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Expansions for Eisenstein integrals on semisimple symmetric spaces

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1. Introduction

Let G/H be a semisimple symmetric space. Related to the (minimal) principal series for G/H there is a series of Eisenstein integrals on G/H. These are K-finite joint eigenfunctions for the G-invariant differential operators on G/H. Here K is a maximal compact subgroup of G. The Eisenstein integrals are generalizations of the elementary spherical functions for a Riemannian symmetric space (and more generally of the generalized spherical functions in [9, §III.2]), and of Harish-Chandra's Eisenstein integrals associated to a minimal parabolic subgroup of a semisimple Lie group.

In this paper we develop a theory of asymptotic (in fact, converging) expansions towards infinity for the Eisenstein integrals. The theory generalizes Harish-Chandra's theory (see [8, Thm. IV.5.5], and [13, Thm. 9.1.5.1]) in the two cases mentioned above (see also [9, Thm. III.2.7]). The main results are Theorems 9.1 and 11.1. The first of these states the convergence on an open Weyl chamber of the series expansion whose coefficients are derived recursively from the differential equations satisfied by the Eisenstein integrals. The sum Φ_λ of the series is an eigenfunction which behaves regularly at infinity but in general is singular at the walls of the chamber. The basic estimates which ensure the convergence of the series also provide an estimate for Φ_λ, which is a generalization of Gangolli's estimates ([7]) in the Riemannian case. As in Gangolli's case, our estimates are derived by a modification of the Φ_λ with the square root of a certain Jacobian function.

The second main result expresses the Eisenstein integral as a linear combination of the Φ_λ; the coefficients are the c-functions (defined in previous work by one of us) related to the Eisenstein integrals.

The results of this paper are used for the Plancherel and Paley–Wiener type results obtained in [5] for the Fourier transform corresponding to the minimal principal series, just as Gangolli's estimates in the Riemannian case play a crucial role in Helgason's and Rosenberg's work for the spherical transform (see [8, §IV.7]). In the case of a semisimple Lie group, considered as a symmetric space, estimates sufficient for the application to the Paley–Wiener theorem are given in [1]. The present, stronger, estimates were in this case obtained in [6].

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2. Notation

Widening the generality a bit let G/H now be a reductive symmetric space of Harish-Chandra's class, that is, G is a real reductive Lie group of Harish-Chandra's class, σ an involution of G, and H an open subgroup of the group G^σ of its fixed points. Let θ be a Cartan involution of G commuting with σ, and let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}=\mathfrak{h}\oplus\mathfrak{q}$ be the ± 1 eigenspace decompositions of the Lie algebra $\mathfrak{g}$ of G, corresponding to θ and σ, respectively. Let $K=G^\theta$, then K is a maximal compact subgroup of G. As usual, the Killing form on $[\mathfrak{g},\mathfrak{g}]$ is extended to an invariant bilinear form B on $\mathfrak{g}$, for which the inner product $\langle X,Y\rangle:=-B(X,\theta Y)$ is positive definite, and which is compatible with σ, that is, $B(\sigma X,Y)=B(X,\sigma Y)$ for all $X,Y\in\mathfrak{g}$.

Let $\mathfrak{a}_{\mathrm{q}}$ be a fixed maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$, Σ the root system of $\mathfrak{a}_{\mathrm{q}}$ in $\mathfrak{g}$, and W the group $N_K(\mathfrak{a}_{\mathrm{q}})/Z_K(\mathfrak{a}_{\mathrm{q}})$, which is naturally identified with the reflection group of Σ. Let $\mathfrak{a}_{\mathrm{q}}^*$ and $\mathfrak{a}_{\mathrm{qc}}^*$ denote the real and complex linear dual spaces of $\mathfrak{a}_{\mathrm{q}}$. The inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathfrak{a}_{\mathrm{q}}$ is transferred to real and complex bilinear forms on $\mathfrak{a}_{\mathrm{q}}^*$ and $\mathfrak{a}_{\mathrm{qc}}^*$ by duality.

Let $A_{\mathrm{q}}=\exp\mathfrak{a}_{\mathrm{q}}$ and let $\mathcal{P}_\sigma(A_{\mathrm{q}})$ denote the set of parabolic subgroups $P=MAN_P$ (with the indicated Langlands decomposition) whose Levi part $M_1=MA$ is the centralizer in G of $\mathfrak{a}_{\mathrm{q}}$. Let $\mathfrak{m}_1$, $\mathfrak{m}$ and $\mathfrak{a}$ denote the Lie algebras of M_1, M and A, then $\mathfrak{a}=(\mathfrak{a}\cap\mathfrak{h})\oplus\mathfrak{a}_{\mathrm{q}}$, $\mathfrak{a}_{\mathrm{q}}=\mathfrak{a}\cap\mathfrak{q}$, and with $\mathfrak{m}_\sigma:=\mathfrak{m}+(\mathfrak{a}\cap\mathfrak{h})$ we have $\mathfrak{m}_1=\mathfrak{m}\oplus\mathfrak{a}=\mathfrak{m}_\sigma\oplus\mathfrak{a}_{\mathrm{q}}$. Notice that M is invariant under both involutions θ and σ, and hence that the quotient $M/M\cap H$ is a symmetric space. It follows from the maximality of $\mathfrak{a}_{\mathrm{q}}$ that this quotient space is compact.

There is a natural bijective correspondence $Q\mapsto\Sigma(Q)$ of the set $\mathcal{P}_\sigma(A_{\mathrm{q}})$ with the set of positive systems for Σ. We denote by $\Delta=\Delta(Q)$ the set of simple roots corresponding to a given $\Sigma^+=\Sigma(Q)$, by $\varrho=\varrho_Q\in\mathfrak{a}_{\mathrm{q}}^*$ the corresponding half sum (with multiplicities) of the positive roots, by $\mathfrak{a}_{\mathrm{q}}^+=\mathfrak{a}_{\mathrm{q}}^+(Q)$ the corresponding positive chamber in $\mathfrak{a}_{\mathrm{q}}$, and by $A_{\mathrm{q}}^+=A_{\mathrm{q}}^+(Q)$ the set $\exp\mathfrak{a}_{\mathrm{q}}^+$. Let $\mathbf{N}\Delta\subset\mathfrak{a}_{\mathrm{q}}^*$ denote the set of linear combinations $\nu=\sum_{\alpha\in\Delta}\nu_\alpha\alpha$ with coefficients ν_α in $\mathbf{N}=\{0,1,\ldots\}$.

Let (τ, V_τ) be a fixed finite dimensional unitary representation of K. A V_τ-valued function f on G/H is called τ-spherical if $f(kx)=\tau(k)f(x)$ for all $k\in K$, $x\in G/H$. The space of smooth τ-spherical functions on G/H is denoted $C^\infty(G/H{:}\,\tau)$. Notice that if $f\in C^\infty(G/H{:}\,\tau)$, then f restricts to a smooth $V_\tau^{M\cap K\cap H}$-valued function on A_{q}. Here $V_\tau^{M\cap K\cap H}$ is the space of $M\cap K\cap H$-fixed vectors in V_τ.

3. Radial components of differential operators

Let $\mathbf{D}(G/H)$ denote the algebra of G-invariant differential operators on G/H. In particular, these operators act on $C^\infty(G/H{:}\,\tau)$. In this section we recall the concept of the τ-radial component of the elements in $\mathbf{D}(G/H)$ (cf. [2, §3]).

Let $Q\in\mathcal{P}_\sigma(A_{\mathrm{q}})$ be fixed and let $A_{\mathrm{q}}^+=A_{\mathrm{q}}^+(Q)$ as above. From the Cartan decomposition (see for example [3, §1]) it follows that KA_{q}^+H is an open subset of G, and that the map $(k,h,a)\mapsto kah$ induces a diffeomorphism from $K\times_{M\cap K\cap H}H\times A_{\mathrm{q}}^+$ onto KA_{q}^+H. Let $T^\downarrow=T_Q^\downarrow$ be the restriction map $f\mapsto f|_{A_{\mathrm{q}}^+}$, then

$$T^\downarrow\colon C^\infty(G/H{:}\,\tau)\to C^\infty(A_{\mathrm{q}}^+, V_\tau^{M\cap K\cap H})\simeq C^\infty(A_{\mathrm{q}}^+)\otimes V_\tau^{M\cap K\cap H}.$$

We define the map

$$T^\uparrow=T_Q^\uparrow\colon C_c^\infty(A_{\mathrm{q}}^+)\otimes V_\tau^{M\cap K\cap H}\to C_c^\infty(G/H{:}\,\tau)$$

by

(a) $\operatorname{supp} T^\uparrow f\subset KA_{\mathrm{q}}^+H$ for $f\in C_c^\infty(A_{\mathrm{q}}^+)\otimes V_\tau^{M\cap K\cap H}$;

(b) $T^\downarrow\circ T^\uparrow=\mathrm{I}$.

If $D\in\mathbf{D}(G/H)$ then one readily checks that

$$\Pi_\tau(D)=\Pi_{Q,\tau}(D):=T^\downarrow\circ D\circ T^\uparrow$$

defines an element of the ring

$$C^\infty(A_{\mathrm{q}}^+)\otimes S(\mathfrak{a}_{\mathrm{q}})\otimes\operatorname{End}(V_\tau^{M\cap K\cap H}) \tag{1}$$

of differential operators on A_{q}^+, with coefficients in $C^\infty(A_{\mathrm{q}}^+)\otimes\operatorname{End}(V_\tau^{M\cap K\cap H})$. The operator $\Pi_\tau(D)$ is called the τ-radial component of D on A_{q}^+. It is easily seen that Π_τ is an algebra homomorphism from $\mathbf{D}(G/H)$ to the ring (1).

Let $U(\mathfrak{g})$ be the universal enveloping algebra of the complexification $\mathfrak{g}_{\mathbf{c}}$ of $\mathfrak{g}$, and let $U(\mathfrak{g})^H$ be the subalgebra of H-fixed elements. There is a natural map r from $U(\mathfrak{g})^H$ to $\mathbf{D}(G/H)$ defined by $(r(X)f)\circ\pi=R_X(f\circ\pi)$ for $f\in C^\infty(G/H)$; here

$\pi: G \to G/H$ is the natural projection and R denotes the right regular representation on $C^\infty(G)$. The map r induces an isomorphism of algebras $U(\mathfrak{g})^H/[U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}] \to \mathbf{D}(G/H)$, which we also denote by r. In the following we shall sometimes abuse notation by identifying an element $D \in \mathbf{D}(G/H)$ with any $X \in U(\mathfrak{g})^H$ for which $D = r(X)$. In particular, for $X \in U(\mathfrak{g})^H$, we write $\Pi_\tau(X)$ for $\Pi_\tau(r(X))$. Thus we also view Π_τ as an algebra homomorphism from $U(\mathfrak{g})^H$ to the ring (1).

Algebraically the map Π_τ can be described as follows. We denote $u^g = \mathrm{Ad}(g)u$ for $u \in U(\mathfrak{g})$, $g \in G$. Let $X \in U(\mathfrak{g})^H$ and assume (cf. [2, Lemma 3.2]) that we have an expression for X as a finite sum

$$X \equiv \sum_i f_i(a) u_i^{a^{-1}} v_i \quad \text{modulo } U(\mathfrak{g})\mathfrak{h},$$

for all $a \in A_\mathrm{q}^+$, where $f_i \in C^\infty(A_\mathrm{q}^+)$, $u_i \in U(\mathfrak{k})^{M \cap K \cap H}$ and $v_i \in U(\mathfrak{a}_\mathrm{q})$. Then it is easily seen from the definitions above that

$$\Pi_\tau(X) = \sum_i f_i v_i \tau(u_i) \in C^\infty(A_\mathrm{q}^+) \otimes S(\mathfrak{a}_\mathrm{q}) \otimes \mathrm{End}(V_\tau^{M \cap K \cap H}). \tag{2}$$

Here it should be noted that for $u \in U(\mathfrak{k})^{M \cap K \cap H}$ the operator $\tau(u)$ on V_τ preserves the subspace $V_\tau^{M \cap K \cap H}$, and that we henceforth are abusing notations by letting $\tau(u)$ denote the induced endomorphism of $V_\tau^{M \cap K \cap H}$.

