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CLASSICAL FIELD THEORY METHODS IN LASER PHYSICS. III. PROPAGATION OF LASER RADIATION IN VACUUM OUTSIDE A RESONATOR

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Abstract

The third Part of our work continues the analysis of the problem considered in Parts I and II [J. Russ. Laser Res., **17**, 205 (1996); **18**, 2 (1997)]. We continue the search for the answer to the question: What is laser radiation from the viewpoint of the classical theory of wave fields? Here, we give one of the possible interpretations of propagation of laser radiation in vacuum outside a resonator from both the nonrelativistic and relativistic points of view. The relation between the metric and the radiation field structure is noted. Moreover, the third Part contains the conclusion concerning the work as a whole. It presents the main observations that may be useful for further studies of the problem.

1. Introduction

It is evident that the interpretation of the structure of laser radiation inside a resonator is intimately connected with the interpretation of laser radiation outside a resonator. Therefore, in our opinion, it is desirable to study both structures simultaneously and within the framework of equivalent concepts. However, from the viewpoint of the analysis technique, first it makes sense to study thoroughly the radiation structure inside a resonator. This way of studying the problem is demonstrated here. Parts I and II [1, 2] were devoted to the analysis of this component of the general problem.

Let us qualitatively illustrate the factors responsible for the relation between the structures of a laser radiation field inside and outside a resonator and analyze the strength of this relation.

Let us illustrate this relation by the example of a regular structure of laser radiation at the output of a resonator with plane-parallel mirrors. For simplicity, we consider a two-dimensional problem (x, z). Let us give the interpretation of the laser radiation structure within the framework of the method of normal coordinates related to the radiation field and the resonator geometry. This interpretation is based on the physical concepts stated by us in Part I. The coordinate system corresponding to this interpretation is schematically shown in Fig. 1.

One can see that this coordinate system strongly differs from the coordinate system constructed, for example, within the framework of the model of an incompressible liquid. The main difference consists in the fact that Fig. 1 gives an approximate relation between the metric and the field. This relation is determined mainly by the diffraction effect.

One can see from Fig. 1 that the wave field outside a resonator may be interpreted within the framework of the method of normal coordinates in a way similar to the one used for interpretation of the field inside a resonator in Part I.

This makes it possible to represent the wave field outside a resonator in the form of a system of two orthogonal waves whose wave fronts correspond to the orientation of the coordinate system shown in Fig. 1.

This interpretation of the laser radiation structure is very simple and pictorial, and we shall use it later.

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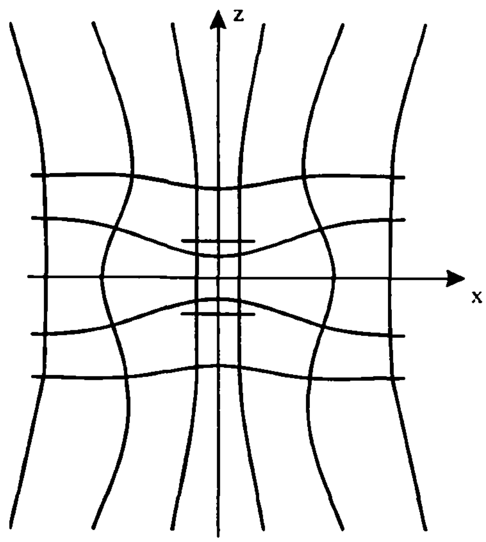

Fig. 1. Coordinate system under consideration (semitransparent resonator mirrors).

Note that within the framework of this interpretation of laser radiation, it is obvious that structures of radiation outside and inside a resonator are coupled. Coupling is realized through the space metric. In the case under consideration, the latter is strongly coupled with the field structure both outside and inside a resonator.

In particular, the field structure outside lateral resonator boundaries largely determines the extent of the area in which transverse waves are reflected into a resonator. (This effect shows itself through the space metric as well.)

