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Graded Rings and Essential Ideals

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Abstract. Let G be a group and A a G-graded ring. A (graded) ideal I of A is (graded) essential if $I \cap J \neq 0$ whenever J is a nonzero (graded) ideal of A. In this paper we study the relationship between graded essential ideals of A, essential ideals of the identity component A_e and essential ideals of the smash product $A\#G^*$. We apply our results to prime essential rings, irredundant subdirect sums and essentially nilpotent rings.

§1. Essential Ideals

Let G be a group with identity e. An associative ring A is G-graded if the additive group of A is a direct sum of subgroups $A_g, g \in G$ which are such that $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$. The grading is faithful if $0 \neq a \in A_g$ implies $aA_h \neq 0$ and $A_h a \neq 0$ for all $g, h \in G$, and the grading is strong if $A_g A_{g^{-1}} = A_e$ for all $g \in G$.

Let A be a G-graded ring. The smash product $A\#G^*$ is an associative ring in which each element can be written uniquely as a finite sum $\sum_g a^g p_g$, $g \in G$, $a^g \in A$, and the multiplication satisfies $ap_g bp_h = ab_{gh^{-1}} p_h$ for $a, b \in A, g, h \in G$ where $b_{gh^{-1}}$ denotes the gh^{-1} component of b. If K is an ideal of $A\#G^*$, then K_e denotes the ideal $\{a \in A_e : ap_e \in K\}$ of A_e (this ideal is used in [8] where the notation is $K_{e,e}$).

Let I be an ideal of a G-graded ring A. For each $g \in G, I_g = I \cap A_g$ and I is graded if $I = \sum_g I_g$. A graded ideal I of A is graded essential if $I \cap J \neq 0$ for each nonzero graded ideal J of A, and an ideal B of A is essential if $B \cap C \neq 0$ for each nonzero ideal C of A.

In this section we prove the following two theorems.

Theorem 1. *Let A be a G-graded ring and let I be a graded ideal of A. The following are equivalent.*

(i) *I is an essential ideal of A.*

(ii) *I is a graded essential ideal of A.*

(iii) *$I\#G^*$ is an essential ideal of $A\#G^*$.*

Theorem 2. *Let A be a G-graded ring with faithful grading, let I be a graded ideal of A and let K be an ideal of $A\#G^*$.*

(i) *K is essential in $A\#G^*$ if and only if K_e is essential in A_e.*

(ii) *I is essential in A if and only if I_e is essential in A_e.*

In subsequent sections we apply these results to prime essential rings, to irredundant subdirect sums and to essential nilpotence.

Let A be a G-graded ring and let I be an ideal of A_e. The associated ideal [I] of $A\#G^*$ is defined in [8] : $[I] = \sum_g AIA_{g^{-1}}p_g$. Note that when A has an identity this is just the ideal of $A\#G^*$ which is generated by I_{p_e}. The results in the following proposition are implicit in [8], but we include a proof for the convenience of the reader.

Proposition 1(Liu). *Let A be a G-graded ring.*

(i) *If K is a nonzero ideal of $A\#G^*$ and the grading on A is faithful, then $K_e \neq 0$.*

(ii) *If I and J are ideals of A_e, then $[I][J] \subseteq [IJ]$.*

Proof. (i) Let $0 \neq \sum_{g,h} a_{g,h}p_g \in K$ where $a_{g,h} \in A_h$, and assume that $a_{\alpha,\beta} \neq 0$. Then

$$A_{(\beta\alpha)^{-1}}p_{\beta\alpha}\left(\sum_{g,h} a_{g,h}p_g\right)A_\alpha p_e = A_{(\beta\alpha)^{-1}}a_{\alpha,\beta}A_\alpha p_e$$

is in K. Since the grading is faithful, $A_{(\beta\alpha)^{-1}}a_{\alpha,\beta}A_\alpha \neq 0$ and so $K_e \neq 0$.