Let $\log: A_\mathrm{q} \to \mathfrak{a}_\mathrm{q}$ denote the inverse of $\exp|_{\mathfrak{a}_\mathrm{q}}$. Then for $a \in A_\mathrm{q}$, $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$ we write $a^\lambda = e^{\lambda(\log a)}$. Moreover, we define the function $e^\lambda: A_\mathrm{q} \to \mathbf{C}$ by

$$e^\lambda(a) = a^\lambda. \tag{3}$$

Let E be a finite dimensional linear space. We are interested in E-valued functions on A_q^+ which admit a series expansion of the form

$$\sum_{\nu \in \mathbf{N}\Delta} c_\nu a^{-\nu} \quad (a \in A_\mathrm{q}^+) \tag{4}$$

with coefficients $c_\nu \in E$, and where $\Delta = \Delta(Q)$. For $\nu = \sum_{\alpha \in \Delta} \nu_\alpha \alpha \in \mathbf{N}\Delta$ and $z = (z_\alpha)_{\alpha \in \Delta} \in \mathbf{C}^\Delta$ we put

$$z^\nu = \prod_{\alpha \in \Delta} (z_\alpha)^{\nu_\alpha}.$$

The map $a \mapsto (a^{-\alpha})_{\alpha \in \Delta}$ maps A_q^+ onto $]0,1[^\Delta$. Hence the series (4) converges if and only if the E-valued power series $\sum_{\nu \in \mathbf{N}\Delta} c_\nu z^\nu$ converges on the polydisc D^Δ; here D denotes the complex unit disk.

4. The radial component of the Laplace operator

Let $\Omega \in U(\mathfrak{g})$ be the Casimir element associated with the bilinear form B, and let $\mathrm{L}=r(\Omega)$ denote its image in $\mathbf{D}(G/H)$, then L is the Laplace–Beltrami operator associated with the natural pseudo-Riemannian structure on G/H induced by B (cf. [8, Exercise II.A.4]). We shall now compute the τ-radial component of L.

For $\alpha \in \Sigma$, let $\mathfrak{g}_\alpha = \mathfrak{g}_\alpha^+ \oplus \mathfrak{g}_\alpha^-$ denote the decomposition of the root space $\mathfrak{g}_\alpha$ into $+1$ and -1 eigenspaces for $\sigma\theta$, and put $m_\alpha^\pm = \dim \mathfrak{g}_\alpha^\pm$. For each space $\mathfrak{g}_\alpha^\varepsilon$ ($\varepsilon=\pm$), we fix a basis $X_{\alpha,i}^\varepsilon$ ($i=1,\dots,m_\alpha^\varepsilon$), orthonormal with respect to the positive definite inner product $\langle\cdot,\cdot\rangle$. Moreover we require that $X_{-\alpha,i}^\varepsilon = -\theta X_{\alpha,i}^\varepsilon$. Let H_α denote the element of $\mathfrak{a}_\mathrm{q}$ determined by $\alpha(Y)=\langle Y, H_\alpha\rangle$ ($Y \in \mathfrak{a}_\mathrm{q}$), then

$$[X_{\alpha,i}^\varepsilon, X_{-\alpha,i}^\varepsilon] = H_\alpha. \tag{5}$$

Let $\Omega_{\mathfrak{m}_\sigma} \in U(\mathfrak{m}_\sigma)$, $\Omega_{\mathrm{mk}} \in U(\mathfrak{m}\cap\mathfrak{k})$ and $\Omega_{\mathfrak{a}_\mathrm{q}} \in U(\mathfrak{a}_\mathrm{q})$ denote the 'Casimir elements' of $\mathfrak{m}_\sigma$, $\mathfrak{m}\cap\mathfrak{k}$ and $\mathfrak{a}_\mathrm{q}$, respectively, defined by means of the restriction of B (i.e. if $X_1,\dots,X_m$ is a basis for $\mathfrak{m}_\sigma$ then $\Omega_{\mathfrak{m}_\sigma} = \sum_{i,j} g^{ij} X_i X_j$, where g^{ij} is the inverse of the matrix $B(X_i,X_j)$; Ω_{mk} and $\Omega_{\mathfrak{a}_\mathrm{q}}$ are defined in the same way). Then we have

$$\Omega = \Omega_{\mathfrak{m}_\sigma} + \Omega_{\mathfrak{a}_\mathrm{q}} + \sum_{\substack{\alpha>0,\, \varepsilon=\pm \\ 1\le i\le m_\alpha^\varepsilon}} (X_{\alpha,i}^\varepsilon X_{-\alpha,i}^\varepsilon + X_{-\alpha,i}^\varepsilon X_{\alpha,i}^\varepsilon). \tag{6}$$

Put

$$Y_{\alpha,i}^\varepsilon = \tfrac{1}{2}(X_{\alpha,i}^\varepsilon + X_{-\alpha,i}^\varepsilon) \quad \text{and} \quad Z_{\alpha,i}^\varepsilon = \tfrac{1}{2}(X_{\alpha,i}^\varepsilon - X_{-\alpha,i}^\varepsilon). \tag{7}$$

Notice that $Y_{\alpha,i}^+ \in \mathfrak{p}\cap\mathfrak{q}$, $Y_{\alpha,i}^- \in \mathfrak{p}\cap\mathfrak{h}$, $Z_{\alpha,i}^+ \in \mathfrak{k}\cap\mathfrak{h}$, and $Z_{\alpha,i}^- \in \mathfrak{k}\cap\mathfrak{q}$; this follows from the fact that $\sigma X_{\alpha,i}^\varepsilon = -\varepsilon X_{-\alpha,i}^\varepsilon$.

Define, for $\varepsilon=+$ and $\varepsilon=-$, the element $\mathrm{L}_\alpha^\varepsilon \in U(\mathfrak{k})$ by:

$$\mathrm{L}_\alpha^\varepsilon = 2\sum_{i=1}^{m_\alpha^\varepsilon} (Z_{\alpha,i}^\varepsilon)^2.$$

Since $M\cap K\cap H$ acts orthogonally on the space $(\mathfrak{g}_\alpha^\varepsilon \oplus \mathfrak{g}_{-\alpha}^\varepsilon)\cap\mathfrak{k}$, for which the elements $\sqrt{2}\, Z_{\alpha,i}^\varepsilon$, ($1\le i\le m_\alpha^\varepsilon$), form an orthonormal basis, we have $\mathrm{L}_\alpha^\varepsilon \in U(\mathfrak{k})^{M\cap K\cap H}$.

Lemma 4.1. *Fix $a=\exp Y \in A$. Then modulo $U(\mathfrak{g})\mathfrak{h}$ we have*

$$\Omega \equiv \Omega_{\mathrm{mk}} + \Omega_{\mathfrak{a}_\mathrm{q}} + \sum_{\alpha>0} [m_\alpha^+ \coth\alpha(Y) + m_\alpha^- \tanh\alpha(Y)] H_\alpha$$
$$+ \sum_{\alpha>0} \big[\sinh^{-2}\alpha(Y) (\mathrm{L}_\alpha^+)^{a^{-1}} - \cosh^{-2}\alpha(Y) (\mathrm{L}_\alpha^-)^{a^{-1}}\big].$$

Proof. Since $\mathfrak{m}_\sigma \subset \mathfrak{k}+\mathfrak{h}$ we have $\Omega_{\mathfrak{m}_\sigma} \equiv \Omega_{\mathrm{mk}}$ modulo $U(\mathfrak{g})\mathfrak{h}$. Hence it remains to consider the summation term in (6).

From (7) we obtain

$$(8) \qquad (Z^\varepsilon_{\alpha,i})^{a^{-1}} = \tfrac{1}{2}[e^{-\alpha(Y)} X^\varepsilon_{\alpha,i} - e^{\alpha(Y)} X^\varepsilon_{-\alpha,i}],$$

which together with (7) gives

$$X^\varepsilon_{\alpha,i} = \sinh^{-1}\alpha(Y)\left[e^{\alpha(Y)} Z^\varepsilon_{\alpha,i} - (Z^\varepsilon_{\alpha,i})^{a^{-1}}\right]$$

and

$$X^\varepsilon_{-\alpha,i} = \sinh^{-1}\alpha(Y)\left[e^{-\alpha(Y)} Z^\varepsilon_{\alpha,i} - (Z^\varepsilon_{\alpha,i})^{a^{-1}}\right].$$

Hence, taking into account that $Z^+_{\alpha,i} \in \mathfrak{h}$, we obtain that

$$X^+_{\alpha,i} X^+_{-\alpha,i} + X^+_{-\alpha,i} X^+_{\alpha,i} \equiv 2\sinh^{-2}\alpha(Y)\left[-\cosh\alpha(Y) Z^+_{\alpha,i} (Z^+_{\alpha,i})^{a^{-1}} + ((Z^+_{\alpha,i})^2)^{a^{-1}}\right],$$

modulo $U(\mathfrak{g})\mathfrak{h}$. Now

$$(9) \qquad \begin{aligned} \left[Z^\varepsilon_{\alpha,i}, (Z^\varepsilon_{\alpha,i})^{a^{-1}}\right] &= \left[\tfrac{1}{2}(X^\varepsilon_{\alpha,i} - X^\varepsilon_{-\alpha,i}), \tfrac{1}{2}(e^{-\alpha(Y)} X^\varepsilon_{\alpha,i} - e^{\alpha(Y)} X^\varepsilon_{-\alpha,i})\right] \\ &= -\tfrac{1}{2}\sinh\alpha(Y)[X^\varepsilon_{\alpha,i}, X^\varepsilon_{-\alpha,i}] = -\tfrac{1}{2}\sinh\alpha(Y) H_\alpha, \end{aligned}$$

and we obtain

$$(10) \qquad X^+_{\alpha,i} X^+_{-\alpha,i} + X^+_{-\alpha,i} X^+_{\alpha,i} \equiv \coth\alpha(Y) H_\alpha + 2\sinh^{-2}\alpha(Y)((Z^+_{\alpha,i})^2)^{a^{-1}}.$$

From (7) and (8) we obtain

$$X^\varepsilon_{\alpha,i} = \cosh^{-1}\alpha(Y)\left[e^{\alpha(Y)} Y^\varepsilon_{\alpha,i} + (Z^\varepsilon_{\alpha,i})^{a^{-1}}\right]$$

and

$$X^\varepsilon_{-\alpha,i} = \cosh^{-1}\alpha(Y)\left[e^{-\alpha(Y)} Y^\varepsilon_{\alpha,i} - (Z^\varepsilon_{\alpha,i})^{a^{-1}}\right].$$

Hence, taking into account that $Y^-_{\alpha,i} \in \mathfrak{h}$, we see that

$$X^-_{\alpha,i} X^-_{-\alpha,i} + X^-_{-\alpha,i} X^-_{\alpha,i} \equiv -2\cosh^{-2}\alpha(Y)\left[\sinh\alpha(Y) Y^-_{\alpha,i} (Z^-_{\alpha,i})^{a^{-1}} + ((Z^-_{\alpha,i})^2)^{a^{-1}}\right].$$

In analogy with (9) we have

$$[Y^{\varepsilon}_{\alpha,i}, (Z^{\varepsilon}_{\alpha,i})^{a^{-1}}] = -\tfrac{1}{2}\cosh\alpha(Y)H_\alpha,$$

and hence

$$X^{-}_{\alpha,i}X^{-}_{-\alpha,i} + X^{-}_{-\alpha,i}X^{-}_{\alpha,i} \equiv \tanh\alpha(Y)H_\alpha - 2\cosh^{-2}\alpha(Y)((Z^{-}_{\alpha,i})^2)^{a^{-1}}. \tag{11}$$

The lemma now follows from (10) and (11) applied to (6). □

From Lemma 4.1 and (2) we obtain

$$\begin{aligned}\Pi_\tau(\mathrm{L}) = \Omega_{\mathfrak{a}_\mathrm{q}} + \tau(\Omega_{\mathrm{mk}}) &+ \sum_{\alpha>0}[m^+_\alpha \coth\alpha + m^-_\alpha \tanh\alpha]H_\alpha \\ &+ \sum_{\alpha>0}[\sinh^{-2}\alpha\,\tau(\mathrm{L}^+_\alpha) - \cosh^{-2}\alpha\,\tau(\mathrm{L}^-_\alpha)],\end{aligned} \tag{12}$$

where the hyperbolic functions $\cosh\alpha$, $\sinh\alpha$ etc. are viewed as functions on A^+_q by means of (3).

5. The recursion formula

Let $\tau_\mathrm{M} = \tau|_{M\cap K}$ denote the restriction to $M\cap K$ of the representation τ, and let $C^\infty(M/M\cap H:\tau_\mathrm{M})$ denote the space of τ_M-spherical smooth functions on the symmetric space $M/M\cap H$. It is easily seen (cf. [4, Lemma 1]) that the evaluation map $f\mapsto f(e)$ yields a linear isomorphism

$$C^\infty(M/M\cap H:\tau_\mathrm{M}) \simeq V_\tau^{M\cap K\cap H}. \tag{13}$$

Via this isomorphism, we view $V_\tau^{M\cap K\cap H}$ as a $\mathbf{D}(M/M\cap H)$-module.

