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On boundary layers

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Abstract. The concept of boundary layers, introduced by A. Volberg in [7], is generalized from subsets of the unit disk to subsets of general non-tangentially accessible (NTA) domains. Capacitary conditions of Wiener type series of both necessary and sufficient type for boundary layers are presented and the connection between boundary layers and minimally thin sets is studied.

1. Introduction

In [7] A. Volberg studied domains in the plane with harmonic measures comparable to the Lebesgue measure for boundary arcs and defined the concept *boundary layer*. More precisely, let U be the unit disk $\{|z|<1\}$. Suppose E is a closed subset of U and $\Omega=U\setminus E$ is a domain containing the origin 0. Volberg [7] said that Ω is a boundary layer if there is a positive constant c such that

(1) $$\omega(0,I)\geq c|I| \quad \text{for all arcs } I\subset\partial U,$$

where $\omega(0,I)$ is the harmonic measure of I in the domain Ω evaluated at 0 and $|I|$ is the length of I. Loosely speaking, a subset Ω of U is a boundary layer if it is sufficiently "big" and sufficiently "connected", seen from the boundary of U, so that a Brownian particle starting in a given point in the subset should be able to hit any arc of ∂U with probability comparable to the length of the arc. For the historical background and the original motivation for studying boundary layers, see [7].

In [7, Propositions 1.1 and 1.2] Volberg presents Wiener type capacitary conditions for boundary layers. Volberg's work was then continued by M. Essén in [4, Chapter 5]. The following formulation is taken from Essén [4]. Let $\{Q_k\}$ be a Whitney decomposition of U and let $t_k=\operatorname{dist}(Q_k,\partial U)$ and $\varrho_k(\xi)=\operatorname{dist}(Q_k,\xi)$. We put

$$W(\xi)=W(\xi,E)=\sum_k \frac{t_k^2}{\varrho_k(\xi)^2}\left(\log\frac{4t_k}{\operatorname{cap}(E\cap Q_k)}\right)^{-1},$$

where cap denotes the logarithmic capacity.

Theorem A. *Let* $\frac{1}{2}U\subset\Omega$. *Then there exist positive constants* M_1, M_2 *and* $q_0<1$ *with the following properties:*

(i) *If* $\sup_{\xi\in\partial U} W(\xi)\le(1-c)/M_1$ *then* (1) *holds, i.e.* Ω *is a boundary layer.*

(ii) *If* (1) *holds with* $c\ge 1-q_0$, *then* $\sup_{\xi\in\partial U} W(\xi)\le M_2(1-c)$.

M. Essén gave in [4, Chapter 5] a relationship between boundary layers and minimally thin sets. Namely, [4, Theorem 3] says: *A necessary but not a sufficient condition for* $\Omega=U\setminus E$ *to be a boundary layer is that* E *is a minimally thin set everywhere on* ∂U. At the International Conference of Potential Theory 1994, [5], Essén raised the following question: *"Can we characterize boundary layers in terms of concepts from potential theory?"* Our motivation of this paper is to give an answer to this question. In fact, Theorem A will be generalized and improved in our Theorem 4.2. We shall characterize boundary layers in terms of capacity.

The paper is organized in the following way: In Section 2 we generalize the notion of boundary layers to general non-tangentially accessible (NTA) domains instead of the unit disk. Since the Martin boundary of an NTA domain is homeomorphic to the Euclidean boundary and every boundary point is minimal ([6]), it is natural to deal with these domains. Section 3 contains the main characterization of boundary layers based on series of reduced functions. We shall use some subtle estimates of the Martin kernels, which can be proved by the boundary Harnack principle. In Section 4 we shall restrict ourselves to smoother domains, namely Liapunov or $C^{1,\alpha}$ domains. For such domains the Martin kernels behave like those for the unit disc. Hence we can give a direct extension of Theorem A. Boundary layers are characterized by Wiener type series based on capacities (analogous series were studied in [2], [4] and [7]). In particular, Theorem 4.2 shows that the constant q_0 in Theorem A may be arbitrarily close to 1. Of course, the constant M_2 tends to ∞ as $q_0\to 1$. We can estimate its growth. In Section 5, we shall discuss a stronger type of boundary layers, which are called good boundary layers. We shall observe that good boundary layers are characterized by the uniform convergence of a certain series involving capacities. In Section 6, we shall discuss a weaker type of boundary layers which turns out to have a precise connection to minimal thinness. See Proposition 6.3. In the last section, relationships among various types of boundary layers will be given.