(ii) Let C and D be ideals of A_e. Then

$$[C] = \sum_g ACA_{g^{-1}}p_g$$

and

$$[D] = \sum_h ADA_{h^{-1}}p_h;$$

so

$$\begin{aligned}[C][D] &= \sum_{g,h} ACA_{g^{-1}}A_gDA_{h^{-1}}p_h \\ &\subseteq \sum_h ACDA_{h^{-1}}p_h = [CD].\end{aligned}$$

We remark that in [8] Liu shows that if the grading is strong then the correspondence $C \leftrightarrow [C]$ is a bijection between the set of ideals of A_e and the set of ideals of $A\#G^*$.

Proof of Theorem 1. (i) implies (ii) is obvious. We now show that (ii) implies (iii).

Let J be a nonzero ideal of $A\#G^*$. Write elements of J in the form

$$\sum_{g,h} a_{g,h}p_g \quad \text{where} \quad a_{g,h} \in A_h, \tag{$*$}$$

and choose $u \neq 0$ in J of this form with a minimum number of coefficients $a_{g,h} \notin I$. Suppose $u = a_{\alpha,\beta}p_\alpha + \cdots$ and $a_{\alpha,\beta} \notin I$.

If $Aa_{\alpha,\beta}A \neq 0$ it has a nonzero intersection with I; and since the intersection is graded there is a nonzero $xa_{\alpha,\beta}y \in I$ where x and y are homogeneous. Suppose $x \in A_\gamma$ and $y \in A_\delta$. Then $xp_m uyp_n = xa_{\alpha,\beta}yp_n$ where $m = \beta\alpha$ and $n = \delta^{-1}\alpha$. Since $xp_m uyp_n \in J$ and its one nonzero coefficient is in I we have reached a contradiction. Hence $u \in I\#G^*$.

Now assume $Aa_{\alpha,\beta}A = 0$ but $Aa_{\alpha,\beta} \neq 0$. Then there is a nonzero $xa_{\alpha,\beta} \in I$ for some homogeneous x, say $x \in A_\gamma$. Hence $xp_m u = xa_{\alpha,\beta}p_\alpha + \cdots$ where $m = \beta\alpha$ is in form (*) and has fewer coefficients not in I than u does. Once again we conclude that $u \in I\#G^*$.

If $Aa_{\alpha,\beta}A = 0$ but $a_{\alpha,\beta}A \neq 0$, an argument similar to the one above shows that $u \in I\#G^*$. Finally, if $Aa_{\alpha,\beta} = a_{\alpha,\beta}A = 0$, then $0 \neq ka_{\alpha,\beta} \in I$ for some integer k and so ku has fewer coefficients not in I than u does.

This completes the proof that (ii) implies (iii).

Now assume (iii) and suppose that K is a nonzero ideal of A. Write elements of K as sums of homogeneous components and choose $0 \neq v = a_{g_1} + \cdots + a_{g_n} \in K$ of shortest length and with a maximum number of homogeneous components in I.

Suppose $a_{g_1} \notin I$. If $Aa_{g_1}A \neq 0$ then it has a nonzero intersection with I (since $(Aa_{g_1}A)\#G^*$ has a nonzero intersection with $I\#G^*$), and because the intersection is graded there is a nonzero element $xa_{g_1}y \in I$ where x and y are homogeneous. Now $xvy \neq 0$ and xvy has more homogeneous components in I than v does. We reach the same conclusion in the cases $Aa_{g_1}A = 0$ but $Aa_{g_1} \neq 0$; $Aa_{g_1}A = 0$ but $a_{g_1}A \neq 0$; and $Aa_{g_1} = a_{g_1}A = 0$. Consequently, we conclude that $v \in I$, and this completes the proof.