Following [4, §3], let

$$`\mu,\ \mu: \mathbf{D}(G/H) \to \mathbf{D}(M/M\cap H)\otimes S(\mathfrak{a}_\mathrm{q})$$

be the algebra homomorphisms defined by the requirement

$$D - `\mu(D) \in \mathfrak{n}_Q U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}$$

for $D\in\mathbf{D}(G/H)$, and by

$$\mu(D:\lambda) = `\mu(D:\lambda+\varrho_Q) \in \mathbf{D}(M/M\cap H)$$

for $\lambda\in\mathfrak{a}_{\mathrm{qc}}^*$; here $\mathfrak{n}_Q$ is the Lie algebra of N_Q. The map $'\mu$ depends on the choice of the parabolic subgroup Q, whereas μ is independent of it.

For $\lambda\in\mathfrak{a}_{\mathrm{qc}}^*$ we denote the endomorphism by which $\mu(D{:}\,\lambda)\in\mathbf{D}(M/M\cap H)$ acts on $V_\tau^{M\cap K\cap H}$ by $\underline{\mu}(D{:}\,\lambda)$. We shall investigate formal $\mathrm{End}(V_\tau^{M\cap K\cap H})$-valued solutions Φ_λ to the differential equation

$$\Pi_\tau(\mathrm{L})\Phi_\lambda=\Phi_\lambda\circ\underline{\mu}(\mathrm{L}{:}\,\lambda) \tag{14}$$

on A_{q}^+. Here λ is a parameter in $\mathfrak{a}_{\mathrm{qc}}^*$, and we assume that Φ_λ is represented by a formal series

$$\Phi_\lambda(a)=a^{\lambda-\varrho}\sum_{\nu\in\mathbf{N}\Delta}a^{-\nu}\Gamma_\nu(\lambda)\quad(a\in A_{\mathrm{q}}^+), \tag{15}$$

with $\Gamma_\nu(\lambda)\in\mathrm{End}(V_\tau^{M\cap K\cap H})$ for $\nu\in\mathbf{N}\Delta$. The application of $\Pi_\tau(L)$ to Φ_λ in (14) is formal. In particular, differentiations are taken term by term. By (12) the resulting formal series is of the same form as (15). The motivation for studying exactly this equation (14) will be clear from Theorem 11.1 below (cf. also Remark 11.2).

The differential equation (14) will yield a recursive relation for the coefficients $\Gamma_\nu(\lambda)$, which will enable us to conclude that for generic λ the power series

$$\sum_{\nu\in\mathbf{N}\Delta}z^\nu\Gamma_\nu(\lambda) \tag{16}$$

actually converges for $z\in D^\Delta$.

As in [7] it is profitable to consider the shifted (at first formally defined) function

$$\widetilde{\Phi}_\lambda(a)=J(a)^{1/2}\Phi_\lambda(a)\quad(a\in A_{\mathrm{q}}^+), \tag{17}$$

where

$$J(a)=\prod_{\alpha>0}[2\sinh\alpha(\log a)]^{m_\alpha^+}[2\cosh\alpha(\log a)]^{m_\alpha^-}$$

is the Jacobian function associated with the $G{=}KA_{\mathrm{q}}H$ decomposition (cf. [11, p. 149]). Write

$$J(a)^{1/2}=a^\varrho\sum_{\xi\in\mathbf{N}\Delta}c_\xi a^{-\xi},\quad J(a)^{-1/2}=a^{-\varrho}\sum_{\xi\in\mathbf{N}\Delta}b_\xi a^{-\xi},$$

with coefficients $c_\xi,b_\xi\in\mathbf{R}$, then $c_0{=}b_0{=}1$. It is easily seen that the coefficients b_ξ and c_ξ have at most polynomial growth in ξ (in fact the c_ξ are bounded). For $\nu\in\mathbf{N}\Delta$ define

$$\widetilde{\Gamma}_\nu(\lambda)=\sum_{\xi\in\mathbf{N}\Delta}c_\xi\Gamma_{\nu-\xi}(\lambda). \tag{18}$$

Here we let $\Gamma_\xi=0$ for $\xi\in\mathbf{Z}\Delta\setminus\mathbf{N}\Delta$. Then the sum (18) is finite. It follows that formally we have

$$\widetilde{\Phi}_\lambda(a)=\sum_{\nu\in\mathbf{N}\Delta}\widetilde{\Gamma}_\nu(\lambda)a^{\lambda-\nu}.$$

Agreeing to write also $\widetilde{\Gamma}_\xi=0$ for $\xi\in\mathbf{Z}\Delta\setminus\mathbf{N}\Delta$ we have the finite sum

$$\Gamma_\nu(\lambda)=\sum_{\xi\in\mathbf{N}\Delta}b_\xi\widetilde{\Gamma}_{\nu-\xi}(\lambda), \tag{19}$$

analogous to (18). Put

$$d(a)=J(a)^{-1/2}\Omega_{\mathfrak{a}_\mathfrak{q}}[J(a)^{1/2}]\quad (a\in A_\mathfrak{q}^+). \tag{20}$$

Lemma 5.1. *Let* $a=\exp Y\in A_\mathfrak{q}^+$. *Then*

$$\begin{aligned}J(a)^{1/2}\,\Pi_\tau(\mathrm{L})\circ J(a)^{-1/2}=\Omega_{\mathfrak{a}_\mathfrak{q}}-d(a)+\tau(\Omega_{\mathrm{mk}})\\ +\sum_{\alpha>0}[\sinh^{-2}\alpha(Y)\tau(\mathrm{L}_\alpha^+)-\cosh^{-2}\alpha(Y)\tau(\mathrm{L}_\alpha^-)].\end{aligned}$$

Proof. The lemma follows from equation (12) combined with the following expression:

$$J(a)^{1/2}\left[\Omega_{\mathfrak{a}_\mathfrak{q}}+\sum_{\alpha>0}[m_\alpha^+\coth\alpha(Y)+m_\alpha^-\tanh\alpha(Y)]H_\alpha\right]\circ J(a)^{-1/2}=\Omega_{\mathfrak{a}_\mathfrak{q}}-d(a).$$

We shall prove this expression in the following equivalent form

$$J(a)^{-1/2}\Omega_{\mathfrak{a}_\mathfrak{q}}\circ J(a)^{1/2}=\Omega_{\mathfrak{a}_\mathfrak{q}}+\sum_{\alpha>0}[m_\alpha^+\coth\alpha(Y)+m_\alpha^-\tanh\alpha(Y)]H_\alpha+d(a). \tag{21}$$

To prove (21), fix an orthonormal basis $H_1,\dots,H_n$ for $\mathfrak{a}_\mathfrak{q}$. Then $\Omega_{\mathfrak{a}_\mathfrak{q}}=\sum_{j=1}^n H_j^2$, and we obtain

$$\Omega_{\mathfrak{a}_\mathfrak{q}}\circ J(a)^{1/2}=J(a)^{1/2}\Omega_{\mathfrak{a}_\mathfrak{q}}+2\sum_{j=1}^n H_j(J(a)^{1/2})H_j+\Omega_{\mathfrak{a}_\mathfrak{q}}(J(a)^{1/2}). \tag{22}$$

Now

$$\begin{aligned}J(a)^{-1/2}H_j(J(a)^{1/2})&=\tfrac12 H_j(\log J(a))\\ &=\tfrac12\sum_{\alpha>0}[m_\alpha^+\coth\alpha(Y)+m_\alpha^-\tanh\alpha(Y)]\alpha(H_j),\end{aligned} \tag{23}$$

and since $\sum_{j=1}^{n} \alpha(H_j)H_j = H_\alpha$ we conclude that

$$2\sum_{j=1}^{n} H_j(J(a)^{1/2})H_j = J(a)^{1/2}\sum_{\alpha>0}[m_\alpha^+ \coth\alpha(Y) + m_\alpha^- \tanh\alpha(Y)]H_\alpha.$$

Inserting this expression in (22) and using (20) we obtain (21). □

The function $d(a)$ has a converging power series expansion of the form

$$d(a) = \sum_{\xi\in\mathbf{N}\Delta} d_\xi a^{-\xi} \quad (a\in A_{\mathrm{q}}^+)$$

with $d_\xi \in \mathbf{R}$. Notice that from the asymptotic behavior of J it follows that

$$d_0 = \langle \varrho, \varrho\rangle. \tag{24}$$

Later we shall give an explicit expression for the coefficients d_ξ, see (36).

We also have the converging expansions

$$\sinh^{-2}\alpha(Y) = 4\sum_{n=1}^{\infty} n a^{-2n\alpha}, \quad \cosh^{-2}\alpha(Y) = -4\sum_{n=1}^{\infty}(-1)^n n a^{-2n\alpha}.$$

Inserting these expansions in the equation of Lemma 5.1 we obtain:

$$\begin{aligned} J(a)^{1/2}\Pi_\tau(\mathrm{L})\circ J(a)^{-1/2} = \Omega_{\mathfrak{a}_{\mathrm{q}}} - \sum_{\xi\in\mathbf{N}\Delta} d_\xi a^{-\xi} + \tau(\Omega_{\mathrm{mk}}) \\ +4\sum_{\alpha>0}\sum_{n=1}^{\infty} n[\tau(\mathrm{L}_\alpha^+) + (-1)^n\tau(\mathrm{L}_\alpha^-)]a^{-2n\alpha}. \end{aligned} \tag{25}$$

Let the operator $\gamma\in\mathrm{End}(\mathrm{End}(V_\tau^{M\cap K\cap H}))$ be defined as the commutator

$$\gamma = [\tau(\Omega_{\mathrm{mk}}), \cdot\,], \tag{26}$$

then we have the following:

Proposition 5.2. *Let $\lambda\in\mathfrak{a}_{\mathrm{qc}}^*$ and suppose that Φ_λ is a formal solution* (15) *to the equation* (14)*. Then for every $\nu\in\mathbf{N}\Delta$ we have:*

$$\begin{aligned} [\langle\nu-2\lambda,\nu\rangle + \gamma]\widetilde{\Gamma}_\nu(\lambda) = \sum_{\xi\in\mathbf{N}\Delta\setminus\{0\}} d_\xi \widetilde{\Gamma}_{\nu-\xi}(\lambda) \\ -4\sum_{\alpha>0}\sum_{n\geq 1} n[\tau(\mathrm{L}_\alpha^+) + (-1)^n\tau(\mathrm{L}_\alpha^-)]\widetilde{\Gamma}_{\nu-2n\alpha}(\lambda). \end{aligned}$$

For the proof of this proposition, we need the following lemma.