2. Equivalent definitions of boundary layers

In [6] Jerison and Kenig introduced the notion of non-tangentially accessible domains, NTA domains. Hereafter, we let D be a bounded domain in the Euclidean space $\mathbf{R}^d$ with $d\geq 2$. By $\delta(x)$ we denote the distance $\operatorname{dist}(x, \partial D)$. We say that D is an NTA domain if there exist positive constants M and r_0 such that:

(a) For any $\xi\in\partial D$ and $r<r_0$ there exists a point $A_r(\xi)\in D$ such that $M^{-1}r<|A_r(\xi)-\xi|<r$ and $\delta(A_r(\xi))>M^{-1}r$. (Corkscrew condition.)

(b) The complement of D satisfies the corkscrew condition.

(c) If $\varepsilon>0$ and x_1 and x_2 belong to D, $\delta(x_j)>\varepsilon$ and $|x_1-x_2|<C\varepsilon$, then there exists a Harnack chain from x_1 to x_2 whose length depends on C, but not on ε. (Harnack chain condition.)

In this and the next sections we let D be an NTA domain. As mentioned above, it is known that the Martin boundary of D is homeomorphic to the Euclidean boundary ∂D and every boundary point is minimal ([6]). To be precise, we fix a point $x_0\in D$. Let $G(x,y)$ be the Green function for D and put $g(x)=G(x,x_0)$. Let $K(x,y)=G(x,y)/g(y)$. Then $K(x,y)$ has a continuous extension to $D\times\overline{D}$. We denote the continuous extension by the same symbol. Sometimes we write K_ξ for $K(\cdot,\xi)$. The kernel K is referred to as the Martin kernel for D. For each $\xi\in\partial D$ the Martin kernel K_ξ is a minimal harmonic function with $K_\xi(x_0)=1$.

Throughout this paper we let E be a relatively closed subset in D and assume that $\Omega=D\setminus E$ is a domain. We fix $x_0\in\Omega$. In general, we denote by $\omega(x,I,V)$ the harmonic measure for an open set V of $I\subset\partial V$ evaluated at $x\in V$. For simplicity we let, for $I\subset\partial D$,

$$\omega(x,I)=\omega(x,I,\Omega),$$
$$\widetilde{\omega}(x,I)=\omega(x,I,D).$$

Definition 2.1. Let $c\in(0,1)$. We say that Ω is a c-boundary layer (at x_0) if

$$\omega(x_0,I)\geq c\widetilde{\omega}(x_0,I)\quad\text{for every Borel set } I\subset\partial D.$$

We sometimes drop the prefix "c-" if Ω is a c-boundary layer for some $c>0$.

Remark 2.2. Let D be the unit disc U and $x_0=0$. Then $\widetilde{\omega}(0,I)=(2\pi)^{-1}|I|$. Hence our definition generalizes Volberg's boundary layer.

Let $E\subset D$ and let u be a nonnegative superharmonic function on D. We put

$$R_u^E(x)=\inf v(x),$$

where the infimum is taken over all nonnegative superharmonic functions v such that $v\geq u$ on E. It is known that the lower regularization

$$\widehat{R}_u^E(x)=\liminf_{y\to x} R_u^E(y)$$

is superharmonic in D and $R_u^E=\widehat{R}_u^E$ q.e. on D, i.e. the equality holds outside a polar set. Moreover, $\widehat{R}_u^E=u$ q.e. on E. The function $\widehat{R}_u^E$ is called the (regularized) reduced function of u with respect to E.

Proposition 2.3. *The following statements are equivalent:*
(i) Ω *is a c-boundary layer.*
(ii) $\widehat{R}^E_{K_\xi}(x_0)\le 1-c$ *for every* $\xi\in\partial D$.
(iii) $(1/h(x_0))\widehat{R}^E_h(x_0)\le 1-c$ *for every positive harmonic function* h *in* D.

