exists a constant $K>0$ such that, for all $u$ and $v$,

$$|U(u+v)-\left(U(u)+U'(u)\,v+\frac{1}{2}U''(u)v^2\right)|\leq K|v|^3. \qquad (14)$$

It is easy to see that this inequality will continue to hold when $U$ is replaced by $U_{i+1}$: we can repeatedly replace $u$ by $u+w$ in (14) and then average the left-hand side with respect to $\mathcal{N}_{0,(b-a)/L}(dw)$ (Kolmogorov, 1950, Ch. IV, Section 5, Theorem 1). Thus, (14) implies that

$$|\int U_{i+1}(\bar{W}_i+v)\mathcal{N}_{0,(b-a)/L}(dv)-\left(U_{i+1}(\bar{W}_i)+\frac{1}{2}U''_{i+1}(\bar{W}_i)\frac{b-a}{L}\right)|$$

$$\leq K\int |v|^3\mathcal{N}_{0,(b-a)/L}(dv).$$

Now noting that

$$K\int |v|^3\,\mathcal{N}_{0,(b-a)/L}(dv)=K\left(\frac{b-a}{L}\right)^{3/2}\int |v|^3\,\mathcal{N}_{0,1}(dv)$$

$$\leq 3K\left(\frac{b-a}{L}\right)^{3/2},$$

we can see that (13) reduces to

$$\left(U'_{i+1}(\bar{W}_i)-U''_{i+1}(\bar{W}_i)\tilde{W}_i\right)\left(\bar{W}_{i+1}-\bar{W}_i\right)$$

$$+\frac{1}{2}U''_{i+1}(\bar{W}_i)\left(\tilde{W}_{i+1}^2-\tilde{W}_i^2-\frac{b-a}{L}\right)+K|\bar{W}_{i+1}-\bar{W}_i|^3$$

$$\geq U_{i+1}(\bar{W}_{i+1})-U_{i+1}(\bar{W}_i)-\frac{1}{2}U''_{i+1}(\bar{W}_i)\frac{b-a}{L},$$

i.e.,

$$\left(U'_{i+1}(\bar{W}_i) - U''_{i+1}(\bar{W}_i)\tilde{W}_i\right)(\bar{W}_{i+1} - \bar{W}_i)$$
$$+\frac{1}{2}U''_{i+1}(\bar{W}_i)\left(\tilde{W}^2_{i+1} - \tilde{W}^2_i\right) + K|\bar{W}_{i+1} - \bar{W}_i|^3$$
$$\geq U_{i+1}(\bar{W}_{i+1}) - U_{i+1}(\bar{W}_i). \tag{15}$$

Since

$$\tilde{W}^2_{i+1} - \tilde{W}^2_i = \left((\tilde{W}_{i+1} - \tilde{W}_i) + \tilde{W}_i\right)^2 - \tilde{W}^2_i$$
$$= (\tilde{W}_{i+1} - \tilde{W}_i)^2 + 2(\tilde{W}_{i+1} - \tilde{W}_i)\tilde{W}_i$$
$$= (\bar{W}_{i+1} - \bar{W}_i)^2 + 2\tilde{W}_i(\bar{W}_{i+1} - \bar{W}_i),$$

we may rewrite (15) as

$$U'_{i+1}(\bar{W}_i)\left(\bar{W}_{i+1} - \bar{W}_i\right) + \frac{1}{2}U''_{i+1}(\bar{W}_i)\left(\bar{W}_{i+1} - \bar{W}_i\right)^2 + K|\bar{W}_{i+1} - \bar{W}_i|^3$$
$$\geq U_{i+1}(\bar{W}_{i+1}) - U_{i+1}(\bar{W}_i).$$

The last inequality reduces to (14).

As in the proof of Theorem 5, there is a problem with the inequality $M \geq \inf U$ (this problem is now less serious because $M$ is continuous). Let $t \in ]\tau_i, \tau_{i+1} \wedge \sigma[$ and $|U''| \leq c$. It is easy to see that (12) will hold when $\bar{W}_{i+1}$ and $\tilde{W}_{i+1}$ are replaced by $W_t - W_{\tau_0}$ and $W_t$ respectively. So

$$\left(U'_{i+1}(\bar{W}_i) - U''_{i+1}(\bar{W}_i)\tilde{W}_i\right)\left(W_t - \tilde{W}_i\right)$$
$$+\frac{1}{2}U''_{i+1}(\bar{W}_i)\left((W^2_t - A_t) - (\tilde{W}^2_i - c_i)\right) + K|W_t - \tilde{W}_i|^3$$
$$\geq U_{i+1}(W_t - W_{\tau_0}) - U_i(\bar{W}_i) - 3K\left(\frac{b-a}{L}\right)^{3/2} - \frac{c(b-a)}{2L},$$

i.e.,

$$M_t - M_{\tau_i} + K|W_t - \tilde{W}_i|^3$$
$$\geq U_{i+1}(W_t - W_{\tau_0}) - U_i(\bar{W}_i) - 3K\left(\frac{b-a}{L}\right)^{3/2} - \frac{c(b-a)}{2L}.$$