Lemma 5.3. *Consider $V_\tau^{M\cap K\cap H}$ as a $\mathbf{D}(M/M\cap H)$-module as before. Then on $V_\tau^{M\cap K\cap H}$ we have:*

$$\underline{\mu}(\mathrm{L}\colon\lambda)=\tau(\Omega_{\mathrm{mk}})+\langle\lambda,\lambda\rangle-\langle\varrho,\varrho\rangle.$$

Proof. Using (5), it follows straightforwardly from (6) that

$$\Omega\equiv\Omega_{\mathrm{mk}}+\Omega_{\mathfrak{a}_\mathrm{q}}-\sum_{\alpha>0}m_\alpha H_\alpha$$

modulo $\mathfrak{n}_Q U(\mathfrak{g})+U(\mathfrak{g})\mathfrak{h}$. Hence $'\mu(\mathrm{L})$ equals the right-hand side of the above congruence, and it follows that

$$\underline{\mu}(\mathrm{L}\colon\lambda)=r(\Omega_{\mathrm{mk}})+\langle\lambda,\lambda\rangle-\langle\varrho,\varrho\rangle,$$

where r indicates that the image in $\mathbf{D}(M/M\cap H)$ has been taken. One readily verifies that $r(\Omega_{\mathrm{mk}})$ acts on $V_\tau^{M\cap K\cap H}$ in (13) by the same endomorphism as $\tau(\Omega_{\mathrm{mk}})$. □

Proof of Proposition 5.2. In view of the above lemma it follows from (14) that

$$\Pi_\tau(\mathrm{L})\Phi_\lambda=\Phi_\lambda\circ[\tau(\Omega_{\mathrm{mk}})+\langle\lambda,\lambda\rangle-\langle\varrho,\varrho\rangle].$$

In view of (17) and (24) this leads to:

$$[J^{1/2}\,\Pi_\tau(\mathrm{L})\circ J^{-1/2}-\langle\lambda,\lambda\rangle+d_0]\widetilde{\Phi}_\lambda=\widetilde{\Phi}_\lambda\circ\tau(\Omega_{\mathrm{mk}}).$$

Using (25) we finally obtain that

$$\left[\Omega_{\mathfrak{a}_\mathrm{q}}-\langle\lambda,\lambda\rangle+\gamma-\sum_{\xi\in\mathbf{N}\Delta\setminus\{0\}}d_\xi a^{-\xi}+4\sum_{\alpha>0}\sum_{n=1}^{\infty}n[\tau(\mathrm{L}_\alpha^+)+(-1)^n\tau(\mathrm{L}_\alpha^-)]a^{-2n\alpha}\right]\widetilde{\Phi}_\lambda=0.$$

By insertion of the series for $\widetilde{\Phi}_\lambda$, the proposition now follows from a comparison of coefficients, since $(\Omega_{\mathfrak{a}_\mathrm{q}}-\langle\lambda,\lambda\rangle)e^{\lambda-\nu}=\langle\nu-2\lambda,\nu\rangle e^{\lambda-\nu}$. □

6. The singular set S

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{m}\cap\mathfrak{k}$, $\Sigma^+(\mathfrak{t})\subset i\mathfrak{t}^*$ a positive system for the root system of $\mathfrak{t}$ in $(\mathfrak{m}\cap\mathfrak{k})_\mathbf{c}$, and $\varrho_\mathfrak{t}$ the associated half sum of the positive roots. Moreover, let $\Lambda(\tau)\subset i\mathfrak{t}^*$ be the set of infinitesimal characters, viewed as a subset of the set of dominant weights in $i\mathfrak{t}^*$, of the $(\mathfrak{m}\cap\mathfrak{k})$-types which occur in τ_M and have a non-zero $(\mathfrak{m}\cap\mathfrak{k}\cap\mathfrak{h})$-fixed vector. Then $\tau(\Omega_{\mathrm{mk}})$ diagonalizes on $V_\tau^{M\cap K\cap H}$ with the eigenvalues $\langle\xi-\varrho_{\mathfrak{t}},\xi-\varrho_{\mathfrak{t}}\rangle$, $\xi\in\Lambda(\tau)$.

It follows that the commutator γ in (26) diagonalizes and has the following set of eigenvalues:

$$\mathcal{N}:=\{\langle\xi_1-\varrho_{\mathfrak{t}},\xi_1-\varrho_{\mathfrak{t}}\rangle-\langle\xi_2-\varrho_{\mathfrak{t}},\xi_2-\varrho_{\mathfrak{t}}\rangle\mid\xi_1,\xi_2\in\Lambda(\tau)\}\subset\mathbf{R}.$$

Notice that $\mathcal{N}=-\mathcal{N}$.

When $\lambda\in\mathfrak{a}_{\mathrm{qc}}^*$ is outside the set

$$S:=\{\lambda\in\mathfrak{a}_{\mathrm{qc}}^*\mid\exists(\nu\in\mathbf{N}\Delta\setminus\{0\}):\langle\nu-2\lambda,\nu\rangle\in\mathcal{N}\},$$

the formula in Proposition 5.2 allows the recurrent determination of all the coefficients $\widetilde{\Gamma}_\nu(\lambda)$, once $\widetilde{\Gamma}_0(\lambda)$ is given. We shall now investigate this singular set.

We first notice that S is the countable union of the hyperplanes $\mathcal{H}_{\nu,d}$ in $\mathfrak{a}_{\mathrm{qc}}^*$ defined by

$$\mathcal{H}_{\nu,d}=\{\lambda\in\mathfrak{a}_{\mathrm{qc}}^*\mid\langle\nu-2\lambda,\nu\rangle=d\},$$

for $\nu\in\mathbf{N}\Delta\setminus\{0\}$, $d\in\mathcal{N}$.

We shall need the following notation. If $\nu=\sum_{\alpha\in\Delta}\nu_\alpha\alpha\in\mathbf{R}\Delta$ we write

$$m(\nu)=\sum|\nu_\alpha|.$$

By equivalence of norms on $\mathbf{R}\Delta$, there exists a constant $c_1>0$ such that

$$c_1m(\nu)\le|\nu|\le c_1^{-1}m(\nu)\tag{27}$$

for all $\nu\in\mathbf{R}\Delta$. For $R\in\mathbf{R}$, let

$$\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)=\{\lambda\in\mathfrak{a}_{\mathrm{qc}}^*\mid\mathrm{Re}\langle\lambda,\alpha\rangle\le R\text{ for }\alpha\in\Sigma(Q)\}.$$

Moreover, let X_R be the subset of $\mathbf{N}\Delta\setminus\{0\}$ defined by

$$X_R=\{\nu\in\mathbf{N}\Delta\setminus\{0\}\mid|\nu|^2-2Rm(\nu)\le\max\mathcal{N}\}.\tag{28}$$

Notice that X_R is finite, in view of (27). Finally, if $R\in\mathbf{R}$ and $\nu\in\mathbf{N}\Delta\setminus\{0\}$, let

$$\mathcal{N}_{R,\nu}=\{d\in\mathcal{N}\mid|\nu|^2-2Rm(\nu)\le d\},\tag{29}$$

then $\nu\in X_R$ if and only if $\mathcal{N}_{R,\nu}\neq\emptyset$.

Lemma 6.1. *Let $R\in\mathbf{R}$ and $\lambda\in\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)$. If $\langle\nu-2\lambda,\nu\rangle=d$ for some $\nu\in\mathbf{N}\Delta\setminus\{0\}$ and $d\in\mathcal{N}$, then $\nu\in X_R$ and $d\in\mathcal{N}_{R,\nu}$. In particular, the set $S\cap\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)$ equals the intersection of $\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)$ with the finite union of the hyperplanes $\mathcal{H}_{\nu,d}$, where $\nu\in X_R$, $d\in\mathcal{N}_{R,\nu}$. If $R\leq\frac{1}{2}\min\mathcal{N}$, then X_R and $S\cap\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)$ are both empty.*

Proof. Since $\mathrm{Re}\langle\lambda,\alpha\rangle\leq R$ for all $\alpha\in\Delta$, we have

$$\mathrm{Re}\langle\nu-2\lambda,\nu\rangle\geq|\nu|^2-2Rm(\nu) \tag{30}$$

for all $\nu\in\mathbf{N}\Delta$. Assume that $\lambda\in\mathcal{H}_{\nu,d}$ for some $\nu\in\mathbf{N}\Delta\setminus\{0\}$ and $d\in\mathcal{N}$, then we see immediately from (30) that $d\in\mathcal{N}_{R,\nu}$ and $\nu\in X_R$. This proves the first statement. The assertion about $S\cap\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)$ is an easy consequence.

Finally if $R\leq\frac{1}{2}\min\mathcal{N}$, then $R\leq0$, hence $|\nu|^2-2Rm(\nu)>-2R\geq\max\mathcal{N}$ for all $\nu\in\mathbf{N}\Delta\setminus\{0\}$, and we have $X_R=\emptyset$. Hence also $S\cap\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)=\emptyset$, by the previous assertion. □

Remark 6.2. Notice that when τ is the trivial K-type, then $\mathcal{N}=\{0\}$, and it follows from Lemma 6.1 that $S\cap\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,0)=\emptyset$. Moreover, when G/H is split, that is when $\mathfrak{a}_{\mathrm{q}}$ is a maximal abelian subspace of $\mathfrak{q}$, then $\mathfrak{m}\subset\mathfrak{h}$ so that $\Lambda(\tau)=\{\varrho_{\mathfrak{t}}\}$ and the same conclusion holds, for all τ for which $V_\tau^{M\cap K\cap H}\neq\{0\}$. In particular this is the case when G/H has rank 1 or is of 'K_ε-type' (see [10] for the latter notion).

7. The fundamental estimate

Let $R\in\mathbf{R}$ be fixed, and let the set X_R be defined by (28).

Lemma 7.1. *Let $\nu\in\mathbf{N}\Delta\setminus(X_R\cup\{0\})$, and let $\gamma\in\mathrm{End}(\mathrm{End}(V_\tau^{M\cap K\cap H}))$ be the commutator given by (26). Then the operator $[\langle\nu-2\lambda,\nu\rangle+\gamma]^{-1}$ depends holomorphically on λ in a neighborhood of $\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)$. Moreover, we have the following uniform estimate for its operator norm:*

$$\|(\langle\nu-2\lambda,\nu\rangle+\gamma)^{-1}\|\leq(|\nu|^2-2Rm(\nu)+\min\mathcal{N})^{-1}\quad(\lambda\in\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)).$$

Proof. Let $\lambda\in\mathfrak{a}_{\mathrm{qc}}^*$. The operator $\langle\nu-2\lambda,\nu\rangle+\gamma$ diagonalizes with respect to an orthonormal basis of $\mathrm{End}(V_\tau^{M\cap K\cap H})$, with eigenvalues $\langle\nu-2\lambda,\nu\rangle+d$, $d\in\mathcal{N}$. For $\lambda\in\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)$ we obtain from (30) the estimate

$$\mathrm{Re}(\langle\nu-2\lambda,\nu\rangle+d)\geq|\nu|^2-2Rm(\nu)-\max\mathcal{N}.$$

Since $\nu\notin X_R$ the right-hand side of this inequality is positive, and the result follows. □

If $\nu \in \mathbf{N}\Delta\setminus\{0\}$, we define the following polynomial function of $\lambda \in \mathfrak{a}^*_{\mathrm{qc}}$:

$$p_\nu(\lambda) = \prod_{d \in \mathcal{N}_{R,\nu}} (\langle \nu - 2\lambda, \nu\rangle + d),$$

where $\mathcal{N}_{R,\nu}$ is the set given in (29). Notice that $p_\nu = 1$ if $\nu \notin X_R$, since then $\mathcal{N}_{R,\nu} = \emptyset$.

Corollary 7.2. *Let $\nu \in \mathbf{N}\Delta\setminus\{0\}$. Then the* $\mathrm{End}(\mathrm{End}(V_\tau^{M\cap K\cap H}))$*-valued expression $p_\nu(\lambda)[\langle \nu - 2\lambda, \nu\rangle + \gamma]^{-1}$ depends holomorphically on λ in a neighborhood of $\bar{\mathfrak{a}}^*_{\mathrm{q}}(Q,R)$. Moreover, there exists a constant $C>0$ such that*

$$|p_\nu(\lambda)|\, \|(\langle \nu - 2\lambda, \nu\rangle + \gamma)^{-1}\| \le C(1+|\lambda|)^{\deg p_\nu} \quad (\lambda \in \bar{\mathfrak{a}}^*_{\mathrm{q}}(Q,R)).$$

Proof. If $\nu \notin X_R$, then $p_\nu = 1$, and the result is an immediate consequence of the previous lemma. On the other hand, if $\nu \in X_R$, then it follows from the above mentioned fact that $\langle \nu - 2\lambda, \nu\rangle + \gamma$ is diagonalizable with eigenvalues $\langle \nu - 2\lambda, \nu\rangle + d$, $d \in \mathcal{N}$, that

$$\lambda \mapsto \prod_d (\langle \nu - 2\lambda, \nu\rangle + d)[\langle \nu - 2\lambda, \nu\rangle + \gamma]^{-1} \tag{31}$$

is holomorphic on a neighborhood of $\bar{\mathfrak{a}}^*_{\mathrm{q}}(Q,R)$, where the product is taken over those $d \in \mathcal{N}$ for which $\langle \nu - 2\lambda', \nu\rangle + d = 0$ for some $\lambda' \in \bar{\mathfrak{a}}^*_{\mathrm{q}}(Q,R)$. By the definition this implies that $\lambda' \in S$, and hence by Lemma 6.1 that $d \in \mathcal{N}_{R,\nu}$. Thus the product term in (31) equals p_ν, and the corollary follows. □

We define the polynomial function p_R on $\mathfrak{a}^*_{\mathrm{qc}}$ by

$$p_R(\lambda) = \prod_{\nu \in X_R} p_\nu(\lambda) = \prod_{\substack{\nu \in X_R \\ d \in \mathcal{N}_{R,\nu}}} (\langle \nu - 2\lambda, \nu\rangle + d),$$

then by Lemma 6.1 we have $S \cap \bar{\mathfrak{a}}^*_{\mathrm{q}}(Q,R) = p_R^{-1}(0) \cap \bar{\mathfrak{a}}^*_{\mathrm{q}}(Q,R)$.