Proof. For a moment, we fix a Borel set I on the boundary ∂D and write $\omega=\omega(\,\cdot\,,I)$ and $\widetilde{\omega}=\widetilde{\omega}(\,\cdot\,,I)$. Since

$$\widetilde{\omega}-\omega=\begin{cases} 0 & \text{q.e. on } \partial D,\\ \widetilde{\omega} & \text{q.e. on } E,\end{cases}$$

it follows that

$$\widetilde{\omega}-\omega=\widehat{R}^E_{\widetilde{\omega}} \quad \text{on } \Omega.$$

Hence Ω is a c-boundary layer if and only if

$$c\widetilde{\omega}(x_0)\le\widetilde{\omega}(x_0)-\widehat{R}^E_{\widetilde{\omega}}(x_0),$$

or equivalently

$$\widehat{R}^E_{\widetilde{\omega}}(x_0)\le(1-c)\widetilde{\omega}(x_0) \quad \text{for every Borel set } I\subset\partial D. \tag{2}$$

In general, a positive harmonic function h is called a kernel function with respect to x_0 at $\xi\in\partial D$ if h vanishes continuously on $\partial D\setminus\{\xi\}$ and $h(x_0)=1$. It is known that a kernel function at ξ is unique and coincides with K_ξ (cf. [6, Theorem 5.5]). Hence if $r_n\to 0$ and $\widetilde{\omega}_n=\widetilde{\omega}(\,\cdot\,,B(\xi,r_n)\cap\partial D)$, then the limit of the ratio $\widetilde{\omega}_n/\widetilde{\omega}_n(x_0)$ exists and is equal to K_ξ. Hence (2) yields

$$\widehat{R}^E_{K_\xi}(x_0)\le 1-c \quad \text{for every } \xi\in\partial D. \tag{3}$$

Thus (i) $\Rightarrow$ (ii). The Martin representation theorem (e.g. [3, 1.XII.9]) yields the equivalence (ii) $\Leftrightarrow$ (iii). Letting $h=\widetilde{\omega}(\,\cdot\,,I)$ in (iii), we observe that (2) follows. Thus (iii) $\Rightarrow$ (i). Proposition 2.3 follows. □

3. Series of reduced functions and boundary layers

In this and the next sections we give more concrete characterizations of boundary layers. We shall need many positive constants. So, for simplicity, by the symbol M we denote a positive constant whose value is unimportant and may change from line to line. If necessary, we use $M_1, M_2, \dots$, to specify them. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1\approx f_2$, if and only if there exists a constant $M\geq 1$ such that $M^{-1}f_1\leq f_2\leq Mf_1$. The constant M will be called the constant of comparison.

Since our Martin kernel $K(x,y)$ has a reference point x_0, it is necessary to assume that the set E is apart from x_0. In this and the next sections we assume that

$$E\subset D_0=D\setminus B(x_0,r_1)\quad \text{with } r_1>0. \tag{4}$$

This assumption corresponds to $\frac{1}{2}U\subset\Omega$ in Theorem A. For a boundary point ξ, let us define a Wiener type series of reduced functions.

Definition 3.1. Let $I_j(\xi)=\{x:2^{-j}\leq|x-\xi|<2^{1-j}\}$ and $E_j(\xi)=E\cap I_j(\xi)$. We define

$$\Phi(\xi):=\sum_{j=1}^{\infty}\widehat{R}_{K_\xi}^{E_j(\xi)}(x_0).$$

We have the following theorem.

Theorem 3.2. *There exists a positive constant M_3 depending only on D, x_0 and r_1 with the following property:*

(i) *If $\sup_{\xi\in\partial D}\Phi(\xi)\leq q<1$, then $\Omega=D\setminus E$ is a $(1-q)$-boundary layer.*

(ii) *If $\Omega=D\setminus E$ is a $(1-q)$-boundary layer, then*

$$\sup_{\xi\in\partial D}\Phi(\xi)\leq M_3\frac{q}{1-q}\log\frac{2}{1-q}.$$

Theorem 3.2(ii) has an immediate corollary.

Corollary 3.3. *Let $0<q_0<1$. Then there is a positive constant M_{q_0} depending only on D, r_1 and q_0 such that if Ω is a $(1-q)$-boundary layer with $0<q\leq q_0$, then*

$$\sup_{\xi\in\partial D}\Phi(\xi)\leq M_{q_0}q.$$

Moreover, $M_{q_0}\approx(1-q_0)^{-1}\log[2/(1-q_0)]$.