Within the framework of this interpretation of the laser radiation field, it cannot be divided into two parts (inside the resonator and outside it). The field is represented as a unit. This is a substantial refinement of the interpretation of laser radiation, and one should always bear it in mind. The division of the laser field into two parts is convenient only from the viewpoint of gradual perception of the problem as a whole. We followed this viewpoint in the course of presenting the material and in the diagram of the structure describing the study of the problem mentioned in Part I. In this case, the laser field is divided into two parts by a very artificial method, a certain indeterminate specification of boundary conditions of the problem. In this sense, the perception of the problem is simplified to a certain extent. This simplification is convenient for understanding individual characteristics of the problem, but not the problem as a whole. In what follows, we use this simplified method for understanding the propagation of laser radiation outside a resonator.

Our interpretation of laser radiation outside a resonator, which is presented below, strongly differs from the interpretations known to us from the literature. As noted above, it is based on the use of the method of normal coordinates, which was considered in close detail in Part I.

In what follows, the material is presented in a way that strongly differs from the one used in the previous parts. It is given in a very brief and mathematically formalized form.

A reader acquainting himself with this part of the work should pay attention to the physical meaning of those mathematical procedures that we used for solving this part of the problem.

2. General Relations. Nonrelativistic Theory

The wave equation for the electric vector $\vec{E}$ of a radiation field will be used as the initial equation for further analysis. It readily follows from the Maxwell equations and has the form

$$\frac{\varepsilon(\vec{r},t)}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} - \nabla^2\vec{E} = \vec{\nabla}\left[\frac{\vec{E}}{\varepsilon}(\vec{\nabla}\varepsilon)\right], \tag{1}$$

where ε is the permittivity of a medium, c is the speed of light, t is time, and $\vec{r}$ is the space radius vector.

Equation (1) was obtained in the approximation where the frequency of field oscillations ω was much higher than the rate of variation of $\varepsilon(\vec{r},t)$ in time.

In what follows, we restrict our consideration to the case where the inertial properties of polarization of a medium may be neglected in relation to the radiation pulse length. It is obvious that this is a rather rough assumption. Nevertheless, as will be seen from further reasoning, this model appears to be useful for simulating propagation of laser radiation in vacuum (in vacuum, $\varepsilon = 1$, but for the present we set $\varepsilon \neq 1$).

Further, we use the scalar approximation according to which one may neglect the right part of Eq. (1) and consider the scalar component E instead of the electric field vector $\vec{E}$. Then in the nonrelativistic case and curvilinear coordinates (X^1, X^2, X^3), Eq. (1) can be written in the form

$$\frac{\varepsilon}{c^2}\frac{\partial^2 E}{\partial t^2} - \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}g^{jk}\frac{\partial E}{\partial x^k}\right) = 0, \tag{2}$$

where g^{jk} are metric tensor components, $g = \det(g_{jk})$, and $j,k = 1,2,3$.

The sense of the method of normal coordinates consists in the representation of oscillation of E in the form of three orthogonal noninteracting normal oscillations: longitudinal w, azimuthal v, and radial u:

$$E = uvw\,. \tag{3}$$

These oscillations satisfy the equations written in orthogonal normal coordinates

$$\frac{\varepsilon}{\tilde{v}_1{}^2}\frac{\partial^2 u}{\partial t^2} - \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}g^{jj}\frac{\partial u}{\partial x^j}\right) = 0; \tag{4a}$$

$$\frac{\varepsilon}{\tilde{v}_2{}^2}\frac{\partial^2 v}{\partial t^2} - \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}g^{jj}\frac{\partial v}{\partial x^j}\right) = 0; \tag{4b}$$

$$\frac{\varepsilon}{\tilde{v}_3{}^2}\frac{\partial^2 w}{\partial t^2} - \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}g^{jj}\frac{\partial w}{\partial x^j}\right) = 0, \tag{4c}$$

where $\tilde{v}_k$ is the phase velocity of the oscillatory wave k. Here, $j = 1,2,3$.

Let us analyze properties of an individual normal wave within the framework of the hydrodynamic model. For this purpose, we represent, for example, w in the form

$$w = \rho_w(\vec{r},t)\exp[iS_w(\vec{r},t)]\,. \tag{5}$$

Here, the amplitude ρ_w and the phase S_w are real functions of time and coordinates. Moreover, let us introduce the complex permittivity of a medium for the wave w in the form

$$\varepsilon_w = \varepsilon_{rw} + i\tilde{\varepsilon}'_w\,,$$

where ε_{rw} and $\tilde{\varepsilon}'_w$ are real values and $\tilde{\varepsilon}'_w > 0$ corresponds to wave absorption.