Proof of Theorem 2. (i) First suppose that K is essential in $A\#G^*$. Let T be a nonzero ideal of A_e. Since the grading is faithful, $[T] \neq 0$; and so since K is essential, $[T] \cap K \neq 0$. From Proposition 1 (i), $([T] \cap K)_e \neq 0$; and since it is clear that $([T] \cap K)_e \subseteq [T]_e \cap K_e$ and that $[T]_e \subseteq T$ we conclude that K_e is essential in A_e.

Conversely, suppose that K_e is essential in A_e, and let J be a nonzero ideal of $A\#G^*$. From Proposition 1 (i), $J_e \neq 0$ and so $K_e \cap J_e \neq 0$. Hence $0 \neq (K_e \cap J_e)p_e \subseteq K \cap J$ and so K is essential in $A\#G^*$.

(ii) If I is a graded ideal of A, then $(I\#G^*)_e = I_e$; and so the result follows from Theorem 1 and (i) of this theorem.

§2. Prime Essential Rings

A ring A is prime essential if A is semiprime and each prime ideal is an essential ideal. Prime essential rings were introduced by Rowen [10] and have recently been studied in [6]. By analogy with the ungraded case, we say that a G-graded ring A is graded prime essential if A is graded semiprime and every graded prime ideal of A is graded essential.

Theorem 3. *Let A be G-graded with faithful grading. The following are equivalent.*

(i) *A_e is prime essential .*

(ii) *$A\#G^*$ is prime essential .*

If G is finite and A has an identity, these are also equivalent to

(iii) *A is graded prime essential.*

Proof. First suppose that A_e is prime essential. If $I \triangleleft A\#G^*$ and $I^2 = 0$, then $(I_e)^2 = 0$. Also, if $I \neq 0$, Proposition 1 (i) implies that $I_e \neq 0$; and so it follows that $A\#G^*$ is semiprime because A_e is semiprime. Let P be a prime ideal of $A\#G^*$ and suppose that C and D are ideals of A_e such that $CD \subseteq P_e$. It follows from Proposition 1 (ii) that $[C][D] \subseteq [CD]$; and since $CD \subseteq P_e, [CD] \subseteq P$. Thus $[C] \subseteq P$ or $[D] \subseteq P$, and so $(A\#G^*)Cp_e(A\#G^*) \subseteq P$ or $(A\#G^*)Dp_e(A\#G^*) \subseteq P$. Since P is prime, $Cp_e \subseteq P$ or $Dp_e \subseteq P$. Hence $C \subseteq P_e$ or $D \subseteq P_e$; and we see that P_e is a prime ideal of A_e. Since A_e is prime essential, P_e is essential in A_e, and hence P is essential in $A\#G^*$ by Theorem 2 (i).

Now suppose that $A\#G^*$ is prime essential. If I is a nonzero ideal of A_e then $[I] \neq 0$ because the grading is faithful and $[I]^2 \subseteq [I^2]$ by Proposition 1 (ii). Hence it follows from the semiprimeness of $A\#G^*$ that A_e is semiprime. Let Q be a prime ideal of A_e. Choose M maximal in $\{K | K \triangleleft A\#G^* \text{ and } K_e = Q\}$. It is straightforward to check that M is prime and so M is essential. Hence Q is essential in A_e by Theorem 2 (i).

We now assume that G is finite and that A has an identity.

Suppose that A_e is prime essential. It follows from [4, Theorem 2.9] that A is graded semiprime. Let P be a graded prime ideal of A. From [4, Lemma 5.1 and Theorem 7.3] we see that $P \cap A_e$ is a finite intersection of prime ideals of A_e. Hence $P \cap A_e = P_e$ is essential in A_e and so P is graded essential in A by Theorem 2 (ii).

