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An asymptotic Cauchy problem for the Laplace equation

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Abstract. The Cauchy problem for the Laplace operator

$$\Delta u(x,y)=0,$$
$$u(x,0)=f(x), \quad \frac{\partial u}{\partial y}(x,0)=g(x)$$

is modified by replacing the Laplace equation by an asymptotic estimate of the form

$$\Delta u(x,y)=O[h(|y|)], \quad y\to 0,$$

with a given majorant h, satisfying $h(+0)=0$. This *asymptotic Cauchy problem* only requires that the Laplacian decay to zero at the initial submanifold. It turns out that this problem has a solution for smooth enough Cauchy data f, g, and this smoothness is strictly controlled by h. This gives a new approach to the study of smooth function spaces and harmonic functions with growth restrictions. As an application, a Levinson-type normality theorem for harmonic functions is proved.

0. Introduction

It is well-known that the Cauchy problem for the Laplace equation in $\mathbf{R}^{d+1}=\mathbf{R}^d\times\mathbf{R}$, $d\geq 2$, which is given by

$$\Delta u(x,y)=0, \tag{0.1}$$

$$u(x,0)=f(x), \quad \frac{\partial u}{\partial y}(x,0)=g(x), \tag{0.2}$$

is ill-posed. In particular, it only has a solution u for real-analytic Cauchy data f and g.

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Suppose that the Laplace equation is replaced by an asymptotic estimate of the form

$$\Delta u(x,y) = O[h(|y|)], \quad y \to 0 \tag{0.3}$$

where the majorant h is an increasing function with $h(+0)=0$. Instead of requiring that Δu vanish, the condition (0.3) demands its decay at a prescribed rate near the initial submanifold $\mathbf{R}^d$. It turns out that *the asymptotic Cauchy problem* given by (0.3), (0.2) has a nice solution u for smooth enough Cauchy data f, g. The degree of this smoothness is strictly determined by the given majorant h. The main subject of this article is this connection between the decay of the Laplacian and the smoothness of the data, which is a multidimensional analogue for the theory of the pseudoanalytic extension (see, e.g., [7]). This connection gives a new approach to the study of both harmonic functions and smooth function spaces.

The article is devoted specifically to the case of Carleman classes of infinitely differentiable functions (for the Cauchy data) and rapidly vanishing Laplacians.

Throughout the paper an *admissible solution* refers to a function u which is continuously differentiable in the whole of $\mathbf{R}^{d+1}$, uniformly bounded together with its gradient ∇u, and twice continuously differentiable outside $\mathbf{R}^d$.

In Section 1, some necessary preliminary information on the Newton kernel, regular majorants and the Carleman classes is given.

In Section 2, we prove the existence of an admissible solution of the asymptotic Cauchy problem (0.3), (0.2) for a regular majorant h when the Cauchy data f and g belong to the corresponding Carleman class.

Of course, the solution of the asymptotic Cauchy problem is not unique.

In Section 3, the converse theorem is proved, i.e., if the asymptotic Cauchy problem has an admissible solution, then the Cauchy data belong to the corresponding Carleman class.

In particular, in the C^∞-case the general constructions of Sections 2 and 3 give the complete result: *the Cauchy data f and g belong to $C^\infty(\mathbf{R}^d)$ if and only if there exists an admissible function u satisfying* (0.2) *such that for any N*

$$\Delta u(x,y) = O(|y|^N), \quad y \to 0.$$

Sections 4 and 5 are devoted to applications to harmonic functions of rapid growth.

Suppose now that our Carleman class is non-quasianalytic (see, e.g., [9]), which in terms of the majorant h means that

$$\int_0^1 \log\log \frac{1}{h(r)}\, dr < +\infty. \tag{0.4}$$

In Section 4 a special reproducing kernel $K(\zeta, z)$ is constructed by means of the construction of Section 2. This kernel is a truncated version of the standard Newton kernel. It is compactly supported in ζ, harmonic in z, and its Laplacian in ζ differs from the δ-function by an $O(h)$ term only.

There is a natural duality between compactly supported solutions u of (0.3) and harmonic functions F in $\mathbf{R}^{d+1}\setminus\mathbf{R}^d$ satisfying the growth condition

$$F(x,y)=O\left[\frac{1}{h(|y|)}\right], \quad y\to 0. \tag{0.5}$$

The corresponding coupling is

$$\langle F,u\rangle=\int_{\mathbf{R}^{d+1}\setminus\mathbf{R}^d} F(z)\Delta u(z)\,dx\,dy.$$

In Section 5, we apply this duality and the kernel K of Section 4 to the study of properties of F. The possibility of a harmonic extension of F across $\mathbf{R}^d$ (cf. [2]) is the first question under consideration. We prove the following generalization of the well-known Weyl lemma [9]: *A harmonic function F defined on $\mathbf{R}^{d+1}\setminus\mathbf{R}^d$, which satisfies* (0.5), *has a harmonic extension across $\mathbf{R}^d$ if and only if*

$$\int F(z)\Delta v(z)\,dx\,dy=0,$$

for any $v\in\mathcal{D}$ such that $\Delta v(x,y)=O(h(|y|))$. This criterion is local, i.e., it acts on functions harmonic in any domain G in $\mathbf{R}^{d+1}$ intersecting $\mathbf{R}^d$. Of course, the test function v must be supported in G in this case.