Lemma 7.3. *Let the endomorphisms $\widetilde{\Gamma}_\nu(\lambda)$ of $V_\tau^{M\cap K\cap H}$ be defined by the recursion formula of Proposition* 5.2 *with $\widetilde{\Gamma}_0(\lambda) = \mathrm{I}_{V_\tau^{M\cap K\cap H}}$, and let the endomorphisms $\Gamma_\nu(\lambda)$ be given by* (19)*. Let $\nu \in \mathbf{N}\Delta$ be fixed. The functions $\lambda \mapsto p_R(\lambda)\Gamma_\nu(\lambda)$ and $\lambda \mapsto p_R(\lambda)\widetilde{\Gamma}_\nu(\lambda)$ are holomorphic on a neighborhood of $\bar{\mathfrak{a}}^*_{\mathrm{q}}(Q,R)$, and moreover there exists a constant $C>0$ such that for all $\lambda \in \bar{\mathfrak{a}}^*_{\mathrm{q}}(Q,R)$:*

$$|p_R(\lambda)|\, \|\Gamma_\nu(\lambda)\| \le C(1+|\lambda|)^{\deg p_R} \quad \text{and} \quad |p_R(\lambda)|\, \|\widetilde{\Gamma}_\nu(\lambda)\| \le C(1+|\lambda|)^{\deg p_R}.$$

Proof. By (19) it suffices to prove the statements for $\widetilde{\Gamma}_\nu(\lambda)$. For $\nu \in \mathbf{N}\Delta$ we put:

$$q_\nu(\lambda) = \prod_{\substack{\xi \in \mathbf{N}\Delta \setminus \{0\} \\ \xi \preceq \nu}} p_\xi(\lambda).$$

Then every q_ν is a divisor of the polynomial p_R. Therefore, it suffices to prove the estimate

$$|q_\nu(\lambda)| \, \|\widetilde{\Gamma}_\nu(\lambda)\| \le C(1+|\lambda|)^{\deg q_\nu} \quad (\lambda \in \bar{\mathfrak{a}}_{\mathrm{q}}^*(Q, R)),$$

with C a constant independent of λ. We will prove this estimate by induction along the natural ordering $\preceq$ on $\mathbf{N}\Delta$. Since by definition $\widetilde{\Gamma}_0 = \mathrm{I}$ and $q_0 = 1$ it clearly holds for $\nu = 0$. Therefore, let $\nu \neq 0$ and suppose the estimate has been established for all elements $\eta \in \mathbf{N}\Delta$ strictly smaller than ν. From the recurrence relation in Proposition 5.2 it follows that $q_\nu(\lambda)\widetilde{\Gamma}_\nu(\lambda)$ can be written as a finite sum of terms of the form

$$\frac{q_\nu(\lambda)}{p_\nu(\lambda) q_\eta(\lambda)} [p_\nu(\lambda)(\langle \nu - 2\lambda, \nu \rangle + \gamma)^{-1}] A_\eta \, [q_\eta(\lambda)\widetilde{\Gamma}_\eta(\lambda)],$$

where $\eta \in \mathbf{N}\Delta$, $\eta \prec \nu$, and where $A_\eta \in \mathrm{End}(\mathrm{End}(V_\tau^{M \cap K \cap H}))$ is independent of λ. The rational factor in front is a polynomial of degree $\deg q_\nu - \deg p_\nu - \deg q_\eta$; therefore the required estimate follows from the induction hypothesis combined with Corollary 7.2. $\square$

The constant C in the above estimate can in turn be estimated uniformly in the parameter ν.

Theorem 7.4. *Fix $R \in \mathbf{R}$, and let $\Gamma_\nu(\lambda)$ and $\widetilde{\Gamma}_\nu(\lambda)$ be as above. There exist constants $C, \varkappa > 0$ (depending on τ, R), such that*

$$|p_R(\lambda)| \, \|\Gamma_\nu(\lambda)\| \le C(1+|\nu|)^\varkappa (1+|\lambda|)^{\deg p_R} \tag{32}$$

and

$$|p_R(\lambda)| \, \|\widetilde{\Gamma}_\nu(\lambda)\| \le C(1+|\nu|)^\varkappa (1+|\lambda|)^{\deg p_R}, \tag{33}$$

for all $\nu \in \mathbf{N}\Delta$ and $\lambda \in \bar{\mathfrak{a}}_{\mathrm{q}}^*(Q, R)$.

Notice that the existence of C and $\varkappa$ such that (32) holds is equivalent to the existence of C and $\varkappa$ such that (33) holds, by the polynomial estimates of the coefficients in (18) and (19), and the fact that the number of terms in (18) and (19) is bounded by a polynomial in $|\nu|$. The estimate (33) is proved in the following section.

8. Proof of Theorem 7.4

The following two lemmas will be needed in the proof of Theorem 7.4.

Lemma 8.1. *There exists a constant $c_2>0$ such that for all $N\in\mathbf{N}\setminus\{0\}$*

$$\sum_{m(\nu)=N} |d_\nu| \le c_2 N. \tag{34}$$

Proof. Recall (20) to motivate the following calculation. Let $H_1, \dots, H_n$ be an orthonormal basis for $\mathfrak{a}_\mathrm{q}$, then $\Omega_{\mathfrak{a}_\mathrm{q}}=\sum_j H_j^2$. Write $Y=\log a$ and recall from (23) that

$$H_j(J(a)^{1/2}) = \tfrac{1}{2} J(a)^{1/2} \sum_{\alpha>0} [m_\alpha^+ \coth\alpha(Y) + m_\alpha^- \tanh\alpha(Y)]\alpha(H_j).$$

Hence

$$\begin{aligned} H_j^2(J(a)^{1/2}) &= \tfrac{1}{4} J(a)^{1/2} \Big(\sum_{\alpha>0} [m_\alpha^+ \coth\alpha(Y) + m_\alpha^- \tanh\alpha(Y)]\alpha(H_j) \Big)^2 \\ &\quad + \tfrac{1}{2} J(a)^{1/2} \sum_{\alpha>0} [m_\alpha^+ \coth'\alpha(Y) + m_\alpha^- \tanh'\alpha(Y)]\alpha(H_j)^2. \end{aligned} \tag{35}$$

Let $M_\alpha^k = m_\alpha^+ + (-1)^k m_\alpha^-$ for $\alpha\in\Sigma^+$, $k\in\mathbf{N}$. Using the power series

$$\coth\alpha = 1+2\sum_{k=1}^\infty e^{-2k\alpha}, \quad \tanh\alpha = 1+2\sum_{k=1}^\infty (-1)^k e^{-2k\alpha},$$

for coth and tanh, we obtain

$$m_\alpha^+ \coth\alpha + m_\alpha^- \tanh\alpha = m_\alpha + 2\sum_{k=1}^\infty M_\alpha^k e^{-2k\alpha} = 2\sum_{k=0}^\infty \chi_k M_\alpha^k e^{-2k\alpha},$$

where for simplicity we have introduced the notation $\chi_k := \frac{1}{2}$ if $k=0$ and $\chi_k:=1$ otherwise. Moreover, by differentiation

$$m_\alpha^+ \coth'\alpha + m_\alpha^- \tanh'\alpha = -4\sum_{k=1}^\infty k M_\alpha^k e^{-2k\alpha}.$$

We insert these expressions in (35) and sum over j. Since $\sum_j \alpha(H_j)\beta(H_j) = \langle\alpha,\beta\rangle$, we conclude

$$\begin{aligned} J(a)^{-1/2}\Omega_{\mathfrak{a}_\mathrm{q}}(J(a)^{1/2}) &= \sum_{\alpha,\beta>0} \langle\alpha,\beta\rangle \Big[\sum_{k,l=0}^\infty \chi_k\chi_l M_\alpha^k M_\beta^l a^{-2k\alpha-2l\beta} \Big] \\ &\quad -2\sum_{\alpha>0} \langle\alpha,\alpha\rangle \sum_{k=1}^\infty k M_\alpha^k a^{-2k\alpha}. \end{aligned}$$

Hence

$$d_\nu = \sum_{\substack{\alpha,\beta\in\Sigma^+,\, k,l\geq 0\\ 2k\alpha+2l\beta=\nu}} \chi_k\chi_l M_\alpha^k M_\beta^l \langle\alpha,\beta\rangle - \sum_{\substack{\alpha\in\Sigma^+,\, k\geq 1\\ 2k\alpha=\nu}} 2kM_\alpha^k\langle\alpha,\alpha\rangle. \tag{36}$$

Since for each pair (α,β) the cardinality of the set

$$\{(k,l)\in\mathbf{N}^2 \mid 2km(\alpha)+2lm(\beta)=N\}$$

is at most linear in N, the lemma easily follows. □

Lemma 8.2. *There exist constants $m_0\in\mathbf{N}$ and $c_3>0$ (both depending on R), such that for all $\lambda\in\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)$ and $\nu\in\mathbf{N}\Delta$ with $m(\nu)\geq m_0$ we have:*

$$\|(\langle\nu-2\lambda,\nu\rangle+\gamma)^{-1}\| \leq \frac{c_3}{m(\nu)^2}. \tag{37}$$

Proof. This is an immediate consequence of (27) and Lemma 7.1. □

Proof of Theorem 7.4. Let m_0 be as in Lemma 8.2. Then by Proposition 5.2 we have, for $\lambda\in\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)$ and $m(\nu)\geq m_0$,

$$\|\widetilde{\Gamma}_\nu(\lambda)\| \leq \frac{c_3}{m(\nu)^2}\left[\sum_{\xi\in\mathbf{N}\Delta\setminus\{0\}} |d_\xi|\,\|\widetilde{\Gamma}_{\nu-\xi}(\lambda)\| + c_4\sum_{\substack{\alpha>0\\ n\geq 1}} n\|\widetilde{\Gamma}_{\nu-2n\alpha}(\lambda)\|\right], \tag{38}$$

where $c_4=4\max_{\alpha>0}(\|\tau(\mathrm{L}_\alpha^+)\|+\|\tau(\mathrm{L}_\alpha^-)\|)$.

From Lemma 7.3 it follows that there exists a constant $C>0$ such that

$$|p_R(\lambda)|\,\|\widetilde{\Gamma}_\nu(\lambda)\| \leq C(1+|\lambda|)^{\deg p_R},$$

for $\lambda\in\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)$ and ν with $m(\nu)\leq m_0$. This implies, for any $\varkappa\geq 0$, the estimate

$$|p_R(\lambda)|\,\|\widetilde{\Gamma}_\nu(\lambda)\| \leq C(1+m(\nu))^{\varkappa}(1+|\lambda|)^{\deg p_R}, \tag{39}$$

for $\lambda\in\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)$ and ν with $m(\nu)\leq m_0$.

Under the assumption that

$$\varkappa \geq 2c_3\big(c_2+\tfrac{1}{2}c_4|\Sigma^+|\big), \tag{40}$$

we shall now prove the estimate (39) for all ν, using induction on $m(\nu)$. Here c_2, c_3 and c_4 are the constants of (34), (37) and (38), respectively. Theorem 7.4 is an immediate consequence.

Fix $m>m_0$, suppose that (39) has already been established for all ν with $m(\nu)<m$, and suppose a ν with $m(\nu)=m$ is given. We claim that (39) holds for this ν, with unchanged constant C. By (38) and (39) we have

$$\begin{aligned}
&|p_R(\lambda)|\,\|\widetilde{\Gamma}_\nu(\lambda)\|\\
&\leq \frac{c_3C}{m(\nu)^2}\Bigg[\sum_{0\prec\xi\preceq\nu}|d_\xi|(1+m(\nu-\xi))^{\varkappa}+c_4\sum_{\substack{\alpha>0,\,n\geq 1\\ 2n\alpha\preceq\nu}}n(1+m(\nu-2n\alpha))^{\varkappa}\Bigg](1+|\lambda|)^{\deg p_R}\\
&\leq \frac{c_3C}{m^2}\sum_{N=1}^{m}(1+m-N)^{\varkappa}\Bigg[\sum_{m(\xi)=N}|d_\xi|+c_4|\Sigma^+|\frac{N}{2}\Bigg](1+|\lambda|)^{\deg p_R}\\
&\leq \frac{\varkappa C}{2m^2}\sum_{N=1}^{m}N(1+m-N)^{\varkappa}(1+|\lambda|)^{\deg p_R},
\end{aligned}$$

where the last estimate has been obtained using Lemma 8.1 and (40). From

$$\sum_{N=1}^{m}N(1+m-N)^{\varkappa}\leq m\sum_{N=1}^{m}N^{\varkappa}\leq m\int_0^{m+1}t^{\varkappa}\,dt=m\frac{(m+1)^{\varkappa+1}}{\varkappa+1}\leq 2m^2\frac{(m+1)^{\varkappa}}{\varkappa}.$$

we obtain (39). □

Remark 8.3. As mentioned in Remark 6.2 we have $S\cap\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,0)=\emptyset$ in several important cases. In these cases it follows that $p_0=1$ and the estimates in Theorem 7.4 are simplified. In particular, in the special case where $H=K$ and τ is trivial, these estimates were obtained by Gangolli ([7, Lemma 3.1]—in fact, when adapted to this case, our proof simplifies slightly that of [7]). For the group case, [1, Lemma 5.1] gives a weaker estimate with an exponential bound in ν, instead of the polynomial bound in (32). For this case the polynomial estimates were obtained in [6].