Conversely, suppose that A is graded prime essential. Then A_e is semiprime [4, Theorem 2.9]. Let P be a prime ideal of A_e. From [4, Theorem 7.3] there is a prime ideal Q of A with $Q \cap A_e \subseteq P$. Let $Q_G = \sum_g Q \cap A_g$. Then Q_G is a graded prime ideal of A [4, Lemma 5.1], and so Q_G is graded essential in A. Hence $Q_G \cap A_e$ is essential in A_e by Theorem 2 (ii), and since $Q_G \cap A_e \subseteq P$ it follows that P is an essential ideal of A_e.

We note that if A is a ring with identity which is graded by a finite group G where A has no $|G|$-torsion, then it follows from [6, Remark 3] that A is prime essential if and only if $A\#G^*$ is prime essential.

Let S be a ring and let P be the product $\Pi\{S_i | i \in \mathbb{Z} \text{ and } S_i = S \text{ for all } i\}$ (here $\mathbb{Z}$ denotes the ring of integers). We will use the notation $(s_i) = (\cdots s_{-1}, s_0, s_1, \cdots)$ for the elements of P. The subring of P, consisting of all (s_i) such that there is a positive integer n (depending on (s_i)) such that $s_i = s_j$ if $i \equiv j \bmod(n)$, will be denoted by $\overline{S}$.

The next proposition will be used to construct examples to show that (i) and (ii) of Theorem 3 are not equivalent to (iii) for arbitrary groups.

Proposition 2. *If S is a semiprime (respectively, G-graded semiprime) ring, then $\overline{S}$ is prime essential (respectively, G-graded prime essential).*

Proof. We first assume S is semiprime. Clearly $\overline{S}$ is also semiprime. Let P be a prime ideal of $\overline{S}$ and let I be a nonzero ideal of $\overline{S}$. Choose $0 \neq (s_i) \in I$ where $s_i = s_j$ if $i \equiv j \bmod(n)$ and $s_k \neq 0$ for some $k, 0 \leq k < n$. Since S is semiprime, $Ss_k \neq 0$. Select $s \in S$ such that $ss_k \neq 0$, and define (u_i) and (v_i) as follows: $u_i = s$ if $i \equiv k \mod(2n)$, and $u_i = 0$ if otherwise; $v_i = s$ if $i \equiv k+n \mod(2n)$, and $v_i = 0$ if otherwise. Now, $(v_i)(s_i)$ and $(u_i)(s_i)$ are nonzero elements of I and at least one of these is in P because $(v_i)(s_i)\overline{S}(u_i)(s_i) = 0$. Hence P is essential and so $\overline{S}$ is a prime essential ring.

Now assume that S is G-graded. We first show that $\overline{S}$ is G-graded. For each $g \in G$ let $\overline{S}_g = \{(s_i) \in \overline{S} | a_i \in S_g \text{ for all } i \in \mathbb{Z}\}$. Clearly $\overline{S}_g \cap \overline{S}_h = 0$ if $g \neq h$. Let $(s_i) \in \overline{S}$. Since there is a positive integer n such that $s_i = s_j$ if $i \equiv j \mod(n)$, we see that $(s_i) \in \sum\{\overline{S}_g | g \in G\}$.

Now the argument given above to show that S is semiprime implies $\overline{S}$ is prime essential can be adapted to show that S is G-graded semiprime implies $\overline{S}$ is G-graded prime essential.

Example 1. As in [3], let R be the polynomial ring over a field k with commuting indeterminates $\{X_i | i \in \mathbb{Z}\}$. Let $A = R/I$ where I is the ideal of R generated by $\{X_i^2 | i \in \mathbb{Z}\}$, and set $x_i = X_i + I$. Then $G = \mathbb{Z}$ acts as automorphisms on A with $n(x_i) = x_{i+n}$ for each $n \in G$. It is clear that the product of any two nonzero G-invariant ideals of A is nonzero, and so it follows from [9, Theorem II] that the skew group ring $S = A * G$ is prime, hence graded prime. By Proposition 2, $\overline{S}$ is prime essential and graded prime essential. However, $\overline{S}_e = \{(s_i) | s_i \in S_e = A \text{ for all } i \in \mathbb{Z}\}$ is not prime essential since it is not even semiprime.