In the rest of Section 5, a harmonic version of the classical Levinson normality theorem is discussed. Let B be an open ball in $\mathbf{R}^{d+1}$, centered at a point of $\mathbf{R}^d$. Consider the family $\mathcal{F}$ of all harmonic functions F on B, satisfying the estimate

$$|F(x,y)|\le\frac{1}{h(|y|)}.$$

We consider under which conditions on h the family is normal in the usual sense [1], i.e., all the functions in $\mathcal{F}$ are *uniformly* bounded on any compact subset of B. The answer is well known for analytic functions in the plane: the family is normal if and only if the non-quasianalyticity condition (0.4) holds (with slight regularity assumptions on h). This is the famous Levinson normality theorem [10]. For harmonic functions in the plane the answer is the same, which follows easily from the analytic result. However, for harmonic functions of more than two variables the question is open. Several proofs of the Levinson theorem have been given (see, e.g., [2]–[6],

[8], [10], [11]). Most of them are of a complex analytic nature, except Domar's proof [3], [4] (see also [2]). Domar considers a family of *subharmonic* functions v in B, satisfying the estimate

$$v(x,y) \le m(|y|),$$

where m is a given majorant, and proves that it is normal if and only if

$$\int_0^1 \log m(r)\, dr < +\infty.$$

In the classical setting of analytic functions in the plane ($d=1$) one can construct the subharmonic logarithm $v=\log|F|$ of an analytic function F and apply Domar's criterion to the family of such v. In this case $m=\log 1/h$, giving the Levinson theorem. But, in the spatial case there is no analogue of the subharmonic logarithm of a harmonic function (or harmonic vector field), and so Domar's proof does not work in the multidimensional harmonic setting.

Our result in the end of Section 5 is as follows: *For a regular majorant h, the family $\mathcal{F}$ is normal if and only if the non-quasianalyticity condition* (0.4) *holds.* Thus, the answer in the multidimensional harmonic case coincides with the classical one. The proof is based on the consideration of the reproducing kernel K of Section 4.

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1. Preliminaries

1.0. *Notation.* Points of $\mathbf{R}^{d+1}=\mathbf{R}^d\times\mathbf{R}$ will be denoted $z=(x,y)$ or $\zeta=(\xi,\eta)$, where $x,\xi\in\mathbf{R}^d$ and $y,\eta\in\mathbf{R}$, with $dx\,dy$ or $d\xi\,d\eta$ being the corresponding volume element. Let $\mathbf{R}^{d+1}_+$ and $\mathbf{R}^{d+1}_-$ be the upper and lower open halfspaces of $\mathbf{R}^{d+1}$ given by $\{z:y>0\}$ and $\{z:y<0\}$ respectively.

We consider $\mathbf{R}^d$ as a subspace in $\mathbf{R}^{d+1}$ and so identify $x=(x_1,x_2,\dots,x_d)\in\mathbf{R}^d$ with $(x,0)\in\mathbf{R}^{d+1}$.

We need multi-indices $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_d)$ or $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_{d+1})$ in $\mathbf{R}^d$ and $\mathbf{R}^{d+1}$ with the usual notation

$$|\alpha|=\alpha_1+\dots+\alpha_{d+1},\qquad \alpha!=\alpha_1!\alpha_2!\dots\alpha_{d+1}!,$$
$$D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\dots\partial x_d^{\alpha_d}\partial y^{\alpha_{d+1}}}.$$

Let $D_x^\alpha u$, for d-index α, denote the corresponding derivative of a function $u(x,y)$ in x only, with $\Delta_x u$ the Laplacian of the function u on $\mathbf{R}^{d+1}$ in the x variables.

C and c, with or without index, are various constants—not necessarily the same throughout a formula.

$B=B(z,r)$ is the open ball in $\mathbf{R}^{d+1}$ of the center z and radius r. λB, $\lambda>0$, is the ball of the same center but with radius λr.

$\mathcal{D}$ is the Schwartz space of all compactly supported infinitely differentiable functions.

Let $\operatorname{supp}_\zeta u$ be the support of the function $u(\zeta,z)$ as a function of ζ for z fixed, this set depends on z.

$A\asymp B$ means that the ratio A/B lies between two positive constants.

χ is a standard cut-off function, that is, $\chi\in C^\infty(\mathbf{R})$, $\chi(t)=1$ for $|t|\le 1$ and $\chi(t)=0$ for $|t|\ge 2$.

1.1. Newton kernel

The Newton kernel

$$k(z)=\frac{\omega_d}{|z|^{d-1}},\quad \omega_d=-\frac{\Gamma((d-1)/2)}{4\pi^{d+1/2}}$$

is the fundamental solution for the Laplace operator in $\mathbf{R}^{d+1}$, i.e. $\Delta k=\delta$. For any compactly supported C^2-function u in $\mathbf{R}^{d+1}$

$$u(z)=\int k(z-\zeta)\Delta u(\zeta)\,d\xi\,d\eta. \tag{1.1}$$

Here we need a slightly more general version of this formula, where $u\in C^2$ outside the subspace $\mathbf{R}^d$ only.

Lemma 1. *If u is a compactly supported C^1-function in $\mathbf{R}^{d+1}$, $u\in C^2$ outside $\mathbf{R}^d$, and*

$$\Delta u(x,y)=O(|y|^{-1+\varepsilon}),\quad \varepsilon>0,\ y\to 0,$$

then (1.1) *holds.*

Proof. Under our assumptions both sides of (1.1) are C^1-functions on $\mathbf{R}^{d+1}$ (since we can differentiate under the integral sign). Their difference is a C^1-function which is harmonic outside of $\mathbf{R}^d\cap\operatorname{supp} u$. Such a singularity is removable (see [1]), so the difference is harmonic in the whole of $\mathbf{R}^{d+1}$, and obviously vanishes at infinity. Therefore, it is identically zero. $\square$

Now we need some estimates of the derivatives of k.