Proof of Theorem 3.2(i). We note that the constant M_3 is not involved in this part. This is straightforward from the countable subadditivity of reduced functions. We have

$$\widehat{R}_{K_\xi}^{E}(x_0)\leq\sum\widehat{R}_{K_\xi}^{E_j(\xi)}(x_0).$$

Hence by Proposition 2.3, we see that if $\sup_{\xi\in\partial D}\Phi(\xi)\leq q<1$, then Ω is a $(1-q)$-boundary layer. $\square$

The second part of Theorem 3.2 is not so obvious. We need several lemmas about the estimates of the Martin kernels.

Lemma 3.4. *There are positive constants* α *and* M_4 *such that if* $\xi\in\partial D$, $x,y\in D_0$ *and* $2|y-\xi|\le|x-\xi|$, *then*

$$\left|\frac{K(x,y)}{K(x,\xi)}-1\right|\le M_4\left(\frac{|y-\xi|}{|x-\xi|}\right)^{\alpha}.$$

We have in particular,

$$K(x,y)\le\left(1+M_4\left(\frac{|y-\xi|}{|x-\xi|}\right)^{\alpha}\right)K(x,\xi).$$

Proof. If $y\in\partial D$, then this is the Hölder continuity of $K(x,y)/K(x,\xi)$ of order α given in [6, Theorem 7.1]. The same proof works, provided $y\in D$ and $2|y-\xi|\le|x-\xi|$. □

Lemma 3.5. *There are positive constants* β *and* M_5 *such that if* $\xi\in\partial D$, $x,y\in D_0$ *and* $2|x-\xi|\le|y-\xi|$, *then*

$$K(x,y)\le M_5\left(\frac{|x-\xi|}{|y-\xi|}\right)^{\beta}K(x,\xi).$$

Proof. Let $r=|x-\xi|$ and $R=|y-\xi|$. Since g is a positive harmonic function outside x_0 and vanishes on the boundary, it follows from [6, Lemmas 4.1 and 4.4] that there is $\beta>0$ such that

$$g\le M\left(\frac{r}{R}\right)^{\beta}g(A_R(\xi))\quad\text{on } B(\xi,r)\cap D.$$

Hence, in particular

$$\frac{g(A_r(\xi))}{g(A_R(\xi))}\le M\left(\frac{r}{R}\right)^{\beta}. \tag{5}$$

Next we show

$$K(y,x)\approx K(y,\xi). \tag{6}$$

Observe that $G(\cdot,y)$ and g are both positive and harmonic on $B(\xi,Mr)\cap D$ and vanish on $B(\xi,Mr)\cap\partial D$. The boundary Harnack principle [6, Lemma 4.10] yields that

$$\frac{G(z,y)}{G(A_r(\xi),y)}\approx\frac{g(z)}{g(A_r(\xi))}\quad\text{for } z\in B(\xi,r)\cap D.$$

This is equivalent to

$$K(y,z)=\frac{G(z,y)}{g(z)}\approx\frac{G(A_r(\xi),y)}{g(A_r(\xi))}.$$

Since the above comparison holds uniformly for $z\in D\cap B(\xi,r)$, we obtain (6) by letting $z\to x$ and $z\to\xi$.

By the maximum principle we have

$$\sup_{D\cap\partial B(\xi,R)} K(\cdot,\xi)\le \sup_{D\cap\partial B(\xi,r)} K(\cdot,\xi).$$

Hence the boundary Harnack principle yields

$$K(A_R(\xi),\xi)\le AK(A_r(\xi),\xi). \tag{7}$$

Once more, we use the boundary Harnack principle to get

$$\frac{K(x,\xi)}{K(A_r(\xi),\xi)}\approx\frac{g(x)}{g(A_r(\xi))},\quad \frac{K(y,\xi)}{K(A_R(\xi),\xi)}\approx\frac{g(y)}{g(A_R(\xi))},$$

or equivalently,

$$\frac{K(x,\xi)}{g(x)}\approx\frac{K(A_r(\xi),\xi)}{g(A_r(\xi))},\quad \frac{K(y,\xi)}{g(y)}\approx\frac{K(A_R(\xi),\xi)}{g(A_R(\xi))}. \tag{8}$$