9. The functions Φ_λ

The estimate we have obtained for the coefficients in the series (15) has the following consequence for the sum of the series.

Theorem 9.1. *Let the coefficients* $\Gamma_\nu(\lambda)$ *in the series* (15) *be defined as in Lemma* 7.3. *For* $\lambda\in\mathfrak{a}_{\mathrm{qc}}^*\setminus S$ *this series converges and represents an analytic* $\mathrm{End}(V_\tau^{M\cap K\cap H})$*-valued function* Φ_λ *on* A_{q}^+*, satisfying the radial differential equation* (14). *Moreover*:

(a) *If* $a\in A_{\mathrm{q}}^+$*, then the function* $\lambda\mapsto p_R(\lambda)\Phi_\lambda(a)$ *is holomorphic in an open neighbourhood of* $\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)$*, for all* $R\in\mathbf{R}$*. In particular,* $\lambda\mapsto\Phi_\lambda(a)$ *is meromorphic in* $\mathfrak{a}_{\mathrm{qc}}^*$.

(b) *Fix $R\in\mathbf{R}$. There exist constants $C, \varkappa>0$ (depending on τ, R), such that*

$$|p_R(\lambda)|\,\|\Phi_\lambda(a)\| \le C(1+|\lambda|)^{\deg p_R}\Big[\prod_{\alpha\in\Delta}(1-a^{-\alpha})\Big]^{-\varkappa} a^{\operatorname{Re}\lambda-\varrho} \tag{41}$$

for all $a\in A_{\mathrm{q}}^+$ and $\lambda\in\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)$. In particular, given $\varepsilon>0$, there exists a constant $C'>0$ (depending on τ, R and ε), such that

$$|p_R(\lambda)|\,\|\Phi_\lambda(a)\| \le C'(1+|\lambda|)^{\deg p_R} a^{\operatorname{Re}\lambda-\varrho} \tag{42}$$

for all $\lambda\in\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)$, and all $a\in A_{\mathrm{q}}$ with $\alpha(\log a)>\varepsilon$ ($\forall\alpha\in\Sigma^+$).

Proof. From Theorem 7.4 we derive, for $\lambda\notin S$, the convergence of the power series (16) for $z\in D^\Delta$. This implies the first assertion of the theorem. Assertion (a) follows from the observation that the convergence of the series for $p_R(\lambda)\Phi_\lambda(a)$ is locally uniform in the variables $\lambda\in\bar{\mathfrak{a}}_{\mathrm{q}}^*(Q,R)$ and $a\in A_{\mathrm{q}}^+$.

To see that (b) holds, notice that (41) implies (42) with $C'=C(1-e^{-\varepsilon})^{-\varkappa|\Delta|}$. Thus it remains to prove (41). To obtain this from Theorem 7.4 it suffices to prove

$$\sum_{\nu\in\mathbf{N}\Delta}(1+|\nu|)^{\varkappa}a^{-\nu} \le \widetilde{C}\Big[\prod_{\alpha\in\Delta}(1-a^{-\alpha})\Big]^{-\widetilde{\varkappa}}, \tag{43}$$

for suitable constants $\widetilde{C}, \widetilde{\varkappa}>0$. Let $\Delta=\{\alpha_1,\dots,\alpha_n\}$. Then by (27) we may estimate the left-hand side of the above inequality by a constant times

$$\begin{aligned}\sum_{\nu\in\mathbf{N}\Delta}(1+m(\nu))^{\varkappa}a^{-\nu} &= \sum_{\nu\in\mathbf{N}^n}(1+\nu_1+\dots+\nu_n)^{\varkappa}a^{-(\nu_1\alpha_1+\dots+\nu_n\alpha_n)}\\ &\le \sum_{\nu\in\mathbf{N}^n}(1+\nu_1)^{\varkappa}\dots(1+\nu_n)^{\varkappa}a^{-\nu_1\alpha_1}\dots a^{-\nu_n\alpha_n}\\ &= \prod_{\alpha\in\Delta}\sum_{k=0}^{\infty}(1+k)^{\varkappa}a^{-k\alpha}.\end{aligned}$$

We may assume that $\varkappa$ is a positive integer. Then $(1+k)^{\varkappa}\le(k+\varkappa)!/k!$, hence

$$\sum_{k=0}^{\infty}(1+k)^{\varkappa}a^{-k\alpha} \le \sum_{k=0}^{\infty}\frac{(k+\varkappa)!}{k!}a^{-k\alpha} = \varkappa!(1-a^{-\alpha})^{-\varkappa-1},$$

and (43) follows. □

The functions Φ_λ are defined by means of the series (15) where the coefficients are recursively obtained from $\Gamma_0(\lambda)=\mathrm{I}_{V_\tau^{M\cap K\cap H}}$. The following lemma describes the sum of the series obtained from using a different first term.

Lemma 9.2. *Fix* $A \in \mathrm{End}(V_\tau^{M\cap K\cap H})$ *and* $\lambda \in \mathfrak{a}_{\mathrm{qc}}^* \setminus S$. *Assume* $\gamma(A)=0$. *Let the endomorphisms* $\widetilde{\Gamma}'_\nu(\lambda)$ *of* $V_\tau^{M\cap K\cap H}$ *be defined by means of the recursion formula of Proposition* 5.2 *starting from* $\widetilde{\Gamma}'_0(\lambda)=A$, *and let the endomorphisms* $\Gamma'_\nu(\lambda)$ *be given as in* (19). *With these coefficients the series* $a^{\lambda-\varrho}\sum_{\nu\in\mathbf{N}\Delta} a^{-\nu}\Gamma'_\nu(\lambda)$ *converges for all* $a \in A_{\mathrm{q}}^+(Q)$; *let* $\Phi'_\lambda(a) \in \mathrm{End}(V_\tau^{M\cap K\cap H})$ *denote its sum. Then*

$$\Phi'_\lambda(a) = \Phi_\lambda(a) \circ A.$$

Proof. Obviously $\widetilde{\Gamma}'_\nu(\lambda) = \widetilde{\Gamma}_\nu(\lambda) \circ A$ for all ν. The lemma follows easily. □

The functions Φ_λ are constructed as eigenfunctions for the radial component of L, but in fact they are joint eigenfunctions for the radial components of all the invariant differential operators (cf. [8, Prop. IV.5.4], in the case $H=K$ and $\tau=1$, and [13, Thm. 9.1.4.1], in the group case):

Corollary 9.3. *Let* $\lambda \in \mathfrak{a}_{\mathrm{qc}}^* \setminus S$. *Then for all* $D \in \mathbf{D}(G/H)$:

$$\Pi_\tau(D)\Phi_\lambda = \Phi_\lambda \circ \underline{\mu}(D\colon \lambda). \tag{44}$$

Proof. Let $D \in \mathbf{D}(G/H)$ and consider the function

$$\Phi'_\lambda := \Pi_\tau(D)\Phi_\lambda$$

defined on A_{q}^+. It follows from the commutativity of $\mathbf{D}(G/H)$ that Φ'_λ satisfies the same differential equation (14) as does Φ_λ. Moreover, since term by term differentiations are allowed in the series (15) for Φ_λ, the function Φ'_λ has a converging series of this type as well. It follows from [5, Lemma 12.2], that the coefficient in the $a^{\lambda-\varrho}$ term is $\underline{\mu}(D\colon\lambda)$. The identity (44) now results from Lemma 9.2 by noting that $\gamma(\underline{\mu}(D\colon\lambda))=0$, cf. Lemma 5.3. □

The singularities of Φ_λ lie along hyperplanes of the form $\langle\lambda,\nu\rangle=c$, where $\nu \in \mathbf{N}\Delta\setminus\{0\}$ and $c\in\mathbf{R}$. Using the full system of differential equations in Corollary 9.3 we can now show that only root hyperplanes occur, that is, hyperplanes of the above form, but with $\nu\in\Sigma$.

Proposition 9.4. *Assume* $\lambda \mapsto \Phi_\lambda$ *is singular along the hyperplane* $\langle\lambda,\nu_0\rangle=c$, *where* $\nu_0 \in \mathbf{N}\Delta\setminus\{0\}$ *and* $c\in\mathbf{R}$. *Then* ν_0 *is a multiple of a root* $\beta\in\Sigma$.

Proof. Inserting the series (15) for Φ_λ in (44) and using [5, eqn. (98)], one obtains a recursion formula for the $\Gamma_\nu(\lambda)$ of the following form:

$$\underline{\mu}(D\colon\lambda-\nu)\circ\Gamma_\nu(\lambda) - \Gamma_\nu(\lambda)\circ\underline{\mu}(D\colon\nu) = \sum_{\eta\prec\nu} A_\eta\Gamma_\eta(\lambda) \tag{45}$$

for all $D\in\mathbf{D}(G/H)$, $\nu\in\mathbf{N}\Delta$ and $\lambda\in\mathfrak{a}_{\mathrm{qc}}^*\setminus S$. Here the A_η are endomorphisms of $\mathrm{End}(V_\tau^{M\cap K\cap H})$ which depend on D, ν and λ. Moreover, the dependence on the latter is holomorphic and extends as such to $\mathfrak{a}_{\mathrm{qc}}^*$.

Let $\mathfrak{b}\subset\mathfrak{q}$ be a Cartan subspace for G/H containing $\mathfrak{a}_{\mathrm{q}}$, and write $\mathfrak{b}=\mathfrak{b}_{\mathrm{k}}\oplus\mathfrak{a}_{\mathrm{q}}$ with $\mathfrak{b}_{\mathrm{k}}=\mathfrak{b}\cap\mathfrak{k}$. The complexified dual spaces $\mathfrak{b}_{\mathrm{kc}}^*$ and $\mathfrak{a}_{\mathrm{qc}}^*$ are viewed as subspaces of $\mathfrak{b}_{\mathrm{c}}^*$ in the obvious fashion. It follows from [4, Lemma 4], that there exists a finite set $L_\tau\subset\mathfrak{b}_{\mathrm{kc}}^*$ such that the endomorphisms $\underline{\mu}(D\colon\lambda)$ are simultaneously diagonalizable for all $D\in\mathbf{D}(G/H)$ and $\lambda\in\mathfrak{a}_{\mathrm{qc}}^*$, with eigenvalues of the form $\gamma(D\colon\Lambda+\lambda)$, $\Lambda\in L_\tau$. Here $\gamma\colon\mathbf{D}(G/H)\to S(\mathfrak{b})^{W(\mathfrak{b})}$ is the Harish-Chandra isomorphism; $W(\mathfrak{b})$ is the reflection group of the root system $\Sigma(\mathfrak{b})$ of $\mathfrak{b}_{\mathrm{c}}$ in $\mathfrak{g}_{\mathrm{c}}$.