Lemma 2. *There exists a constant c_1 depending on d, such that for any multi-index α*

$$|D^\alpha k(z)| \le c_1^{|\alpha|}\alpha! \frac{1}{|z|^{d-1+|\alpha|}}. \tag{1.2}$$

Proof. After an appropriate rotation and scaling, we may suppose that $z=(1,0,\dots,0)\in\mathbf{R}^{d+1}$. The analytic function in $\mathbf{C}^{d+1}$ given by

$$f(\zeta_1,\dots,\zeta_{d+1}) = [(1+\zeta_1)^2+\zeta_2^2+\dots+\zeta_{d+1}^2]^{-(d-1)/2}, \quad f(0)=1,$$

is obviously bounded in the ball of radius $\frac{1}{4}$ centered at the origin. So, by the usual Cauchy inequalities, $|D^\alpha f(0)|\le \text{const}\cdot 4^{|\alpha|}\alpha!$. But, $f(\zeta)=k(z+\zeta)$ for real ζ. $\square$

For $z=(x,y)\in\mathbf{R}^{d+1}$, consider the Taylor expansion

$$k(\zeta-z) = \sum_{p=0}^{n-1}\left(\frac{\partial}{\partial y}\right)^p k(\zeta-z)\Big|_{z=(x,0)} \frac{y^p}{p!} + R_n(\zeta,z),$$

where $\zeta\in\mathbf{R}^{d+1}$, and $R_n(\zeta,z)$ is the remainder term. The following estimates are immediate corollaries of the last lemma:

$$|R_n(\zeta,z)| \le k(\zeta-z) + c_2^n \frac{|y|^n}{|\zeta-x|^{d-1+n}}, \quad |\zeta-z|<2|y|, \tag{1.4}$$

and

$$|R_n(\zeta,z)| \le c_2^n \frac{|y|^n}{|\zeta-x|^{d-1+n}}, \quad |\zeta-z|>2|y|. \tag{1.5}$$

Here, c_2 depends on d only.

1.2. Regular sequences and majorants

A positive sequence

$$M_n = n!m_n, \quad n=0,1,\dots, \tag{1.6}$$

is called regular if

$$m_n^2 \le m_{n-1}m_{n+1}, \tag{1.7}$$

$$m_n \le m_{n+1} \quad \text{and} \quad m_n^{1/n} \to +\infty, \tag{1.8}$$

$$\sup(m_{n+1}/m_n)^{1/n} < +\infty. \tag{1.9}$$

Thus the ratios $r_n = m_{n-1}/m_n$, $n=1,2,\dots$, decrease, and $r_n \to 0$.

For a given regular sequence define its associated majorant

$$h(r) = \inf_n m_n r^n, \quad 0 < r < 1,$$

and its central index

$$N(r) = \min\{n : h(r) = m_n r^n\}, \quad 0 < r < 1.$$

Because of the convexity condition (1.7), the sequence $\{m_n r^n\}$ is convex for any r, it decreases and then increases. Clearly,

$$h(r) = m_p r^p, \quad r_{p+1} \le r \le r_p.$$

The initial sequence may be recovered from h by the formula

$$m_n = \sup_{0<r<1} \frac{h(r)}{r^n}. \tag{1.10}$$

A strictly positive increasing function h on $[0,1]$, with $h(+0)=0$, and with logarithm $\varphi = \log 1/h$, is called a regular majorant if

$$\varphi(e^{-t}) \text{ is a convex function of } t \ge 0, \tag{1.11}$$

and

$$h(r)/r \le h(Q_0 r), \quad 0 < r \le 1, \text{ for some constant } Q_0. \tag{1.12}$$

It is easy to check that the associated majorant for a regular sequence is regular in this sense.

Let now h be an arbitrary regular majorant. Define a positive sequence $\{M_n\}$ by (1.10) and (1.6), and let $\tilde{h}$ be its associated majorant.

Lemma 3. *The sequence $\{M_n\}$ is regular and*

$$h(r) \le \tilde{h}(r) \le h(r)/r, \quad 0 < r < 1. \tag{1.13}$$

Proof. Step 1. The convexity condition (1.7) is evident, because

$$m_n^2 = \sup \frac{h(r)^2}{r^{2n}} = \sup \frac{h(r)^2}{r^{n-1} r^{n+1}} \le \sup \frac{h(r)}{r^{n-1}} \cdot \sup \frac{h(r)}{r^{n+1}}.$$

Because $0<r<1$ in (1.10), m_n increases. Because of the strict positivity of h for any $r_0>0$,

$$m_n=\sup\frac{h(r)}{r^n}\geq\frac{h(r_0)}{r_0^n},$$

whence (1.8) holds. As for (1.9), we have

$$m_{n+1}=\sup\frac{h(r)}{r^{n+1}}\leq\sup\frac{h(Q_0r)}{r^n}\leq Q_0^n m_n.$$

Thus the sequence $\{M_n\}$ is regular.

Step 2. The left hand inequality in (1.13) holds since

$$\tilde{h}(r)=\inf m_n r^n=\inf_n\sup_{0<s<1}h(s)(r/s)^n\geq h(r).$$

Next, for the logarithm $\tilde{\varphi}=\log 1/\tilde{h}$ we have

$$\tilde{\varphi}(e^{-s})=\sup_n\inf_{t>0}[n(s-t)+\varphi(e^{-t})].$$

But $\varphi(e^{-t})$ is convex, so there exists $A>0$ such that $\varphi(e^{-t})\geq\varphi(e^{-s})+A(t-s)$. Therefore

$$\tilde{\varphi}(e^{-s})\geq\varphi(e^{-s})+\sup_n\inf_{t>0}[(n-A)(s-t)].$$

However,

$$\sup_n\inf_{t>0}[(n-A)(s-t)]=-\{A\}s\geq -s,$$

where $\{A\}$ is the fractional part of A. Thus, $\tilde{\varphi}(e^{-s})\geq\varphi(e^{-s})-s$. □

1.3. Carleman classes

Let $\{M_n\}$ be a regular sequence. The Carleman class $C\{M_n\}$ on $\mathbf{R}^d$ is the set of all C^∞-functions f on $\mathbf{R}^d$, such that for some C and A

$$|D^\alpha f(x)|\leq CA^{|\alpha|}M_{|\alpha|},\quad x\in\mathbf{R}^d, \tag{1.14}$$

for any multi-index α. The $C\{M_n\}$ class is isotropic, because this definition involves the total order $|\alpha|$ of the multi-index only.