Now (5), (6), (7) and (8) imply

$$\begin{aligned}K(x,y)&=\frac{K(y,x)}{g(y)}g(x)\approx\frac{K(y,\xi)}{g(y)}g(x)\approx\frac{K(A_R(\xi),\xi)}{g(A_R(\xi))}g(x)\\&\le M\frac{K(A_r(\xi),\xi)}{g(A_r(\xi))}\frac{g(A_r(\xi))}{g(A_R(\xi))}g(x)\le M\frac{K(x,\xi)}{g(x)}\left(\frac{r}{R}\right)^{\beta}g(x)\\&=M\left(\frac{|x-\xi|}{|y-\xi|}\right)^{\beta}K(x,\xi),\end{aligned}$$

which finishes the proof of the lemma. □

For a positive integer k and $\xi\in\partial D$ we let

$$I_{j,k}(\xi)=\{x\in D: 2^{-j-k}\le|x-\xi|<2^{k+1-j}\}.$$

Lemma 3.6. *Let α, β, M_4 and M_5 be as in Lemmas* 3.4 *and* 3.5. *For $\varepsilon>0$ we define*

$$k_0(\varepsilon)=\max\left\{\frac{1}{\alpha\log 2}\log\frac{M_4}{\varepsilon},\frac{M_5}{\beta\log 2}\right\}.$$

If k is an integer such that $k\ge k_0(\varepsilon)$, then

$$K(x,y)\le(1+\varepsilon)K(x,\xi)\quad \textit{for } x\in I_j(\xi) \textit{ and } y\in D_0\setminus I_{j,k}(\xi).$$

Proof. Let $x\in I_j(\xi)$ and $y\in D\setminus I_{j,k}(\xi)$. Then one of (a) or (b) below holds,

(a) $|y-\xi|<2^{-j-k}$,

(b) $|y-\xi|\ge 2^{k+1-j}$.

Case (a). Since $|y-\xi|/|x-\xi|<2^{-k}$, it follows from Lemma 3.4 that

$$K(x,y)\le\left(1+M_4\left(\frac{|y-\xi|}{|x-y|}\right)^{\alpha}\right)K(x,\xi)\le(1+M_4 2^{-k\alpha})K(x,\xi)\le(1+\varepsilon)K(x,\xi).$$

Case (b). Since $|x-\xi|/|y-\xi|<2^{-k}$, it follows from Lemma 3.5 that

$$K(x,y)\le M_5 2^{-k\beta}K(x,\xi)\le K(x,\xi).$$

Thus in both cases we obtain the required inequality. The proof is complete. □

Proof of Theorem 3.2(ii). Let $k_0(\varepsilon)$ be as in Lemma 3.6. For $\varepsilon=\frac{1}{2}(1-q)$ we can choose and fix a positive integer k such that

$$k_0(\varepsilon)\le k\le M\log\frac{2}{1-q}.$$

Take an arbitrary boundary point $\xi\in\partial D$. For simplicity we will use the notation $I_j^*(\xi)=I_{j,k}(\xi)$. Lemma 3.6 gives us that

$$K(x,y)\le\left(1+\tfrac{1}{2}(1-q)\right)K(x,\xi)=\tfrac{1}{2}(3-q)K(x,\xi)\tag{9}$$

for $x\in I_j(\xi)$ and $y\in D_0\setminus I_j^*(\xi)$. Let us now use the distribution μ defined by

$$\widehat{R}^E_{K_\xi}=K\mu.$$

By (4) μ is concentrated on D_0. Since $K(x_0,y)=1$ and since Ω is a $(1-q)$-boundary layer, it follows that

$$\|\mu\|=K\mu(x_0)=\widehat{R}^E_{K_\xi}(x_0)\le 1-(1-q)=q.\tag{10}$$

We have from (9)

$$\int_{D\setminus I_j^*(\xi)} K(x,y)\,d\mu(y) \le \tfrac{1}{2}q(3-q)K(x,\xi).$$

On the other hand, since $K\mu \ge K_\xi$ q.e. on E, it follows that for q.e. $x\in E_j(\xi)$

$$\int_{I_j^*(\xi)} K(x,y)\,d\mu(y) \ge \bigl(1-\tfrac{1}{2}q(3-q)\bigr)K(x,\xi) \ge \tfrac{1}{2}(1-q)K(x,\xi).$$

The last inequality comes simply from the fact that $0<q<1$. Hence, by putting $\mu_j=\mu|_{I_j^*(\xi)}$, we obtain

$$K\mu_j \ge \tfrac{1}{2}(1-q)\widehat{R}_{K_\xi}^{E_j(\xi)} \quad \text{on } D.$$

Evaluating both sides at x_0, we see that

$$\|\mu_j\| = K\mu_j(x_0) \ge \tfrac{1}{2}(1-q)\widehat{R}_{K_\xi}^{E_j(\xi)}(x_0).$$