The sense of introducing the imaginary component of ε consists in the following. The hydrodynamic model has a substantial drawback. It does not take into account the effect of wave reflection. On the other hand, as shown in Part I, this effect is quite possible, for example, in the case of propagation of transverse waves in the space. It is natural that it is closely connected with the geometry of such waves. This effect can be taken into account within the framework of the hydrodynamic model by introducing the energy loss factor for the incident wave and the gain factor for the reflected one. These factors should depend on the coordinate along the directions of the incident and reflected waves. In this case, the quantity $\tilde{\varepsilon}'_w$ fulfills the role of these factors. It can be determined within the framework of the wave model, which will be considered below.

Let us substitute (5) into (4c) and separate the real and imaginary parts. This gives the system of two equations

$$\frac{\varepsilon_{rw}}{\tilde{v}_3{}^2}\frac{\partial}{\partial t}(\rho_w^2\omega_w) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}(\sqrt{g}\rho_w^2 v_w^j) = \frac{\rho_w\tilde{\varepsilon}'_w}{\tilde{v}_3{}^2}\left(\frac{\partial^2\rho_w}{\partial t^2} - \rho_w\omega_w^2\right); \tag{6}$$

$$g_{jj}v_w^j v_w^j = \frac{\varepsilon_{rw}\omega_w^2}{\tilde{v}_3{}^2} - \frac{\tilde{\varepsilon}'_w}{\rho_w^2\tilde{v}_3{}^2}\frac{\partial}{\partial t}(\omega_w\rho_w^2) - \frac{\varepsilon_{rw}}{\tilde{v}_3{}^2\rho_w}\frac{\partial^2\rho_w}{\partial t^2} + \frac{1}{\rho_w\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}g^{jj}\frac{\partial\rho_w}{\partial x^j}\right), \tag{7}$$

where $v_w^j = g^{jj}\partial S_w/\partial x^j$ is the quasi-particle velocity, and $\omega = -\partial S_w/\partial t$ is the frequency.

Equation (6) is the continuity equation, and Eq. (7) describes the energy of the flux of quasi-particles. The last term on the right part of Eq. (7) describes the diffraction effect, the next to last one describes the dynamic effect, the quantity $g_{jj}v_w^j v_w^j$ is proportional to the kinetic energy, $\omega_w^2(1-\varepsilon_{rw})/\tilde{v}_3{}^2$ is proportional to the potential energy, and $\omega_w^2/\tilde{v}_3{}^2$ is proportional to the total energy of quasi-particles.

Systems of equations similar to (6) and (7) can also be derived for the waves u, v and for the wave E. The last fact is convenient to use for determining the metric and, hence, the system of normal coordinates. Indeed, if E is represented in the form $E = \rho\exp(iS)$, then assuming $\varepsilon = \varepsilon_r + i\tilde{\varepsilon}'$ one obtains, in view of (7),

$$g_{jj}v^j v^j = \varepsilon_* k_0^2, \tag{8}$$

where $v^j = v_u^j + v_v^j + v_w^j$ is the velocity of quasi-particles, $k_0 = \omega/c$ is the module of the wave vector of radiation in vacuum (in the general case, ω is a variable quantity), and

$$\varepsilon_* = \varepsilon_r + \frac{1}{k_0^2}\left[\frac{1}{\rho\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}g^{jj}\frac{\partial\rho}{\partial x^j}\right) - \frac{\varepsilon_r}{c^2}\frac{1}{\rho}\frac{\partial^2\rho}{\partial t^2} - \frac{\tilde{\varepsilon}'}{\rho^2c^2}\frac{\partial}{\partial t}(\omega\rho^2)\right]. \tag{9}$$

Here, $\rho = \rho_u\rho_v\rho_w$ and ε_* has the meaning of permittivity modeling diffraction, dynamic, and absorption effects described by the first, second, and third terms in square brackets of (9), respectively.