As usual (see [9]), the Carleman class is called quasianalytic if it does not contain a nonzero function of compact support. The quasianalyticity in our case, of isotropic regular classes, does not depend on the dimension d, but on $\{M_n\}$ only. The quasianalyticity condition is given by the classical Denjoy–Carleman theorem.

In terms of the associated majorant h, it is as follows (see [2], [5], [6]): the Carleman class is quasianalytic if and only if

$$\int_0^1 \log\log\frac{1}{h(r)}\,dr=+\infty.$$

Now, if h is an arbitrary regular majorant, then it defines the corresponding regular sequence $\{M_n\}$ given by (1.10) and (1.6), and therefore it defines the corresponding Carleman class $C\{M_n\}$. According to Section 1.2, this correspondence between regular majorants and Carleman classes is almost one-to-one.

2. Asymptotic Cauchy problem: Existence of solutions

Let h be a regular majorant, and let $C\{M_n\}$ be the corresponding Carleman class as in Section 1.3. Here we consider the following asymptotic Cauchy problem in $\mathbf{R}^{d+1}=\mathbf{R}^d\times\mathbf{R}$:

$$\Delta u(x,y)=O[h(Q|y|)], \quad y\to 0, \tag{2.1}$$

$$u(x,0)=f(x), \quad \frac{\partial u}{\partial y}(x,0)=g(x), \tag{2.2}$$

where $x\in\mathbf{R}^d$, $y\in\mathbf{R}$, and Q is a positive constant. The Cauchy data f and g are given bounded, continuous functions on $\mathbf{R}^d$. An admissible solution u must be a C^1-function on $\mathbf{R}^{d+1}$, which is C^2 outside the $\mathbf{R}^d$ subspace, and such that both u and its gradient ∇u are uniformly bounded.

Theorem 1. *If $f,g\in C\{M_n\}$, then there exists a solution u of the problem given by* (2.1) *and* (2.2), *for some Q. This solution is infinitely differentiable on $\mathbf{R}^{d+1}$, and all its derivatives are bounded.*

Remarks. 1. If f, g are compactly supported, then so is u.

2. The allowable constant Q depends on the corresponding constant A in the $C\{M_n\}$-condition (1.14) for f and g.

3. The solution u is in no way unique. In particular, an admissible solution of the problem need not be infinitely differentiable in $\mathbf{R}^{d+1}$.

Proof. Without loss of generality, we may suppose that for any α

$$|D^\alpha f|+|D^\alpha g|\le M_{|\alpha|},$$

and that the given h is exactly the associated majorant for the sequence $\{M_n\}$.

Step 1. We begin with the following partial solutions:

$$u_n(x,y)=\sum_{p=0}^{L}(-\Delta_x)^p f(x)\frac{y^{2p}}{(2p)!}+\sum_{p=0}^{L}(-\Delta_x)^p g(x)\frac{y^{2p+1}}{(2p+1)!}, \tag{2.3}$$

where $L=[n/2]-1$. Next, a straightforward calculation gives

$$\Delta u_n(x,y)=-(-\Delta_x)^{L+1}f(x)\frac{y^{2L}}{(2L)!}-(-\Delta_x)^{L+1}g(x)\frac{y^{2L+1}}{(2L+1)!},$$

and so

$$|\Delta u_n(x,y)|\le Q_1^n m_n |y|^{n-3}, \quad |y|<1, \tag{2.4}$$

where the constant Q_1 depends only on d.

In order to obtain a solution, we sew these partial solutions together with the help of an appropriate partition of unity. Let $\chi_k\in C^\infty(0,+\infty)$, such that

$$1=\sum_{-\infty}^{\infty}\chi_k, \quad \operatorname{supp}\chi_k\subset[2^{-k-1},2^{-k+1}],$$

and

$$|\chi_k^{(p)}|\le c_p 2^{kp}, \quad p=0,1,\dots,$$

where c_p depends on p only.

Fix a constant $Q_2>0$, and define the sequence of integers

$$n_k=N(2^{-k}Q_2),$$

where $N(r)$ is the corresponding central index. This definition applies for $2^{-k}Q_2<1$, i.e., for $k\ge k_0=\log_2 Q_2$. Now, set

$$u(x,y)=\sum_{k=k_0}^{\infty}\chi_k(|y|)u_{n_k}(x,y), \quad x\in\mathbf{R}^d,\ y\in\mathbf{R}. \tag{2.5}$$

After a suitable choice of Q_2, this is our desired solution.

Step 2. First of all, $u\in C^\infty(\mathbf{R}^{d+1}\setminus\mathbf{R}^d)$ because of the local finiteness of the sum (2.5). Clearly, $\operatorname{supp} u$ is bounded in the y direction, and is compact for compactly supported f and g.