The "annuli" $\{I_j^*(\xi)\}$ overlap each $I_j^*(\xi)$ at most $2k+1$ times. By (10)

$$\tfrac{1}{2}(1-q)\sum \widehat{R}_{K_\xi}^{E_j(\xi)}(x_0) \le \sum \|\mu_j\| \le (2k+1)q.$$

Therefore

$$\Phi(\xi) \le \frac{2q}{1-q}(2k+1) \le M\frac{q}{1-q}\log\frac{2}{1-q}.$$

Theorem 3.2(ii) is proved. □

Remark 3.7. We have actually proved a pointwise estimate: for each fixed $\xi\in\partial D$

$$\widehat{R}_{K_\xi}^{E}(x_0) \le q < 1 \quad\Longrightarrow\quad \Phi(\xi) \le M_3\frac{q}{1-q}\log\frac{2}{1-q}.$$

We say that E is minimally thin at $\xi\in\partial D$ if $\widehat{R}_{K_\xi}^{E}(x)\ne K_\xi(x)$ for some $x\in D$. The minimal thinness can be characterized by $\Phi(\xi)$.

Proposition 3.8. *Let $\xi\in\partial D$. Then the following statements are equivalent*:
(i) *E is minimally thin at ξ.*
(ii) $\widehat{R}_{K_\xi}^{E}(x_0)<1$.
(iii) $\Phi(\xi)<\infty$.
(iv) $\sum_{j=1}^{\infty}\widehat{R}_{K_\xi}^{E_j(\xi)}$ *is a Green potential.*

As an immediate corollary to Theorem 3.2 and this proposition, we have the following, which is a generalization of part of Theorem 3(a) in [4].

Corollary 3.9. *If $\Omega = D\setminus E$ is a boundary layer, then E is minimally thin at every $\xi \in \partial D$.*

Proof of Proposition 3.8. (i) $\Rightarrow$ (ii): We know that $\widehat{R}^E_{K_\xi} = K_\xi$ q.e. on E and hence (i) implies that there is $x_1 \in \Omega = D\setminus E$ such that $\widehat{R}^E_{K_\xi}(x_1) \neq K_\xi(x_1)$. Since Ω is a domain, it follows from the minimum principle that $\widehat{R}^E_{K_\xi}(x_0) < K_\xi(x_0) = 1$.

(ii) $\Rightarrow$ (iii): By Remark 3.7 we have $\Phi(\xi) < \infty$.

(iii) $\Rightarrow$ (iv): It is easy to see that each $\widehat{R}^{E_j(\xi)}_{K_\xi}$ is a Green potential. By assumption the summation is convergent at x_0 and hence $\sum_{j=1}^\infty \widehat{R}^{E_j(\xi)}_{K_\xi}$ is a Green potential.

(iv) $\Rightarrow$ (i): Since $\sum_{j=1}^\infty \widehat{R}^{E_j(\xi)}_{K_\xi}$ is a Green potential, which majorizes K_ξ over $\bigcup_{j=1}^\infty E_j(\xi)$, it follows that $\widehat{R}^E_{K_\xi}$ is a Green potential, and in particular $\widehat{R}^E_{K_\xi} \neq K_\xi$. Thus E is minimally thin at ξ. □

4. Wiener type criterion for boundary layers

In this section we study boundary layers in Liapunov or $C^{1,\alpha}$ domains instead of NTA domains. In view of Widman [8] we have the following estimates

$$g(x) \approx \delta(x), \quad K(x,\xi) \approx g(x)|x-\xi|^{-d} \quad \text{for } x \in D_0,\ \xi \in \partial D. \tag{11}$$

From these estimates and the quasiadditivity of the Green energy we will obtain a Wiener type criterion for boundary layers in terms of capacity. The following series was introduced in [7] and considered in [4], [1] and [2] also.