This example shows that Theorem 3, (iii) implies (i), do not hold for arbitrary groups.

Example 2. Let S be a semiprime ring with identity. Then $G = \mathbb{Z}$ acts on $\overline{S}$ via $n((s_i)) = (t_i)$ where $t_{i+n} = s_i$ for all $i \in \mathbb{Z}$. Since the product of any two nonzero G-invariant ideals is nonzero, it follows from [9, Theorem II] that the skew group ring $A = \overline{S} * G$ is prime, hence graded prime. Thus A is not prime essential or graded prime essential even though $A_e = \overline{S}$ is prime essential.

This examples shows that Theorem 3, (i) implies (iii), do not hold for arbitrary groups.

§3. Irredundant Subdirect Sums

A ring A is an irredundant subdirect sum of rings $A_\gamma : \gamma \in \Gamma$ if and only if there are ideals P_γ of A such that $A_\gamma \cong A/P_\gamma$ for all $\gamma \in \Gamma, \cap_\gamma P_\gamma = 0$, and for all $\delta \in \Gamma, \cap_{\gamma \neq \delta} P_\gamma \neq 0$. When the P_γ are prime ideals it is easy to check that $\cap_{\gamma \neq \delta} P_\gamma$ is the annihilator of P_δ; in particular, each P_δ is a minimal prime ideal. Irredundant subdirect sums were introduced by Levy[7], and irredundant subdirect sums of prime rings were studied in [10].

Theorem 4. *Let A be a G-graded ring with faithful grading, and consider the following conditions :*

(i) *A_e is an irredundant subdirect sum of prime rings,*

(ii) *$A\#G^*$ is an irredundant subdirect sum of prime rings,*

(iii) *A is an irredundant subdirect sum of graded prime rings.*

Conditions (i) *and* (ii) *are equivalent and they imply* (iii). *When G is finite and A has an identity, all the three conditions are equivalent.*

Proof. First we suppose that A_e has prime ideals $P_\gamma : \gamma \in \Gamma$ such that $\cap_\gamma P_\gamma = 0$ and $\cap_{\gamma \neq \delta} P_\gamma \neq 0$ for each $\delta \in \Gamma$. For each $\gamma \in \Gamma$ choose M_γ maximal in $\{I | I \text{ is an ideal of } A\#G^* \text{ and } I_e = P_\gamma\}$. Then $M_\gamma : \gamma \in \Gamma$ is a family of prime ideals of $A\#G^*$; and if $\delta \in \Gamma, \cap_{\gamma \neq \delta} M_\gamma \neq 0$ because it contains $(\cap_{\gamma \neq \delta} P_\gamma) p_e$. Also, $(\cap_\gamma M_\gamma)_e = \cap_\gamma (M_\gamma)_e = \cap_\gamma P_\gamma = 0$, and so $\cap_\gamma M_\gamma = 0$ by Proposition 1 (i).

Now assume that $A\#G^*$ is an irredundant subdirect sum of prime rings and has prime ideals $Q_\gamma : \gamma \in \Gamma$ such that $\cap_\gamma Q_\gamma = 0$ but $\cap_{\gamma \neq \delta} Q_\gamma \neq 0$ for each $\delta \in \Gamma$. As in the proof of

Theorem 3, $(Q_\gamma)_e : \gamma \in \Gamma$ is a family of prime ideals of A_e and certainly $\cap_\gamma (Q_\gamma)_e = 0$. In particular, A_e is semiprime.

We now show that if $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$, then $(Q_\alpha)_e \neq (Q_\beta)_e$. Let K be the annihilator in $A\#G^*$ of Q_α. Then $K \neq 0$; and so from Proposition 1 (i), $K_e \neq 0$. Since $K \subseteq Q_\beta, K_e p_e \subseteq (Q_\beta)_e p_e$. If $(Q_\alpha)_e = (Q_\beta)_e$, then $(K_e p_e)^2 \subseteq (K_e)p_e(Q_\alpha)_e p_e \subseteq KQ_\alpha = 0$; and so $K_e^2 = 0$, contradicting the semiprimeness of A_e.