Now, let us estimate the Laplacian of u. If $2^{-k}<|y|<2^{-k+1}$, $k>k_0$, then there are only two nonzero terms in (2.5), and so

$$u(x,y)=u_{n_{k-1}}(x,y)+\chi_k(|y|)[u_{n_k}(x,y)-u_{n_{k-1}}(x,y)]. \tag{2.6}$$

Thus,

$$\Delta u=\Delta u_{n_{k-1}}+\Delta\chi_k\cdot(u_{n_k}-u_{n_{k-1}})+2\nabla\chi_k\cdot\nabla(u_{n_k}-u_{n_{k-1}})+\chi_k\cdot\Delta(u_{n_k}-u_{n_{k-1}}). \tag{2.7}$$

For the first term here, we have by (2.4) that

$$\begin{aligned}|\Delta u_{n_{k-1}}| &\le Q_1^{n_{k-1}} m_{n_{k-1}}|y|^{n_{k-1}-3}\\ &\le \frac{1}{|y|^3} m_{n_{k-1}}(2^{-k+1}Q_2)^{n_{k-1}}(2^{k-1}Q_1Q_2^{-1}|y|)^{n_{k-1}}\\ &\le \frac{1}{|y|^3}h(2^{-k+1}Q_2)\le \frac{1}{|y|^3}h(2Q_2|y|),\end{aligned}$$

for $Q_2>Q_1$. The same estimate holds for the last term in (2.7). Estimates of the second and the third term are quite similar. For example, in the second term we have

$$|\Delta\chi_k|\le \mathrm{const}\cdot 4^k\le \frac{\mathrm{const}}{|y|^2}.$$

Furthermore, by the definition of n_k,

$$\begin{aligned}|u_{n_k}-u_{n_{k-1}}| &\le \sum_{j=L_{k-1}+1}^{L_k} |\Delta_x^j f|\frac{|y|^{2j}}{(2j)!}+|\Delta_x^j g|\frac{|y|^{2j+1}}{(2j+1)!}\\ &\le \sum_{l=n_{k-1}}^{n_k} m_l(Q_1|y|)^l\\ &\le \sum_{n_{k-1}}^{n_k} m_l(2^{-k}Q_2)^l(2^kQ_1Q_2^{-1}|y|)^l\\ &\le m_{n_{k-1}}(2^{-k}Q_2)^{n_{k-1}}\sum_{n_{k-1}}^{n_k}(2Q_1Q_2^{-1})^l\\ &\le 2m_{n_{k-1}}(2^{-k+1}Q_2)^{n_{k-1}}=2h(2^{-k+1}Q_2)\le 2h(2Q_2|y|),\end{aligned} \tag{2.8}$$

for $Q_2>4Q_1$. Collecting all these estimates, we obtain the desired inequality

$$|\Delta u(x,y)|\le \frac{c}{|y|^3}h(2Q_2|y|)\le ch(2Q_0^3Q_2|y|).$$

Step 3. To finish the proof, we have to check the global C^∞-smoothness of u. It is enough to prove the continuity up to $\mathbf{R}^d$ of any derivative $D^\alpha u$. However, if one replaces the Cauchy data f and g, by, say, $\partial f/\partial x_1$ and $\partial g/\partial x_1$, then the construction above gives $\partial u/\partial x_1$. Therefore, we may assume that D^α involves differentiation in the y direction only. So, we have to check the continuity of $\partial^l u/\partial y^l$, for $l=0,1,\dots$.

By (2.6), for $2^{-k}\le|y|<2^{-k+1}$,

$$\frac{\partial^l}{\partial y^l}u=\frac{\partial^l}{\partial y^l}u_{n_{k-1}}+\frac{\partial^l}{\partial y^l}[\chi_k\cdot(u_{n_k}-u_{n_{k-1}})]. \tag{2.9}$$

A straightforward calculation of the first term gives, for, say, l even and $l<n_{k-1}$,

$$\frac{\partial^l}{\partial y^l}u_{n_{k-1}}(x,y)=(-\Delta_x)^{l/2}f(x)+R,$$

where

$$\begin{aligned}|R|&\le\sum_{j=l/2+1}^{L_{k-1}}Q_1^{2j}M_{2j}\frac{|y|^{2j-l}}{(2j-l)!}+\sum_{j=l/2}^{L_{k-1}}Q_1^{2j}M_{2j}\frac{|y|^{2j-l+1}}{(2j-l+1)!}\\&\le Q_1^lM_l|y|+\frac{l!}{|y|^l}\sum_{p=l+2}^{n_{k-1}}m_p(2Q_1|y|)^p.\end{aligned}$$

But

$$m_p(2Q_1|y|)^p=m_p(2^{-k+1}Q_2)^p(2Q_1Q_2^{-1})^p,$$

and by the definition of n_{k-1}, the sequence $\{m_p(2^{-k+1}Q_2)^p\}$ decreases. Therefore, if $Q_2>4Q_1$, then, for $y\to0$,

$$|R|\le\text{const}\cdot|y|+\frac{l!}{|y|^l}\cdot2m_{l+2}(2^{-k+1}Q_2)^{l+2}=O(|y|).$$

So, $\partial^l/\partial y^l u_{n_{k-1}}$ tends to the limit $(-\Delta_x)^{l/2}f(x)$. As for the second term in (2.9), very similar estimates show that it also tends to zero as $y\to0$. Thus, we have proved the continuity of $\partial^l u/\partial y^l$ up to $\mathbf{R}^d$.

The theorem is proved. □

3. Asymptotic Cauchy problem: Converse theorem

3.1. Converse theorem

We have proved above, that the asymptotic Cauchy problem (2.1), (2.2) has an admissible solution if its Cauchy data belongs to $C\{M_n\}$. It turns out that the converse is also true. Recall that a solution u is admissible if $u\in C^1(\mathbf{R}^{d+1})\cap C^2(\mathbf{R}^{d+1}\setminus\mathbf{R}^d)$, and both u and ∇u are uniformly bounded.