In view of the fact that the meaning of the quantity ε_* is established, it is natural now to take into account that $\varepsilon_r = 1$ in vacuum. Operating on Eq. (8) with the gradient operator, one obtains the system of three equations of motion

$$2\left[\frac{k_0}{c}\frac{\partial v_k}{\partial t} + g_{jj}v^j\left(\frac{\partial v^j}{\partial x^k} + \left\{ {j \atop nk} \right\} v^n\right)\right] = \frac{\partial}{\partial x^k}(\varepsilon'_* k_0^2). \tag{10}$$

Here, k is fixed and takes the values $1,2,3$; the summation is performed over n and j; $\varepsilon'_* = \varepsilon_* - 1$.

Equations (10) along with the continuity equation

$$\frac{\varepsilon_r}{c^2}\frac{\partial(\rho^2\omega)}{\partial t} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}(\sqrt{g}\rho^2 v^j) = \frac{\rho\tilde{\varepsilon}'}{c^2}\left(\frac{\partial^2\rho}{\partial t^2} - \rho\omega^2\right) \tag{11}$$

determine the space metric deformed by a radiation pulse. In essence, Eqs. (10) and (11) for the given $\rho = \rho_u\rho_v\rho_w$, $v^j = v_u^j + v_v^j + v_w^j$, and $\omega = \omega_u + \omega_v + \omega_w$ represent the conditions of expansion of the wave E in three normal wave components u, v, and w.

Qualitatively, the metric may also be analyzed by using the electrostatic analogy (see Part I). In this case, the metric is determined by the system of three equations

$$\frac{\partial}{\partial x^j}(\sqrt{g}g^{jj}) = -\sqrt{g}g^{jj}\,\frac{1}{\varepsilon_*}\,\frac{\partial \varepsilon_*}{\partial x^j}\,. \tag{12}$$

Now, if information about ε_* is included into g^{jj}, i.e., a field–metric coupling is taken into account, then it follows from Eqs. (9)–(11) that the metric distortion by a pulse has a wave character and the deviation of normal coordinates from their equilibrium values changes in the course of propagation of a radiation pulse. Let us now describe dynamics of the field of propagating radiation.

3. Field Dynamics

The field dynamics will be described with the help of the formalism developed in Part I. According to this formalism, normal waves are represented in the form of individual partial waves directed along coordinate tubes. In the general case, these partial waves are coupled, and coupling is caused by the transverse transfer of amplitude.

Let us analyze a particular problem, the propagation of a stationary beam of laser radiation in vacuum. We consider an idealized case where the transverse transfer of wave amplitude is not taken into account and one is interested only in the radial structure of a cylindrical beam. Within the framework of the formalism mentioned above, the radial structure of the beam is described by equations of oscillations of the field u_i along radial coordinate tubes i. In the laboratory system of coordinates moving along the optical axis z together with the front of a longitudinal wave, we have the equation

$$\frac{1}{\tilde{v}_i^2}\,\frac{\partial^2 u_i}{\partial t^2} - \frac{\partial^2 u_i}{\partial \xi_i^2} - \frac{1}{\sigma_i}\,\frac{d\sigma_i}{d\xi_i}\,\frac{\partial u_i}{\partial \xi_i} = 0\,, \tag{13}$$

where ξ_i is a coordinate with a uniform scale along the axis of a radial coordinate tube, $\sigma_i = \sigma_i(\xi_i)$ is the cross section of the radial tube i, and $\tilde{v}_i$ is the phase velocity of the radial wave.

The structure of the cylindrical radiation beam is shown in Fig. 2. For simplicity, it is assumed that the curvature of longitudinal coordinate lines may be neglected. This is correct under the assumption that the longitudinal wave is plane, has a constant amplitude over the wave front, and transverse waves do not change the metric. One can see from Fig. 2 that the beam structure is determined by the time evolution of the wave packet corresponding to radial field oscillations. In the figure, the evolution of this packet is shown at moments 1–4. At the initial moment 1, the packet represents a standing wave of radial oscillations on the surface of a resonator mirror. At moments of time following moment 4, the diffraction destruction and the package displacement along the radial coordinate proceed. Note that in the case under consideration it is assumed that the laser generates a highly excited azimuthal oscillation and the field on the z axis vanishes. In this case, the region of maximum beam energy density in the far field is far removed from the optical axis z. In a more general case (for example, in the case of generation of the fundamental Gaussian mode in a resonator with concave mirrors), the density of radiation energy in the far field may have a maximum on the z axis. However, in this case, a beam propagates in a substantially curved space.