Let $\delta \in \Gamma$. Since the annihilator of Q_δ is nonzero, Q_δ is not essential in $A\#G^*$. Hence, by Theorem 1, $(Q_\delta)_e$ is not essential in A_e; and so if I is the annihilator of $(Q_\delta)_e, I \neq 0$. So each $(Q_\gamma)_e, \gamma \in \Gamma$, is a minimal prime because A_e is semiprime and $(Q_\gamma)_e$ has a nonzero annihilator. It follows that $I \subseteq (Q_\gamma)_e$ for all $\gamma \in \Gamma, \gamma \neq \delta$. Hence $\cap_{\gamma\neq\delta}(Q_\gamma)_e \neq 0$, and so A_e is an irredundant subdirect sum of prime rings. We now show that A is an irredundant subdirect sum of graded prime rings.

As in [1], $Q_\gamma^\downarrow = \{a \in A | ap_g \in Q_\gamma \text{ for all } g \in G\}$ is a graded prime ideal of A for each $\gamma \in \Gamma$ and clearly $\cap\{Q_\gamma^\downarrow | \gamma \in \Gamma\} = 0$. However, it may be that $Q_\gamma^\downarrow = Q_\delta^\downarrow$ when $\gamma, \delta \in \Gamma, \gamma \neq \delta$, so we choose $\Delta \subseteq \Gamma$ such that $\cap\{Q_\alpha^\downarrow | \alpha \in \Delta\} = 0$ and $\alpha, \beta \in \Delta, \alpha \neq \beta$ implies $Q_\alpha^\downarrow \neq Q_\beta^\downarrow$. Let $\alpha \in \Delta$ and let J be the annihilator of Q_α in $A\#G^*$. Then $J_e \neq 0$ by Proposition 1(i), and $J_e Q_\alpha^\downarrow = 0$ because $J_e Q_\alpha^\downarrow p_e = (J_e p_e)(Q_\alpha^\downarrow p_e) \subseteq JQ_\alpha = 0$. Let J^* be the ideal of A generated by J_e. Then J^* is a nonzero graded ideal of A and $J^* Q_\alpha^\downarrow = 0$. Since A is graded semiprime it follows that $Q_\alpha^\downarrow$ is a minimal graded prime, and so just as in the ungraded case we see that $\cap\{Q_\beta^\downarrow | \beta \in \Delta, \beta \neq \alpha\} \neq 0$.

Finally, we assume that A has an identity, G is finite and A has graded prime ideals $T_\gamma : \gamma \in \Gamma$ such that $\cap_\gamma T_\gamma = 0$ but $\cap_{\gamma\neq\delta} T_\gamma \neq 0$ for each $\delta \in \Gamma$. From [4; Lemma 5.1 and Theorem 7.3], for each $\gamma \in \Gamma, T_\gamma \cap A_e = \cap_{i=1}^{n_\gamma} P_{\gamma,i}$ where $P_{\gamma,i}$ are prime ideals of A_e and $\cap_{\substack{i=1\\ i\neq j}}^{n_\gamma} P_{\gamma,i} \neq T_\gamma \cap A_e$ for each $j, 1 \leq j \leq n_\gamma$. Clearly $\cap_{\gamma,i} P_{\gamma,i} = 0$. Suppose that for some $\delta \in \Gamma$ and some $j, 1 \leq j \leq n_\delta$,

$$\left(\bigcap_{\gamma\neq\delta}\bigcap_{i=1}^{n_\gamma} P_{\gamma,i}\right) \cap \left(\bigcap_{\substack{i=1\\ i\neq j}}^{n_\delta} P_{\delta,i}\right) = 0.$$