Theorem 2. *If the asymptotic Cauchy problem* (2.1), (2.2) *has an admissible solution u, then its Cauchy data $f, g\in C\{M_n\}$.*

Remark. Of course, the constant A in (1.14) depends on the corresponding Q in (2.1).

Proof. Fix any point $x_0\in\mathbf{R}^d$. We estimate the derivatives of f and g at x_0. Let χ be a cut-off function:

$$\chi\in C^\infty(0,+\infty), \quad \chi(t)=1 \text{ for } 0\le t\le 1, \quad \chi(t)=0 \text{ for } t\ge 2.$$

Now, for any $z\in\mathbf{R}^{d+1}$, set

$$u_1(z)=u(z)\chi(|z-x_0|).$$

Then, the smoothness of u_1 is the same as that of u, u_1 coincides with u in the ball $B_1=B(x_0,1)$, and is supported in the ball $B_2=B(x_0,2)$. Its Laplacian Δu_1 is uniformly bounded. According to Lemma 1, in B_1 we have the representation

$$\begin{aligned} u(z) &= \int_{B_1} \Delta u(\zeta)k(z-\zeta)\,d\xi\,d\eta+\int_{B_2\setminus B_1} \Delta u_1(\zeta)k(z-\zeta)\,d\xi\,d\eta \\ &= v(z)+w(z), \end{aligned} \tag{3.1}$$

where k is the standard Newton kernel. The function w is harmonic in B_1, and bounded by a constant independent of x_0. So, for any α,

$$|D^\alpha w(x_0,0)|\le \text{const}\cdot c_1^{|\alpha|}\alpha!,$$

where the constant c_1 depends on d only. As for v, according to (3.1), we can differentiate $v(x,0)$ in x under the integral sign without any restriction because of the estimate (2.1). So, $v(x,0)$ is C^∞, and Lemma 2 gives, for any multi-index α in $\mathbf{R}^d$,

$$\begin{aligned} |D_x^\alpha v(x_0)| &\le C\int_{B_1} h(Q|\eta|)\frac{c_1^{|\alpha|}\alpha!}{|\zeta-x_0|^{d-1+|\alpha|}} \\ &\le C|\alpha|!c_1^{|\alpha|}\max_{0<r<1}\frac{h(Qr)}{r^{|\alpha|}}\le C(c_1Q)^{|\alpha|}|\alpha|!m_{|\alpha|}\le C(c_1Q)^{|\alpha|}M_{|\alpha|}. \end{aligned}$$

Therefore, $f(x)=v(x,0)+w(x,0)\in C\{M_n\}$. The corresponding estimate for g is the same, because we can differentiate (3.1) with respect to y once. $\square$

3.2. Remarks on the C^∞-case

Theorems 1 and 2 treat the case of Carleman classes. Now, let f and g be compactly supported functions in $C^\infty(\mathbf{R}^d)$ only, i.e., $f, g \in \mathcal{D}(\mathbf{R}^d)$. Set

$$M_n = \max_{|\alpha| \le n} \max_x (|D^\alpha f(x)| + |D^\alpha g(x)|),$$

and

$$m_n = M_n/n!, \quad n = 0, 1, \dots .$$

Then, define $\{\widetilde{m}_n\}$ to be the least logarithmically convex majorant of the sequence $\{m_n\}$. It satisfies conditions (1.7) and (1.8), which is enough to apply the main construction of Section 2. The construction gives a C^∞-function u with compact support that satisfies the Cauchy initial conditions (2.2), and for any N

$$\Delta u(x,y) = O(|y|^N), \quad y \to 0.$$

Using Theorem 2, and the same argument, it is possible to prove the converse statement. Thus, we obtain the following C^∞-result.

Theorem 3. *The asymptotic Cauchy problem in* $\mathbf{R}^{d+1}$:

$$\Delta u(x,y) = O(|y|^N), \quad y \to 0, \text{ for any } N > 0, \tag{3.2}$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial y}(x,0) = g(x), \tag{3.3}$$

has an admissible solution u of compact support if and only if $f, g \in \mathcal{D}(\mathbf{R}^d)$.

4. A reproducing kernel

Let h be a regular majorant satisfying the condition that

$$\int_0^1 \log\log \frac{1}{h(r)}\, dr < +\infty. \tag{4.1}$$

The corresponding Carleman class $C\{M_n\}$ is non-quasianalytic (see [2], [5], [6]).

Theorem 4. *For any ball B in $\mathbf{R}^{d+1}$, centered at a point of $\mathbf{R}^d$, there exists a function $K_B(\zeta, z)$ of two variables $\zeta \in \mathbf{R}^{d+1}$ and $z \in B$, which is infinitely differentiable unless $\zeta = z$, and such that*

$$\operatorname{supp}_\zeta K_B \subset 16B, \tag{4.2}$$

$$K_B(\zeta, z) = k(\zeta - z), \quad \zeta \in 2B, \tag{4.3}$$

$$\Delta_z K_B(\zeta, z) = 0, \quad \zeta \notin 2B, \tag{4.4}$$

and

$$|\Delta_\zeta K_B(\zeta,z)| \le Ch(|\eta|), \quad \zeta \notin 2B. \tag{4.5}$$

Remark. Thus, K_B is a truncated version of the Newton kernel. It is compactly supported, harmonic in z, and almost harmonic in ζ.