Let us complicate the problem and describe a stationary radiation beam in a more real case of a curvilinear coordinate grid. Then, in contrast to the problem considered above, the propagation of normal waves w, u, and v along orthogonal coordinate axes (longitudinal ζ, radial ξ, and azimuthal η) is described by three equations of oscillations in curved space. For the longitudinal natural wave w_k propagating along an individual coordinate tube k, we have

$$\frac{1}{\tilde{v}_{3k}^2}\,\frac{\partial^2 w_k}{\partial t^2} - \left\{\frac{\partial^2 w_k}{\partial \zeta_k^2} + \frac{1}{\sigma_{3k}}\,\frac{d\sigma_{3k}}{d\zeta_k}\,\frac{\partial w_k}{\partial \zeta_k} + \left[\varepsilon_3' k_0^2 - U_{3k}(\zeta_k)\right] w_k\right\} = 0\,, \tag{14}$$

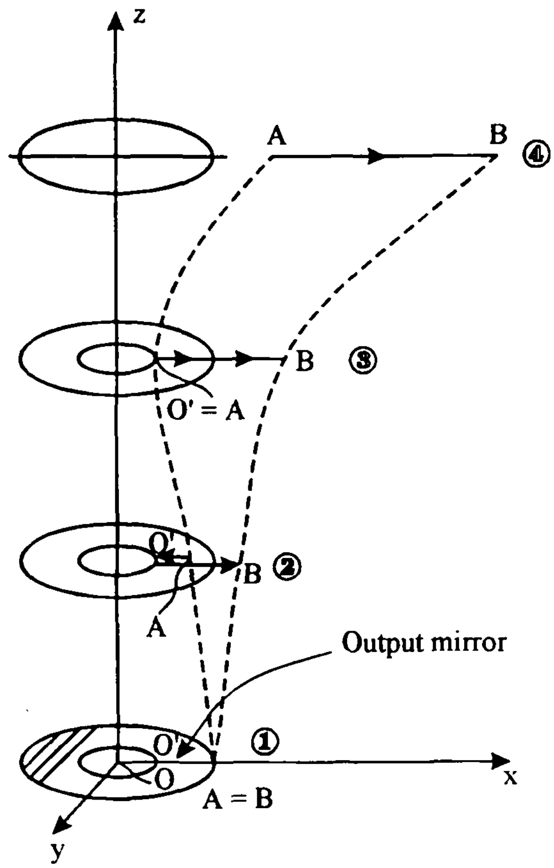

Fig. 2. The structure of a radiation beam outcoupled from a cylindrical flat resonator. O' is the turning point; the points A and B specify boundaries of a transverse wave packet of a radial oscillation. The arrows indicate the directions of propagation of the radial wave.

where $\tilde{v}_{3k}$ is the phase velocity of the wave, σ_{3k} is the cross section of tube k, U_{3k} is the potential caused by the curvature of the axis of the tube k, and ε_3' is the changing component of effective permittivity containing information about the transverse diffraction of the longitudinal wave. The radial and azimuthal waves are conveniently described in the laboratory system of coordinates moving along the optical axis of the beam with velocity of the front of the longitudinal wave. In the general case, the coordinates ξ and η undergo local stretches and compressions in the course of motion. Therefore, it makes sense to pass from these coordinates to the coordinates (x^1, x^2), which do not change in time:

$$d\xi = \sqrt{g_1}\, dx^1\,; \quad d\eta = \sqrt{g_2}\, dx^2\,; \quad dt = \sqrt{g_0}\, dx^0\,.$$