Adding this inequality to $M_{\tau_i} \geq U_i(\bar{W}_i)$ gives

$$M_t \geq U_{i+1}(W_t - W_{\tau_0}) - K|W_t - \tilde{W}_i|^3 - 3K\left(\frac{b-a}{L}\right)^{3/2} - \frac{c(b-a)}{2L}$$

$$\geq \inf U - 9K\varepsilon - 3K\left(\frac{b-a}{L}\right)^{3/2} - \frac{c(b-a)}{2L}.$$

So, by replacing $M_t$ with

$$X_t := M_t + 9K\varepsilon + 3K\left(\frac{b-a}{L}\right)^{3/2} + \frac{c(b-a)}{2L},$$

we achieve $X \geq \inf U$; also, $X_{\tau_0}$ is close to $\int U d\mathcal{N}_{0,b-a}$ (when $\varepsilon$ is small and $L$ is large) and $X_{\tau_L} \geq U(\bar{W}_L)$ when $\tau_L < \sigma$.

Now redefine $\tilde{W}_i := W_{\tau_i}$. It remains for us to prove that, for arbitrarily small constants $\varepsilon, \delta > 0$,

$$\Pr\left[\tau_L < \infty, \sum_{i=0}^{L-1} |\tilde{W}_{i+1} - \tilde{W}_i|^3 \geq 9\varepsilon | \tau_0\right] \leq \delta, \qquad (16)$$

provided that $L$ is sufficiently large. Later, we shall prove the following two assertions:

(a) there exists a constant $C = C(\delta)$ such that, for large $L$, $\Pr[E_1|\tau_0] \leq \delta/2$, where

$$E_1 := \left\{\omega : \tau_L(\omega) < \infty, \sum_{i=0}^{L-1}\left(\tilde{W}_{i+1}(\omega) - \tilde{W}_i(\omega)\right)^2 > C\right\};$$

(b) $\Pr[E_2^c \cap E_3|\tau_0] \leq \delta/2$, where

$$E_2 := \left\{\omega : \tau_L(\omega) < \infty, \operatorname{Card} \jmath(\omega) > C^3/\varepsilon^2\right\},$$

$$\jmath(\omega) := \{i = 0, \dots, L-1 : |\tilde{W}_{i+1}(\omega) - \tilde{W}_i(\omega)| \geq \varepsilon/C\},$$

$$E_3 := \{\omega : \tau_L(\omega) < \infty,$$
$$\exists i \in \jmath(\omega) : |\tilde{W}_{i+1}(\omega) - \tilde{W}_i(\omega)| \geq 2\varepsilon/C\}.$$

We shall now see that (a), (b) imply (16). First note that $E_2 \subseteq E_1$ (since $(C^3/\varepsilon^2)\ (\varepsilon/C)^2 = C$) or, equivalently, $E_1^c \subseteq E_2^c$. Part (b) implies $\Pr[E_1^c \cap E_3 | \tau_0] \le \delta/2$. So, by part (a), $\Pr[E_1 \cup E_3 | \tau_0] \le \delta$. This implies $\Pr[E_1 \cup E_2 \cup E_3 | \tau_0] \le \delta$. It remains to note that on the complement $E_1^c \cap E_2^c \cap E_3^c$ of $E_1 \cup E_2 \cup E_3$ we have either $\tau_L = \infty$ or

$$\begin{aligned} &\sum_{i=0}^{L-1} |\tilde{W}_{i+1}(\omega) - \tilde{W}_i(\omega)|^3 \\ &= \sum_{i \in j(\omega)} |\tilde{W}_{i+1}(\omega) - \tilde{W}_i(\omega)|^3 + \sum_{i \notin j(\omega)} |\tilde{W}_{i+1}(\omega) - \tilde{W}_i(\omega)|^3 \\ &< (C^3/\varepsilon^2)(2\varepsilon/C)^3 + (\varepsilon/C)C = 8\varepsilon + \varepsilon = 9\varepsilon. \end{aligned}$$