Proof. It is enough to prove the theorem for the unit ball $B(0,1)$ only. Indeed, if K is such a kernel for $B(0,1)$, then

$$K_B(\zeta,z) = \frac{1}{r^{d+1}} K\left(\frac{\zeta - x_0}{r}, \frac{z-x_0}{r}\right)$$

satisfies (4.2)–(4.4) for $B=B(x_0,r)$, and

$$|\Delta_\zeta K_B(\zeta,z)| \le \frac{C}{r^{d+1}} h\left(\frac{|\eta|}{r}\right), \quad \zeta \notin 2B.$$

Thus, let B be the unit ball. Because of the non-quasianalyticity of $C\{M_n\}$, there is a function φ on $\mathbf{R}^d$, such that: $\varphi=1$ on $4B\cap\mathbf{R}^d$, $\varphi=0$ outside $8B\cap\mathbf{R}^d$, and

$$|D^\alpha \varphi| \le C_1 q_1^{|\alpha|} M_{|\alpha|},$$

where q_1, $0<q_1<1$, will be chosen later. Fix $z\in B$, and define

$$f(\xi) = \varphi(\xi) k(\xi - z), \quad g(\xi) = \varphi(\xi) \frac{\partial k}{\partial \eta}(\xi - z), \quad \xi \in \mathbf{R}^d. \tag{4.6}$$

We are going to consider f and g as initial data for an asymptotic Cauchy problem. They are C^∞, unless $\zeta=z$, and by (1.2) we get

$$|D^\alpha f(\xi)| + |D^\alpha g(\xi)| \le C_2 q_2^{|\alpha|} M_{|\alpha|}, \quad \xi \notin 2B\cap\mathbf{R}^d, \tag{4.7}$$

where q_2 depends on q_1, and is arbitrarily small for small enough q_1. Although the estimate (4.7) holds for $\xi\notin 2B$ only, we can apply the main construction of Section 2 to f and g because of its pointwise nature with respect to ξ. We obtain a function $v(\zeta)$, defined in $\mathbf{R}^{d+1}$ for $\zeta=(\xi,\eta)$ such that $\xi\notin 2B\cap\mathbf{R}^d$, vanishing outside of $8B$, and such that

$$v(\xi,0) = f(\xi), \quad \frac{\partial v}{\partial \eta}(\xi,0) = g(\xi),$$
$$|\Delta v(\xi,\eta)| \le Ch(|\eta|).$$

Let χ be the standard cut-off function:

$$\chi\in C^\infty(\mathbf{R}),\quad \chi(t)=1,\ |t|\le 1,\quad \chi(t)=0,\ |t|\ge 2.$$

Now define

$$K(\zeta,z)=\chi(|\eta|/2)\{\chi(|\xi|/2)k(\zeta-z)+[1-\chi(|\xi|/2)]v(\zeta)\}. \tag{4.8}$$

This is our desired kernel. Let us check this.

1. If $\zeta\notin 16B$, then either $|\eta|>8$ or $|\xi|>8$, and so $K(\zeta,z)=0$, which is (4.2).

2. According to (2.3) and (2.5), $v(\xi,\eta)$ is a linear combination of derivatives of f and g in ξ, with coefficients depending on η. In view of (4.6), these are, in turn, linear combinations of the derivatives of $k(\xi-z)$ and $\partial/\partial\eta k(\xi-z)$ in ξ, with C^∞-smooth coefficients depending on z. The multiplier before v in (4.8) vanishes for $\xi\in 2B\cap\mathbf{R}^d$. Therefore, K is a C^∞-function of $\zeta\in\mathbf{R}^{d+1}$ and $z\in B$, unless $\zeta=z$, and it is harmonic in z otherwise.

3. It remains to check (4.5). Since the estimate (4.5) is a restriction for small η only, we may suppose $|\eta|<2$. Next, if $|\xi|<2$, then $K(\zeta,z)=k(\zeta-z)$ is harmonic in ζ, and if $|\xi|>4$, then $K(\zeta,z)=v(\zeta)$, and (4.5) holds. Therefore, we suppose that $2<|\xi|<4$. Now

$$K(\zeta,z)=k(\zeta-z)+[1-\chi(|\xi|/2)](v(\zeta)-k(\zeta-z)).$$

This means that

$$\begin{aligned}\Delta_\zeta K(\zeta,z)=[1-\chi(|\xi|/2)]\Delta v(\zeta)&+2\nabla[1-\chi(|\xi|/2)]\cdot\nabla(v(\zeta)-k(\zeta-z))\\ &+\Delta[1-\chi(|\xi|/2)]\cdot(v(\zeta)-k(\zeta-z)).\end{aligned} \tag{4.9}$$

The first term above does not exceed $Ch(|\eta|)$. The multipliers above containing χ are bounded. The only term we need to study is the difference $v-k$. But $2<|\xi|<4$, and so

$$f(\xi)=k(\xi-z),\quad g(\xi)=\frac{\partial k}{\partial\eta}(\xi-z).$$

Therefore, every partial solution u_n, given by (2.3), is really the Taylor polynomial of the harmonic function $k(\zeta-z)$, centered at ξ, of degree $2L+1=2[n/2]-1$, which is $n-1$ or $n-2$. In view of (1.5),

$$|k(\zeta-z)-u_n(\zeta)|\le c_2^{n-1}\frac{|\eta|^{n-1}}{|\xi-z|^{d-2+n}}\le(c_2|\eta|)^{n-1}.$$

For $2^{-k}<|\eta|<2^{-k+1}$, by (2.6) we have

$$|v(\zeta)-k(\zeta-z)|\le C(c_2|\eta|)^{n_{k-1}-1}\le C'm_{n_{k-1}}|\eta|^{n_{k-1}}\le C'h(|\eta|).$$

The gradient $\nabla(v-k)$ is estimated in the same way. We see that the remaining terms of (4.9) are bounded by $Ch(|\eta|)$ also. $\square$

5. Application to harmonic functions

5.1. An integral representation

Let G be a bounded domain in $\mathbf{R}^{d+1}$, which intersects $\mathbf{R}^d$, and let $G_\pm = G \cap \mathbf{R}^d_\pm$. Here, we consider harmonic functions F in $G_+ \cup G_-$, i.e., couples $F=(F_+, F_-)$ of harmonic functions in G_+ and G_-, respectively.