In new coordinates (x^1, x^2), the transverse waves u and v along corresponding individual tubes i and j are described by the equations

$$\frac{g_{1i}}{\tilde{v}_{1i}^{\,2} g_{0i}} \left(\frac{\partial^2 u_i}{\partial (x_i^0)^2} - \frac{\partial \log \sqrt{g_{0i}}}{\partial x_i^0} \frac{\partial u_i}{\partial x_i^0} \right) - \left\{ \frac{\partial^2 u_i}{\partial (x_i^1)^2} + \left[\frac{1}{\sigma_{1i}} \frac{\partial \sigma_{1i}}{\partial x_i^1} - \frac{\partial \log \sqrt{g_{1i}}}{\partial x_i^1} \right] \frac{\partial u_i}{\partial x_i^1} + g_{1i} \left[\varepsilon_1' k_0^2 - U_{1i}(x_i^1, x_i^0) \right] u_i \right\} = 0\,; \tag{15}$$

$$\frac{g_{2j}}{\tilde{v}_{2j}^{\,2} g_{0j}} \left(\frac{\partial^2 v_j}{\partial (x_j^0)^2} - \frac{\partial \log \sqrt{g_{0j}}}{\partial x_j^0} \frac{\partial v_j}{\partial x_j^0} \right) - \left\{ \frac{\partial^2 v_j}{\partial (x_j^2)^2} + \left[\frac{1}{\sigma_{2j}} \frac{\partial \sigma_{2j}}{\partial x_j^2} \right.\right.$$

$$-\frac{\partial\log\sqrt{g_{2j}}}{\partial x_j^2}\Bigg]\frac{\partial v_j}{\partial x_j^2}+g_{2j}\left[\varepsilon_2' k_0^2-U_{2j}(x_j^2,x_j^0)\right]v_j\Bigg\}=0\,. \tag{16}$$

In these equations, the meaning of variables with subscripts corresponds to their meaning in Eq. (14). We emphasize once again that Eqs. (15) and (16) for transverse waves are extremely useful because they permit one to change from the formulation of the problem of wave motion in the time-dependent coordinates (ξ,η) to its formulation in the stationary coordinates (x^1,x^2). (The quantities g_1,g_2 contain the time dependence.)

From Eqs. (15) and (16) follows an interesting conclusion. The effect of change of the tube cross section along the transverse coordinate can be easily simulated by compression and stretching of this coordinate (change of the "metric" g_i). This conclusion follows from the structure of the terms

$$\frac{1}{\sigma_i}\frac{\partial\sigma_i}{\partial x^i}\qquad\text{and}\qquad\frac{\partial\log\sqrt{g_i}}{\partial x^i}=\frac{1}{\sqrt{g_i}}\frac{\partial\sqrt{g_i}}{\partial x^i}\,.$$

One can see that these terms have the same structures, and from Eqs. (15) and (16) it follows that the terms have different signs. Therefore, in this simulation, a decreasing dependence $g_i(x^i)$ will correspond to an increasing dependence $\sigma_i(x^i)$.

In Eqs. (13)–(16) presented above, the effect of amplitude transfer along the front of normal waves was not taken into account. Usually, this effect cannot be ignored in problems that are of practical interest. This conclusion is valid for the problem under consideration as well, especially in the case where the propagation of short pulses of laser radiation is studied. In the nonrelativistic approximation, velocities of diffraction transfer of wave amplitudes may appear to be very high. Because of this, the problems under consideration should be analyzed within the framework of the relativistic theory. The need for taking into account the relativistic effect also arises in the case where one studies the propagation of a pulse and describes its longitudinal component moving with a relativistic velocity. In the general case, it is possible to state that the relativistic effect produces an additional distortion of the metric (similarly to the diffraction effect). This property was noted in Part I of our work during the study of metric properties of the space in empty laser resonators. We noted there that the inclusion of the relativistic effect into analysis was a difficult problem. Below, we try to construct a simple model enabling one to evaluate the influence of the relativistic effect on the problem considered.