Let λ be a generic element of the singular hyperplane $\langle\lambda,\nu_0\rangle=c$ in $\mathfrak{a}_{\mathrm{qc}}^*$, and pick $\nu\in\mathbf{N}\Delta\setminus\{0\}$ minimal such that $\Gamma_\nu(\lambda)$ is singular at λ. Then the right-hand side of the expression (45) is regular at λ, and it follows from the joint diagonalization of $\underline{\mu}(D\colon\lambda)$ that there exist $\Lambda_1,\Lambda_2\in L_\tau$ such that

$$\gamma(D\colon\Lambda_1+\lambda-\nu)=\gamma(D\colon\Lambda_2+\lambda)$$

for all $D\in\mathbf{D}(G/H)$. Hence

$$\Lambda_1+\lambda-\nu=s(\Lambda_2+\lambda)$$

for some $s\in W(\mathfrak{b})$. The fact that λ is generic on the hyperplane $\langle\lambda,\nu_0\rangle=c$ in $\mathfrak{a}_{\mathrm{qc}}^*$ now implies that s leaves the orthocomplement of ν_0 in $\mathfrak{a}_{\mathrm{qc}}^*$ pointwise fixed. Hence s is a product of reflections in roots of $\Sigma(\mathfrak{b})$ orthogonal to this subspace, that is, roots belonging to $\mathfrak{b}_{\mathrm{k}}^*\oplus\mathbf{R}\nu_0$. One of these roots must have a nontrivial restriction to $\mathfrak{a}_{\mathrm{q}}$, since otherwise s would leave $\mathfrak{a}_{\mathrm{q}}^*$ fixed and $\mathfrak{b}_{\mathrm{k}}^*$ invariant, forcing $s\lambda=\lambda$ and $\nu=\Lambda_1-s\Lambda_2\in\mathfrak{b}_{\mathrm{kc}}^*$, a contradiction. Hence there is a root β in the restricted root system Σ which is proportional to ν_0. $\square$

10. Two lemmas

In the following section we shall express the Eisenstein integral in terms of the c-functions and the functions Φ_λ of the previous section. The result below is the first step towards this goal. Let $Q\in\mathcal{P}_\sigma(A_{\mathrm{q}})$ be fixed, and let the notation be as in the previous section.

Lemma 10.1. *Let $\lambda\in\mathfrak{a}_{\mathrm{qc}}^*\setminus S$, and suppose that the formal series*

$$\phi(a)=\sum_{\nu\in\mathbf{N}\Delta}a^{\lambda-\varrho-\nu}v_\nu\quad(a\in A_{\mathrm{q}}^+)$$

with coefficients $v_\nu \in V_\tau^{M\cap K\cap H}$ *is a* $V_\tau^{M\cap K\cap H}$*-valued eigenfunction for* $\Pi_\tau(\mathrm{L})$*. Assume moreover that* $v_0 \neq 0$*. Then the series converges and we have:*

$$\phi(a) = \Phi_\lambda(a)v_0 \quad (a \in A_\mathrm{q}^+).$$

Proof. Let $c\in\mathbf{C}$ be the eigenvalue given by $\Pi_\tau(\mathrm{L})\phi = c\phi$. Consider the shifted series

$$\tilde{\phi}(a) = J^{1/2}(a)\phi(a) = \sum_{\nu\in\mathbf{N}\Delta} a^{\lambda-\nu}\tilde{v}_\nu.$$

Then exactly as in the proof of Proposition 5.2 one obtains the recurrence relations:

$$[\langle \nu-2\lambda, \nu\rangle + \gamma_c]\tilde{v}_\nu = \sum_{\xi\succ 0} d_\xi \tilde{v}_{\nu-\xi} - 4\sum_{\alpha>0}\sum_{n\geq 1} n[\tau(\mathrm{L}_\alpha^+) + (-1)^n\tau(\mathrm{L}_\alpha^-)]\tilde{v}_{\nu-2n\alpha}, \tag{46}$$

where $\gamma_c \in \mathrm{End}(V_\tau^{M\cap K\cap H})$ is the endomorphism $\underline{\mu}(\mathrm{L}:\lambda) - c\,\mathrm{I}$. For $\nu=0$ this yields

$$[\underline{\mu}(\mathrm{L}:\lambda) - c\,\mathrm{I}]\tilde{v}_0 = \gamma_c\tilde{v}_0 = 0, \tag{47}$$

hence, since $\tilde{v}_0 = v_0 \neq 0$, c is an eigenvalue for the action of $\underline{\mu}(\mathrm{L}:\lambda)$ on $V_\tau^{M\cap K\cap H}$. The eigenvalues of γ_c are therefore contained in $\mathcal{N}$, and since $\lambda\notin S$ it follows that $\langle\nu - 2\lambda, \nu\rangle + \gamma_c$ is invertible for all $\nu\in\mathbf{N}\Delta\setminus\{0\}$. Hence the $\tilde{v}_\nu$ are uniquely determined by $\tilde{v}_0$ and the recurrence relations. On the other hand, it follows from Lemma 5.3 that

$$\gamma(A) = [\tau(\Omega_{\mathrm{mk}}), A] = [\underline{\mu}(\mathrm{L}:\lambda), A] = [\gamma_c, A]$$

for all $A\in\mathrm{End}(V_\tau^{M\cap K\cap H})$, and hence by (47) we have $\gamma(A)\tilde{v}_0 = \gamma_c(A\tilde{v}_0)$. Thus

$$\gamma(\widetilde{\Gamma}_\nu(\lambda))\tilde{v}_0 = \gamma_c(\widetilde{\Gamma}_\nu(\lambda)\tilde{v}_0)$$

for all $\nu\in\mathbf{N}\Delta$. By application of both sides of the equation in Proposition 5.2 to $\tilde{v}_0$ it now follows that the coefficients $\tilde{v}'_\nu := \widetilde{\Gamma}_\nu(\lambda)\tilde{v}_0$ satisfy (46). Since $\tilde{v}'_0 = \tilde{v}_0$ we conclude that $\tilde{v}'_\nu = \tilde{v}_\nu$ for all ν, hence $\tilde{\phi} = \widetilde{\Phi}_\lambda(a)\tilde{v}_0$, and the result follows. □

Remark 10.2. Let ϕ be as above, and let $c\in\mathbf{C}$ be the eigenvalue given by $\Pi_\tau(\mathrm{L})\phi = c\phi$. If it is known a priori that c is an eigenvalue for the action of $\underline{\mu}(\mathrm{L}:\lambda)$ on $V_\tau^{M\cap K\cap H}$, then the assumption that $v_0\neq 0$ is not needed in the above proof. The conclusion, if $v_0 = 0$, is then that $\phi(a) = \Phi_\lambda(a)v_0 = 0$ for all $a\in A_\mathrm{q}^+$.

The $\mathrm{End}(V_\tau^{M\cap K\cap H})$-valued function Φ_λ on $A_\mathrm{q}^+(Q)$ depends on the given parabolic subgroup $Q\in\mathcal{P}_\sigma(A_\mathrm{q})$. To express this we also denote it by $\Phi_Q(\lambda:\cdot)$, and we denote its expansion coefficients by $\Gamma_{Q,\nu}(\lambda)$.

Let $w \in N_K(\mathfrak{a}_{\mathrm{q}})$ and consider the involution $w\sigma: x \mapsto w\sigma(w^{-1}xw)w^{-1}$ of G. The group wHw^{-1} is an open subgroup of the fixed point group $G^{w\sigma}$, and we have the decomposition $\mathfrak{g} = \mathrm{Ad}\, w(\mathfrak{h}) \cap \mathrm{Ad}\, w(\mathfrak{q})$ of $\mathfrak{g}$ in ± 1 eigenspaces for $w\sigma$. Moreover $w\sigma$ commutes with θ. In particular, $\mathfrak{a}_{\mathrm{q}}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathrm{Ad}\, w(\mathfrak{q})$. Hence there are functions $A_{\mathrm{q}}^+(Q) \to \mathrm{End}\big(V_\tau^{M \cap K \cap wHw^{-1}}\big)$ defined by series expansions as the Φ_λ in Theorem 9.1, but with respect to the pair (G, wHw^{-1}). We shall use the notation

$$\Phi_{Q,w}(\lambda): A_{\mathrm{q}}^+(Q) \to \mathrm{End}\big(V_\tau^{M \cap K \cap wHw^{-1}}\big)$$

for these functions, with the understanding that $\Phi_Q = \Phi_{Q,e}$ as originally defined. It is easily seen that $\Phi_{Q,w}$ depends on w only through its coset in $N_K(\mathfrak{a}_{\mathrm{q}})/N_{K \cap H}(\mathfrak{a}_{\mathrm{q}})$.

Lemma 10.3. *Let $Q \in \mathcal{P}_\sigma(A_{\mathrm{q}})$ and $w \in N_K(\mathfrak{a}_{\mathrm{q}})$. We have for generic $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$:*

$$\Phi_{Q,w}(\lambda: a) = \tau(w) \circ \Phi_{w^{-1}Qw}(w^{-1}\lambda: w^{-1}aw) \circ \tau(w^{-1}) \quad (a \in A_{\mathrm{q}}^+(Q)).$$

Here it should be noted that $\tau(w)$ maps $V_\tau^{M \cap K \cap H}$ bijectively to $V_\tau^{M \cap K \cap wHw^{-1}}$ with inverse $\tau(w^{-1})$, so that the right-hand side of the above expression yields an element of $\mathrm{End}\big(V_\tau^{M \cap K \cap wHw^{-1}}\big)$.

Proof. For $f: A_{\mathrm{q}}^+(w^{-1}Qw) \to \mathrm{End}(V_\tau^{M \cap K \cap H})$ we define a map ${}^w f$ from $A_{\mathrm{q}}^+(Q)$ to $\mathrm{End}\big(V_\tau^{M \cap K \cap wHw^{-1}}\big)$ by ${}^w f(a) = \tau(w) \circ f(w^{-1}aw) \circ \tau(w^{-1})$. Let

$$\Phi = {}^w[\Phi_{w^{-1}Qw}(w^{-1}\lambda)],$$

then the claim is that $\Phi_{Q,w} = \Phi$.

It follows easily from the expansion (15) for $\Phi_{w^{-1}Qw}(w^{-1}\lambda: a)$, $a \in A_{\mathrm{q}}^+(w^{-1}Qw)$, that $\Phi(a)$ has an expansion of the form required for $\Phi_{Q,w}(\lambda: a)$, $a \in A_{\mathrm{q}}^+(Q)$, with first term $a^{\lambda - \varrho_Q}\,\mathrm{I}$. We claim that

$$[\Pi_{Q,\tau}(\mathrm{L}^w)\Phi](a) = \Phi(a) \circ \underline{\mu}(\mathrm{L}^w: \lambda), \tag{48}$$

where L^w is the Laplace–Beltrami operator on G/wHw^{-1}, and $\Pi_{Q,\tau}(\mathrm{L}^w)$ and $\underline{\mu}(\mathrm{L}^w: \lambda)$ are defined with respect to the pair (G, wHw^{-1}). Since $\underline{\mu}(\mathrm{L}^w: \lambda) \in \mathrm{End}(V_\tau^{M \cap K \cap wHw^{-1}})$ is diagonalizable the lemma is an immediate consequence of Lemma 10.1 and (48).

The right multiplication by w naturally induces a map R_w from $C^\infty(G/H)$ to $C^\infty(G/wHw^{-1})$ which intertwines the actions of L and L^w. Using this fact it can be seen that

$$\Pi_{Q,\tau}(\mathrm{L}^w)({}^w f) = {}^w[\Pi_{w^{-1}Qw,\tau}(\mathrm{L})(f)]$$

for any smooth $\mathrm{End}(V_\tau^{M \cap K \cap H})$-valued function f on $A_{\mathrm{q}}^+(w^{-1}Qw)$.

Furthermore we have

$$\underline{\mu}(\mathrm{L}^w\colon\lambda)=\tau(w)\circ\underline{\mu}(\mathrm{L}\colon\lambda)\circ\tau(w^{-1})$$

by Lemma 5.3, because $\operatorname{Ad} w(\Omega_{\mathrm{mk}})=\Omega_{\mathrm{mk}}$. Since

$$\underline{\mu}(\mathrm{L}\colon w^{-1}\lambda)=\underline{\mu}(\mathrm{L}\colon\lambda) \tag{49}$$

by the same lemma, the claimed identity (48) easily follows. □

11. Eisenstein integrals and their expansions

We first recall (from [4, §2]), some notation related to the Eisenstein integrals associated with the K-representation τ. We fix a set $\mathcal{W}$ of representatives in $N_K(\mathfrak{a}_\mathrm{q})$ for the double quotient $Z_K(\mathfrak{a}_\mathrm{q})\backslash N_K(\mathfrak{a}_\mathrm{q})/N_{K\cap H}(\mathfrak{a}_\mathrm{q})$; the image of $\mathcal{W}$ in W is then a set of representatives for $W/W_{K\cap H}$, where $W_{K\cap H}$ is the subgroup $N_{K\cap H}(\mathfrak{a}_\mathrm{q})/Z_{K\cap H}(\mathfrak{a}_\mathrm{q})$ of W. We denote by 1 the representative in $\mathcal{W}$ of $eW_{K\cap H}$.