Let h be a regular majorant satisfying the non-quasianalyticity condition (4.1). Define the space $\mathcal{F}(h)$ consisting of all harmonic functions F in $G_+ \cup G_-$, such that

$$|F(x,y)| \le \frac{\text{const}}{h(|y|)}, \quad (x,y) \in G_+ \cup G_-. \tag{5.1}$$

Let B be a ball in $\mathbf{R}^{d+1}$, centered at a point of $\mathbf{R}^d$, such that $16B \in G$.

Theorem 5. *For any $F \in \mathcal{F}(h)$, and $z \in B \setminus \mathbf{R}^d$,*

$$F(z) = \int_{16B \setminus 2B} F(\zeta) \Delta_\zeta K_B(\zeta, z)\, d\xi\, d\eta - \int_{16B} F(\zeta) \Delta v(\zeta)\, d\xi\, d\eta, \tag{5.2}$$

where K_B is the reproducing kernel from Theorem 4, while $v \in \mathcal{D}(G)$ and

$$\Delta v(\xi, \eta) = O(h(|\eta|)). \tag{5.3}$$

Proof. Let $z=(x,y) \in B \setminus \mathbf{R}^d$. We may suppose $y>0$. Let $\chi \in C^\infty(\mathbf{R})$ be our standard cut-off function. Define

$$v(\zeta) = \chi(4|\eta|/y) K_B(\zeta, z), \quad \zeta \in \mathbf{R}^{d+1}.$$

Obviously, $v \in \mathcal{D}(G)$ and v coincides with K_B for $|\eta| < \frac{1}{4} y$, whence (5.3) holds. The difference $K_B - v$ is compactly supported inside $G_+ \cup G_-$, coincides with $k(\zeta - z)$ near to z, and is C^∞ otherwise. So, by Green's formula, (5.2) holds. □

5.2. Harmonic continuation across the linear boundary

Let $F \in \mathcal{F}(h)$. Under which condition does F admit a harmonic extension onto the whole of G? In other words, when are F_+ and F_- harmonic continuations of one another?

Recall that h satisfies the non-quasianalyticity condition (4.1).

Theorem 6. *$F \in \mathcal{F}(h)$ admits a harmonic extension onto the whole of G if and only if*

$$\int_{G_+ \cup G_-} F(\zeta)\Delta v(\zeta)\, d\xi\, d\eta = 0, \tag{5.4}$$

for any $v \in \mathcal{D}(G)$, such that $\Delta v(\xi,\eta) = O(h(|\eta|))$.

Remarks. 1. If (4.1) fails, then there is no nonzero function $v \in \mathcal{D}(G)$ satisfying (5.3).

2. By the classical Weyl lemma, if F is locally integrable and (5.4) holds for any $v \in \mathcal{D}(G)$, then F is harmonic in G. Functions from $\mathcal{F}(h)$ are not locally integrable, but this version of the Weyl lemma holds when the integral (4.1) converges.

Proof. Let B be, as before, a ball in $\mathbf{R}^{d+1}$, centered at a point of $\mathbf{R}^d$, such that $16B \in G$. It suffices to prove that F has an extension onto the whole of B. Apply to this case the integral representation (5.2). The second term vanishes by assumption, therefore

$$F(z) = \int_{16B \setminus 2B} F(\zeta)\Delta_\zeta K_B(\zeta, z)\, d\xi\, d\eta, \quad z \in B \setminus \mathbf{R}^d. \tag{5.5}$$

But, the right hand side is clearly harmonic in the whole of B, because K_B is harmonic in z. □

5.3. A Levinson type theorem for harmonic functions

Consider the family $\mathcal{F}_0(h)$ of all harmonic functions on the whole of G, such that

$$|F(x,y)| \le \frac{1}{h(|y|)}, \quad (x,y) \in G \setminus \mathbf{R}^d. \tag{5.6}$$

Under which condition is this family normal, i.e., uniformly bounded on any compact subset of G?

Theorem 7. *Let h be a regular majorant. The family $\mathcal{F}_0(h)$ is normal if and only if the non-quasianalyticity condition* (4.1) *holds.*

Proof. 1. Let (4.1) hold. As before, let B be a ball in $\mathbf{R}^{d+1}$, centered at a point of $\mathbf{R}^d$, such that $B \in G$. It suffices to check the uniform boundedness of $\mathcal{F}_0(h)$ on B. Apply the representation (5.2). Now, the second term vanishes because F

is harmonic in the whole of G. The equality (5.5) holds for any $z\in B\setminus\mathbf{R}^d$, and, by continuity, for any $z\in B$ also. Therefore,

$$|F(z)|\leq C\int_{16B\setminus 2B}\frac{1}{h(|\eta|)}h(|\eta|)\,d\xi\,d\eta\leq \text{const}.$$

The family is normal.

2. Suppose (4.1) fails. Let G' be the projection of G onto the plane of the first coordinate x_1 and y. Now, $G'\subset\mathbf{R}^2$, and, by the classical Levinson theorem (see [6], [10]), one can construct a sequence of harmonic functions $\{F_n(x_1,y)\}$ in G', such that

$$|F_n(x_1,y)|\leq\frac{1}{h(|y|)},$$

and $F_n(x_1^0,0)\to\infty$, for a fixed point $(x_1^0,0)\in G'$. The functions $F_n(x_1,y)$ can be viewed as harmonic functions in G, not depending on the other coordinates. Thus, $\{F_n\}$ is a subfamily of $\mathcal{F}_0(h)$, and so $\mathcal{F}_0(h)$ itself is not normal. □

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