Thus, our task has reduced to proving (a) and (b). Let us begin with (a). Define

$$h_i := \left(-\frac{\delta}{b-a}\tilde{W}_i; \frac{\delta}{2(b-a)}\right), i = 0, \dots, L-1, h_{-1} := h_L := (0;0),$$

and let $H_t$ be the elementary gambling strategy associated with $(\tau_i)$ and $(h_i)$ and $X$ be the elementary martingale

$$X_t := \frac{\delta}{2} + (H \cdot \mu)_t.$$

When $\tau_{i+1} < \infty$,

$$\begin{aligned} (H \cdot \mu)_{\tau_{i+1}} - (H \cdot \mu)_{\tau_i} &= h_i \cdot \left(\mu_{\tau_{i+1}} - \mu_{\tau_i}\right) \\ &= -\frac{\delta}{b-a}\tilde{W}_i \left(\tilde{W}_{i+1} - \tilde{W}_i\right) \\ &\quad + \frac{\delta}{2(b-a)} \left(\left(\tilde{W}_{i+1}^2 - A_{\tau_{i+1}}\right) - \left(\tilde{W}_i^2 - A_{\tau_i}\right)\right) \\ &= \frac{\delta}{2(b-a)} \left(\left(\tilde{W}_{i+1} - \tilde{W}_i\right)^2 - \left(A_{\tau_{i+1}} - A_{\tau_i}\right)\right), \end{aligned}$$

and we have $X_{\tau_0} = \delta/2$,

$$t \le \tau_L \Longrightarrow X_t \ge \delta/2 - \frac{\delta}{2(b-a)} \sum_{i=0}^{L-1} \left(A_{\tau_{i+1} \wedge t} - A_{\tau_i \wedge t}\right)$$

$$\geq \delta/2 - \frac{\delta}{2(b-a)} \sum_{i=0}^{L-1} \frac{b-a}{L} = 0,$$

and, on $E_1$,

$$X_{\tau_L} \geq \frac{\delta}{2(b-a)} \sum_{i=0}^{L-1} \left(\tilde{W}_{i+1} - \tilde{W}_i\right)^2 > \frac{\delta}{2(b-a)} C.$$

Therefore, it suffices to take $C = 2(b-a)/\delta$.

It remains to prove (b). The idea behind our construction is quite natural: after detecting that $i \in j(\omega)$, we stake a great deal on the martingale $W_t^2 - A_t$ and, thus, our gain is non-negligible when the oscillation is great, despite the fact that we cancel the stake very soon after. Without loss of generality, we assume that $r := C^3/\varepsilon^2$ is an integer.

Define $\sigma_{-1} := 0, \sigma_0 = \tau_0$, and $\sigma_{2r+1} := \infty$; for $k = 1, \ldots, 2r, \sigma_k$ will be defined inductively. Assume that $\sigma_{2j}$, $j < r$, is already defined. Let

$$\sigma_{2j+1} := \inf\{t : \sigma_{2j} < t \leq \tau_L,$$
$$\exists i : \tau_i < t \leq \tau_{i+1}, |W_t - \tilde{W}_i| \geq \varepsilon/C\}$$

and define $\sigma_{2j+2}$ by (9). We then define

$$h_{-1} := h_{2r} := (0; 0),$$
$$h_{2j} := (0; 0), h_{2j+1} := \left(-\frac{L\delta}{r(b-a)} W_{\sigma_{2j+1}}; \frac{L\delta}{2r(b-a)}\right),$$
$$j = 0, \ldots, r-1.$$

Let $H_t$ be the elementary gambling strategy associated with $(\sigma_k)$ and $(h_k)$ and $X$ be the elementary martingale

$$X_t := \frac{\delta}{2} + (H \cdot \mu)_t.$$

Since, for $\sigma_{2j+2} < \infty$,

$$(H \cdot \mu)_{\sigma_{2j+2}} - (H \cdot \mu)_{\sigma_{2j+1}} = h_{2j+1} \cdot \left(\mu_{\sigma_{2j+2}} - \mu_{\sigma_{2j+1}}\right)$$

$$= -\frac{L\delta}{r(b-a)} W_{\sigma_{2j+1}} \left( W_{\sigma_{2j+2}} - W_{\sigma_{2j+1}} \right)$$

$$+ \frac{L\delta}{2r(b-a)} \left( \left( W^2_{\sigma_{2j+2}} - A_{\sigma_{2j+2}} \right) - \left( W^2_{\sigma_{2j+1}} - A_{\sigma_{2j+1}} \right) \right)$$

$$= \frac{L\delta}{2r(b-a)} \left( \left( W_{\sigma_{2j+2}} - W_{\sigma_{2j+1}} \right)^2 - \left( A_{\sigma_{2j+2}} - A_{\sigma_{2j+1}} \right) \right),$$

we have $X_{\tau_0} = \delta/2$,

$$X_t \geq \delta/2 - \frac{L\delta}{2r(b-a)} \sum_{j=0}^{r-1} \frac{b-a}{L} = 0,$$

and we have, on $E_2^c \cap E_3$,

$$X_{\tau_L} \geq \frac{L\delta}{2r(b-a)} \sum_{j=0}^{r-1} \left( W_{\sigma_{2j+2}} - W_{\sigma_{2j+1}} \right)^2$$

$$\geq \frac{L\delta}{2r(b-a)} \left( \frac{2\varepsilon}{C} - \frac{\varepsilon}{C} \right)^2 = \frac{L\delta\varepsilon^2}{2r(b-a)C^2} \geq 1,$$

provided $L$ is sufficiently large.

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