Notice that the space $M/w(M\cap H)w^{-1}$ is a compact symmetric space for all $w\in\mathcal{W}$ (cf. [4, Lemma 1]). For $w\in\mathcal{W}$ we denote by ${}^\circ\mathcal{C}_w(\tau)=C^\infty(M/w(M\cap H)w^{-1}\colon\tau_\mathrm{M})$ the space of τ_M-spherical functions on $M/w(M\cap H)w^{-1}$. By *loc. cit.* the evaluation at e maps this space isomorphically onto the space $V_\tau^{M\cap K\cap wHw^{-1}}$, in particular ${}^\circ\mathcal{C}_w(\tau)$ is hence finite dimensional. We then define the space ${}^\circ\mathcal{C}(\tau)$ by the following formal sum:

$${}^\circ\mathcal{C}(\tau)=\bigoplus_{w\in\mathcal{W}}{}^\circ\mathcal{C}_w(\tau). \tag{50}$$

The Eisenstein integral $E(P\colon\psi\colon\lambda)$ is defined for $P\in\mathcal{P}_\sigma(A_\mathrm{q})$, $\psi\in{}^\circ\mathcal{C}(\tau)$ and generic $\lambda\in\mathfrak{a}_{\mathrm{qc}}^*$ (see [4, eqn. (29)]); it is a smooth τ-spherical function on G/H, and it depends meromorphically on $\lambda\in\mathfrak{a}_{\mathrm{qc}}^*$.

The Eisenstein integrals are $\mathbf{D}(G/H)$-finite functions on G/H. More precisely we have

$$DE(P\colon\psi\colon\lambda)=E(P\colon\mu(D\colon\lambda)\psi\colon\lambda)\quad(D\in\mathbf{D}(G/H),\psi\in{}^\circ\mathcal{C}(\tau)). \tag{51}$$

Here $\mu(D\colon\lambda)\in\operatorname{End}({}^\circ\mathcal{C}(\tau))$ is the endomorphism defined in [4, above eqn. (43)]. In particular its restriction to ${}^\circ\mathcal{C}_1(\tau)\simeq V_\tau^{M\cap K\cap H}$ coincides with the endomorphism $\underline{\mu}(D\colon\lambda)$ defined earlier. It follows (for details, see [4, §4]) that $E(P\colon\psi\colon\lambda)$ allows a converging asymptotic expansion along any parabolic subgroup $Q\in\mathcal{P}_\sigma(A_\mathrm{q})$ of the form

$$E(P\colon\psi\colon\lambda)(maw)=\sum_{s\in W}\sum_{\nu\in\mathbf{N}\Sigma(Q)}a^{s\lambda-\varrho_Q-\nu}[p_{Q|P,\nu}(s\colon\lambda)\psi]_w(m) \tag{52}$$

for $w \in \mathcal{W}$, $m \in M$, $a \in A_{\mathrm{q}}^{+}(Q)$. Here $p_{Q|P,\nu}(s\colon\lambda) \in \mathrm{End}({}^{\circ}\mathcal{C}(\tau))$, and $[\,\cdot\,]_w$ indicates that the w-component of an element in (50) has been taken. The terms corresponding to $\nu=0$ in this expansion are denoted as follows:

$$C_{Q|P}(s\colon\lambda) := p_{Q|P,0}(s\colon\lambda) \in \mathrm{End}({}^{\circ}\mathcal{C}(\tau)).$$

These are the *c-functions* associated with the minimal principal series for G/H.

We shall now relate the above expansion (52) of the Eisenstein integral to the functions Φ_λ defined in the previous sections. Let $w \in \mathcal{W}$. Recall from Lemma 10.3 that we have

$$\text{(53)} \quad \Phi_{Q,w}(\lambda\colon a) = \tau(w) \circ \Phi_{w^{-1}Qw}(w^{-1}\lambda\colon w^{-1}aw) \circ \tau(w)^{-1} \in \mathrm{End}\big(V_\tau^{M\cap K\cap wHw^{-1}}\big)$$

for $a \in A_{\mathrm{q}}^{+}(Q)$, where $\Phi_{Q,w}$ has been defined in the lines preceding the lemma. Then we have the following.

Theorem 11.1. *Let $P, Q \in \mathcal{P}_\sigma(A_{\mathrm{q}})$, $w \in \mathcal{W}$. Then for every $\psi \in {}^{\circ}\mathcal{C}(\tau)$ we have*

$$\text{(54)} \qquad E(P\colon\psi\colon\lambda)(aw) = \sum_{s\in W} \Phi_{Q,w}(s\lambda\colon a)[C_{Q|P}(s\colon\lambda)\psi]_w(e) \quad (a \in A_{\mathrm{q}}^{+}(Q)),$$

as a meromorphic $V_\tau^{M\cap K\cap wHw^{-1}}$-valued identity in $\lambda \in \mathfrak{a}_{\mathrm{qc}}^{*}$.

Proof. By sphericality we have for the left-hand side of (54):

$$\text{(55)} \qquad E(P\colon\psi\colon\lambda)(aw) = \tau(w) E(P\colon\psi\colon\lambda)(w^{-1}aw).$$

It follows from [4, Lemma 7], that

$$[C_{Q|P}(s\colon\lambda)\psi]_w(e) = \tau(w)[C_{w^{-1}Qw|P}(w^{-1}s\colon\lambda)\psi]_1(e) \quad (\psi \in {}^{\circ}\mathcal{C}).$$

Using this identity as well as (53) we obtain the following expression for the right-hand side of (54):

$$\tau(w) \sum_{s\in W} \Phi_{w^{-1}Qw}(w^{-1}s\lambda\colon w^{-1}aw)[C_{w^{-1}Qw|P}(w^{-1}s\colon\lambda)\psi]_1(e).$$

Replacing $w^{-1}s$ by s in the sum we obtain that this equals

$$\tau(w) \sum_{s\in W} \Phi_{w^{-1}Qw}(s\lambda\colon w^{-1}aw)[C_{w^{-1}Qw|P}(s\colon\lambda)\psi]_1(e).$$

Combined with (55) this shows that it suffices to prove the theorem for $w=1$ and Q arbitrary.

By linearity we may assume that ψ belongs to one of the components of ${}^\circ\mathcal{C}(\tau)$ in (50), say ${}^\circ\mathcal{C}_v(\tau)\simeq V_\tau^{M\cap K\cap vHv^{-1}}$. Now $\tau(v)$ maps $V_\tau^{M\cap K\cap H}$ onto the latter space, and applying once more *loc. cit.*, Lemma 7, we have

$$[C_{Q|P}(s:\lambda)\psi]_1(e)=[C_{Q|v^{-1}Pv}(sv:v^{-1}\lambda)\tau(v)^{-1}\psi]_1(e)$$

for $\psi\in V_\tau^{M\cap K\cap vHv^{-1}}$. On the other hand, by *loc. cit.*, eqn. (69), we have

$$E(P:\psi:\lambda)=E(v^{-1}Pv:\tau(v)^{-1}\psi:v^{-1}\lambda).$$

Combined with the above this shows that it suffices to prove the theorem for $\psi\in {}^\circ\mathcal{C}_1(\tau)$ and P arbitrary.

We now have $\psi\in{}^\circ\mathcal{C}_1(\tau)\simeq V_\tau^{M\cap K\cap H}$. By linearity we may also assume that ψ is a joint eigenvector for all $\underline{\mu}(D:\lambda)$, $D\in\mathbf{D}(G/H)$, $\lambda\in\mathfrak{a}_{\mathrm{qc}}^*$ (cf. *loc. cit.*, Lemma 4). Fix such an eigenvector ψ and let $\gamma(\lambda)$ denote the corresponding eigenvalue of $\underline{\mu}(\mathrm{L}:\lambda)$. Then $\underline{\mu}(\mathrm{L}:\lambda)\psi=\gamma(\lambda)\psi$, and if $U\subset\mathfrak{a}_{\mathrm{qc}}^*$ is a non-empty open set on which the map $\lambda\mapsto E(P:\psi:\lambda)$ is holomorphic for all P, then $\mathrm{L}E(P:\psi:\lambda)=\gamma(\lambda)E(P:\psi:\lambda)$ for all $\lambda\in U$ (cf. (51)). Taking restrictions to $A_{\mathrm{q}}^+(Q)$ we have that $f=T_Q^{\downarrow}E(P:\psi:\lambda)$ satisfies the differential equation

$$\Pi_{Q,\tau}(\mathrm{L})f=\gamma(\lambda)f \tag{56}$$

on $A_{\mathrm{q}}^+(Q)$, for all $\lambda\in U$.

Notice that both sides of (54) are meromorphic in λ. Hence it suffices to establish this identity for λ in U, or any nonempty open subset thereof. Shrinking U if necessary we obtain from (52) that

$$E(P:\psi:\lambda)(a)=\sum_{s\in W}\sum_{\nu\in\mathbf{N}\Sigma(Q)}a^{s\lambda-\varrho_Q-\nu}[p_{Q|P,\nu}(s:\lambda)\psi]_1(e) \tag{57}$$

for all $P,Q\in\mathcal{P}_\sigma(A_{\mathrm{q}})$ and $\lambda\in U$, with absolute convergence for $a\in A_{\mathrm{q}}^+(Q)$. Moreover, we have

$$[p_{Q|P,0}(s:\lambda)\psi]_1(e)=[C_{Q|P}(s:\lambda)\psi]_1(e). \tag{58}$$

Consider the inner sum in (57):

$$\Psi_{Q|P}(s:\lambda)(a):=\sum_{\nu\in\mathbf{N}\Sigma(Q)}a^{s\lambda-\varrho_Q-\nu}[p_{Q|P,\nu}(s:\lambda)\psi]_1(e)\quad(a\in A_{\mathrm{q}}^+(Q)).$$

We claim that

$$\Psi_{Q|P}(s:\lambda)(a)=\Phi_Q(s\lambda:a)[p_{Q|P,0}(s:\lambda)\psi]_1(e)\quad(a\in A_{\mathrm{q}}^+(Q)) \tag{59}$$

for all λ in a non-empty open subset of U, and for all P, Q and s. Once this claim has been established the theorem is immediate from (57) and (58).

From [3, Lemma 13.9], it follows that we may assume (by shrinking U)

$$[s\lambda+\mathbf{Z}\Sigma]\cap[t\lambda+\mathbf{Z}\Sigma]=\emptyset \tag{60}$$

for all $\lambda\in U$ and $s,t\in W$, $s\neq t$. Since the action of $\Pi_{Q,\tau}(\mathrm{L})$ on the series (57) may be computed by insertion of (12) with term by term differentiations, we see from (60) that each of the functions $\Psi_{Q|P}(s{:}\,\lambda)$, $s\in W$, $\lambda\in U$, $Q\in\mathcal{P}_\sigma(A_\mathrm{q})$, satisfies the same differential equation (56) as $T_Q^{\downarrow}\,E(P{:}\,\psi{:}\,\lambda)$.

Shrinking U again we may also assume that $s\lambda$ does not belong to the singular set S for any $s\in W$. Now fix $\lambda\in U$ and let $c=\gamma(\lambda)$. Then by definition c is an eigenvalue for $\underline{\mu}(L{:}\,\lambda)$ on $V_\tau^{M\cap K\cap H}$. The claimed identity (59) now follows from Lemma 10.1 (with λ replaced by $s\lambda$) and Remark 10.2 (notice that $\underline{\mu}(L{:}\,s\lambda)$ has the same eigenvalues as $\underline{\mu}(L{:}\,\lambda)$ by (49)). This completes the proof. □

Remark 11.2. By [4, eqn. (73)], we have

$$C_{Q|P}(s{:}\,\lambda)\circ\mu(D{:}\,\tau{:}\,\lambda)=\mu(D{:}\,\tau{:}\,s\lambda)\circ C_{Q|P}(s{:}\,\lambda)\quad (D\in\mathbf{D}(G/H)).$$

Combining this with Corollary 9.3 we see that each of the summands in the expression (54) for $w=1$, $\Psi_{Q|P}(s{:}\,\psi{:}\,\lambda)(a){:=}\Phi_Q(s\lambda{:}\,a)[C_{Q|P}(s{:}\,\lambda)\psi]_1(e)\in V_\tau^{M\cap K\cap H}$, satisfies the same system of differential equations on $A_\mathrm{q}^+(Q)$ as does the sum (cf. (51)), that is,

$$\Pi_{Q,\tau}(D)\Psi_{Q|P}(s{:}\,\psi{:}\,\lambda)=\Psi_{Q|P}(s{:}\,\mu(D{:}\,\lambda)\psi{:}\,\lambda)\quad (D\in\mathbf{D}(G/H)).$$

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