4. Wave Equation in the Relativistic Theory

The wave equation in the relativistic theory has the form of the Laplace–Beltrami equation [1, 2]

$$g^{\mu\nu}E_{;\nu;\mu}\equiv\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial E}{\partial x^\nu}\right)=0\,. \tag{17}$$

Here, the greek indices μ, ν take the values $0,1,2,3$; $E_{;\nu;\mu}$ is the covariant derivative with respect to x^ν and x^μ, $g_{\mu\mu}=(-,+,+,+)$ is the signature, $g=\det(g_{\mu\nu})$, and x^0 is the time coordinate.

For $E=\rho\exp(iS)$, one obtains in the known way from Eq. (17) the system of energy and continuity equations

$$g_{\mu\nu}v^\mu v^\nu=\frac{1}{\rho\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial\rho}{\partial x^\nu}\right), \tag{18}$$

$$\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\rho^2 v^\mu\right)=0\,, \tag{19}$$

where $v^\mu = g^{\mu\nu}\dfrac{\partial S}{\partial x^\nu}$ are velocities of quasi-particles. Operating on Eq. (18) with the gradient operator, one obtains a system of four equations of motion. In combination with the continuity equation (19), this system is written in the form

$$2g^{\sigma\alpha}g_{\mu\nu}v^\mu\left(\frac{\partial v^\nu}{\partial x^\alpha}+\left\{\begin{matrix}\nu\\ \alpha\kappa\end{matrix}\right\}v^\kappa\right)=g^{\sigma\alpha}\frac{\partial\hat{\varepsilon}'_*}{\partial x^\alpha}\,; \tag{20}$$

$$\frac{\partial}{\partial x^\mu}(\sqrt{-g}\rho^2v^\mu)=0\,, \tag{21}$$

where the index σ is fixed and takes the values $0,1,2,3$, summation is performed over the indices μ,α,ν,κ, and

$$\hat{\varepsilon}'_*=\frac{1}{\rho\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial\rho}{\partial x^\nu}\right). \tag{22}$$

Equations (20) and (21) describe the process of propagation of a laser radiation pulse within the framework of the hydrodynamic model taking into account the relativistic effect, and the information on it is contained in the components of the metric tensor.

In the simplest case where the acceleration of a medium may be neglected, the metric of an uncurved space–time is invariant under the Lorentz transformation. In the case of a linearly accelerated medium, this property is apparently not satisfied (see [3]). The metric of the space–time $g^{\mu\nu}$ and, therefore, the system of normal coordinates x^μ are determined with the help of the method described above.

5. Comparison with Other Theoretical Methods and Discussion of Results

Let us illustrate the advantages of the above method of analysis of the structure of a laser radiation field in comparison with other methods available at present. One of the most widespread methods is based on the Huygens–Fresnel principle [5, 6]. It is mathematically formulated in the form (see [4], p. 194)

$$U(P)=\int_f GU(Q)\,df_n\,, \tag{23}$$

where $U(P)$ is the field at the observation point, $U(Q)$ is the field at a point Q on the wavefront surface, G is the Green function of the Helmholtz equation, and df_n is the projection of an element of the surface f onto the plane arranged perpendicular to the segment $QP = R$ at the point Q. It is common to use the Green function in the form (see [4], p. 194)

$$G(R)=\frac{k_0}{2\pi i}\frac{e^{ik_0R}}{R}\,. \tag{24}$$

The drawback of the method is the difficulty of determining the Green function in curved space. It may considerably differ from the Green function in uncurved space (24). As shown above, laser radiation usually propagates in a substantially curved space. Therefore, this disadvantage may be a serious problem.

The most important conclusion of our work is the following statement. We have shown that metric properties of space–time in which laser radiation propagates were intimately connected with the radiation field structure. Because of this, an independent specification of the metric and, therefore, the coordinate system is nothing more than an approximation to the exact solution of the self-consistent problem. Such a situation is typical of problems of the general relativity in which the concepts of field structure and metric are indivisible.

This statement is usually neglected in the solution of practical problems. In this case, one attempts to select a coordinate system as close as possible to the real one. For this purpose, various approximations are used. For example, in [7] Vainshtein used the model of the harmonic coordinate system.