Then $\cap_{\substack{i=1\\ i\neq j}}^{n_\delta} P_{\delta,i}$ is contained in the annihilator of $\cap_{\gamma\neq\delta}\cap_{i=1}^{n_\gamma} P_{\gamma,i} = (\cap_{\gamma\neq\delta} T_\gamma) \cap A_e$. Since the annihilator in A of $\cap_{\gamma\neq\delta} T_\gamma$ is T_δ and since the grading is faithful, the annihilator in A_e of $(\cap_{\gamma\neq\delta} T_\gamma) \cap A_e$ is $T_\delta \cap A_e$. This contradicts the fact that $\cap_{\substack{i=1\\ i\neq j}}^{n_\delta} P_{\delta,i} \neq T_\delta \cap A_e$, and the proof is complete.

The ring A in Example 2 is graded prime and so is certainly an irredundant subdirect sum of graded prime rings. However A_e is prime essential and so A_e is clearly not an irredundant subdirect sum of prime rings. Hence condition (iii) of Theorem 4 does not imply conditions (i) and (ii) for arbitrary groups.

§4. Essential Nilpotence

A ring A is essentially nilpotent if A contains a nilpotent ideal which is essential. Essential nilpotence was introduced by Fisher[5] and it follows easily from the results in that paper that A is essentially nilpotent if and only if the prime radical of $A, N(A)$, is essential in A. Recall that $N(A)$ is the intersection of the prime ideals of A. If A is a G-graded ring, the graded prime radical of A, $N_G(A)$, is the intersection of the graded prime ideals of A, and we say that A is graded essentially nilpotent if $N_G(A)$ is graded essential in A.

Theorem 5. *Let A be a G-graded ring with faithful grading, and consider the following conditions :*

(i) *A_e is essentially nilpotent,*

(ii) *$A\#G^*$ is essentially nilpotent,*

(iii) *A is graded essentially nilpotent.*

Conditions (i) *and* (ii) *are equivalent and are implies by* (iii). *When G is finite and A has an identity, all the three conditions are equivalent.*

Proof. Suppose that N is a nilpotent ideal of A_e which is essential. Let $[N]$ be the associated ideal of $A\#G^*$. From Proposition 1 (ii) we see that $[N]^2 \subseteq [N^2]$, and so $[N]$ is a nilpotent ideal of $A\#G^*$. Let I be a nonzero ideal of $A\#G^*$. Then $I_e \neq 0$ by Proposition 1 (i), and so $I_e \cap N \neq 0$. Since the grading is faithful, this implies that $I \cap [N] \neq 0$, and hence $A\#G^*$ is essentially nilpotent.

Now suppose that K is a nilpotent ideal of $A\#G^*$ which is essential. Then K_e is a nilpotent ideal of A_e , and it follows from Theorem 2 (i) that K_e is essential.

Let P be a prime ideal of A. Clearly $P_G = \{a \in A | a_g \in P \text{ for all } g \in G\}$ is a graded prime ideal of A, and so $N_G(A) \subseteq N(A)$. Suppose that A satisfies (iii). Then $N_G(A)$ is graded essential in A, and hence $(N_G(A))_e$ is essential in A_e by Theorems 1 and 2. Since $N_G(A) \subseteq N(A), (N_G(A))_e \subseteq (N(A))_e$ and since subrings of prime radical rings are prime radical, we have $(N(A))_e \subseteq N(A_e)$. It follows that $N(A_e)$ is essential in A_e, and so A satisfies (i).

Finally, if G is finite and A has an identity then $(N_G(A))_e = N(A_e)$ by [4, Corollary 5.4], and so (i) implies (iii) by Theorem 2 (ii).

The ring S in Example 1 is graded prime, but S_e is essentially nilpotent, so the three conditions in Theorem 5 are not equivalent for arbitrary groups.

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