In the paraxial model of propagation of laser radiation, which is extensively used at present (see, for example, [8, 9]),

$$2ik_0\frac{\partial A}{\partial z}+\nabla_\perp^2 A=0\,. \tag{25}$$

Here, A is the envelope of the field amplitude, z is the coordinate along the optical axis, and $\nabla_\perp^2$ is the Laplacian with respect to the coordinates x and y in the plane perpendicular to the optical axis z. It is assumed there that radiation propagates in uncurved space–time ($g^{\mu\nu}=0$ for $\mu\neq\nu$, and $g^{00}=-g^{ii}=-1$ for $i=1,2,3$) and the Green function of Eq. (25) under such conditions has the form

$$G(r,z)=\frac{k_0}{2\pi iz}\exp\left(\frac{ik_0r^2}{2z}\right),$$

where $r=\sqrt{x^2+y^2}$.

It should be remembered that such approximations are not exact. In the general case, they may be substantially different from the real picture corresponding to propagation of laser radiation.

6. Conclusion

In conclusion, we note some remarks that seem to be of importance.

(1) In this work, we did not analyze the problem by the conventional method within the framework of Maxwell equations for vector-wave fields. Note that the method is used in almost all modern publications concerning the problem under consideration. This way is difficult for a simple intuitive understanding of the problem at the initial stage because it contains a large number of degrees of freedom (variables). This complexity is particularly strong for the interpretation of processes of interaction of a wave field with a continuous medium.

Meanwhile, we think that our way based on the analysis of scalar-wave fields can be specified in this respect later on.

(2) We think that the theory of Lie groups, topology, and mathematical statistics hold the greatest promise for the physical analysis of the problem under consideration.

Let us enumerate the main advantages of the use of the group-theoretical methods.

(i) These methods provide simple and pictorial forms of representation of solutions even in the case of very complex nonlinear equations and systems of such equations by the algebraic transformations

$$\vec{A}=\hat{L}_{t_1,t_2,\ldots,t_m}\vec{A}_0\,,$$

where $\vec{A}=(x,y,z,t,\varphi,\rho,\ldots)$ is a vector of dependent and independent variables of the problem, $\hat{L}_{t_1,t_2,\ldots,t_m}$ is the algebraic transformation with m parameters $t_1,t_2,\ldots,t_m$, and $\vec{A}_0$ is the vector determined by the boundary condition of the problem.

Varying parameters t_i, one can determine the space $\vec{A}$ by the known vector $\vec{A}_0$. This is simple and clear.

(ii) Several group-theoretical methods are available that can be used to analyze the given problem. This is an additional advantage.

(iii) One can give a simple physical interpretation of the process of wavefront propagation. For example, the change in one of the group parameters corresponds to the wavefront motion in the space (x,y,z,t), and the change in other parameters corresponds to the change in amplitude on this moving front.

(iv) The algebraic form of the construction of spaces of dependent and independent variables of the problem is very convenient for its statistical analysis.

(v) The construction of the space of dependent and independent variables of the problem in a form similar to the one considered in item (i) is suitable for studying global properties of the solution of the problem.

This way led to the use of topological methods of analysis of the problem.

As for the topological methods of research, we agree with the American mathematician M. Morse, who wrote [10] that if a problem is nonlinear in character, more than one system of coordinates or more than one variable is involved in it, or it concerns in a nonlocal manner the structure being determined, the solution of this problem usually requires invoking topology or group theory. The classical analysis, as a rule, is applied to the solution of such problems for a preliminary local study, whereas the following generalization is made with the help of topology or group theory.

As for the mathematical methods of statistical analysis of the problem, the necessity of their use is explained by that fact that real laser radiation has a statistical nature.

(3) In our interpretation of the structure of a laser radiation field, some classical methods of analysis of the statistical aspect of the problem are strongly modified.

For example, the classical method of analysis of statistical properties of laser radiation outside a resonator, formulated in the form of linear differential equations [11] concerning the coherence function of the radiation field, becomes inconvenient because initial wave equations for regular fields are nonlinear in our interpretation. Fortunately, some other mathematical methods that are convenient for the physical analysis of the statistical aspect of the problem within the framework of our representations are currently available.

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