Definition 4.1. Let $\{Q_k\}$ be the Whitney decomposition of D. For the cube Q_k, let $t_k = \operatorname{dist}(Q_k, \partial D)$ and $\varrho_k(\xi) = \operatorname{dist}(Q_k, \xi)$. By cap we denote the logarithmic capacity when $d=2$, and the Newtonian capacity when $d \geq 3$. We put

$$W(\xi) = W(\xi, E) = \begin{cases} \displaystyle\sum_k \frac{t_k^2}{\varrho_k(\xi)^2}\left(\log \frac{4t_k}{\operatorname{cap}(E\cap Q_k)}\right)^{-1} & \text{if } d=2, \\ \displaystyle\sum_k \frac{t_k^2}{\varrho_k(\xi)^d} \operatorname{cap}(E\cap Q_k) & \text{if } d \geq 3. \end{cases}$$

Theorem 4.2. *There exist positive constants M_6 and M_7 depending only on D, x_0 and r_1 with the following properties*:

(i) *If $\sup_{\xi\in\partial D} W(\xi) \leq M_6 q$, then Ω is a $(1-q)$-boundary layer.*

(ii) *If Ω is a $(1-q)$-boundary layer, then*

$$\sup_{\xi\in\partial D} W(\xi) \leq M_7 \frac{q}{1-q} \log \frac{2}{1-q}.$$

Corollary 4.3. *Let $0<q_0<1$. Then there is a positive constant M_{q_0} depending only on D, r_1 and q_0 such that if $\Omega=D\setminus E$ is a $(1-q)$-boundary layer with $0<q\le q_0$, then*

$$\sup_{\xi\in\partial D} W(\xi)\le M_{q_0}q.$$

Moreover, $M_{q_0}\approx(1-q_0)^{-1}\log[2/(1-q_0)]$.

Remark 4.4. In view of Remark 3.7, we have pointwise results in Theorem 4.2 and Corollary 4.3: for each fixed $\xi\in\partial D$

(i) $W(\xi)\le M_6 q \Rightarrow \widehat{R}^E_{K_\xi}(x_0)\le q$.

(ii) $\widehat{R}^E_{K_\xi}(x_0)\le q<1 \Rightarrow W(\xi)\le M_7(q/(1-q))\log(2/(1-q))$.

(iii) $\widehat{R}^E_{K_\xi}(x_0)\le q$ with $0<q\le q_0<1 \Rightarrow W(\xi)\le M_{q_0}q$.

For the proof of the above theorem we use the quasiadditivity of Green energy. For a subset E of D we observe that $\widehat{R}^E_g$ is a Green potential, $G(\,\cdot\,,\lambda_E)$. The energy

$$\gamma(E)=\iint G(x,y)\,d\lambda_E(x)\,d\lambda_E(y)$$

is called the Green energy of E (relative to g). Observe that

$$\gamma(E)=\int \widehat{R}^E_g\,d\lambda_E=\int g\,d\lambda_E=G\lambda_E(x_0)=\widehat{R}^E_g(x_0), \tag{12}$$

where the second equality follows from $\widehat{R}^E_g=g$ q.e. on the support of λ_E. In view of (11), the quasiadditivity of the Green energy [2, Corollary 2] reads as follows.

Theorem B. *Let $E\subset D_0$. Then*

$$\gamma(E)\approx\begin{cases}\displaystyle\sum_k t_k^2\left(\log\frac{4t_k}{\operatorname{cap}(E\cap Q_k)}\right)^{-1} & \textit{if } d=2,\\ \displaystyle\sum_k t_k^2\operatorname{cap}(E\cap Q_k) & \textit{if } d\ge 3.\end{cases}$$

Proof of Theorem 4.2. Let us for a moment consider the case $d\ge3$. We have from (11)

$$K(x,\xi)\approx g(x)|x-\xi|^{-d}\approx 2^{jd}g(x)\quad\text{for } x\in I_j(\xi).$$

Hence we have from (12) and Theorem B

$$\widehat{R}^{E_j(\xi)}_{K_\xi}(x_0)\approx 2^{jd}\widehat{R}^{E_j(\xi)}_g(x_0)=2^{jd}\gamma(E_j(\xi))\approx 2^{jd}\sum_k t_k^2\operatorname{cap}(E_j(\xi)\cap Q_k).$$

Since $\varrho_k(\xi)\approx 2^{-j}$ for $E_j(\xi)\cap Q_k\neq\emptyset$, it follows that

$$\Phi(\xi)\approx\sum_j 2^{jd}\sum_k t_k^2\operatorname{cap}(E_j(\xi)\cap Q_k)\approx\sum_k\frac{t_k^2}{\varrho_k(\xi)^d}\operatorname{cap}(E\cap Q_k)=W(\xi).$$

The same type of arguments hold for the case $d=2$ and we conclude $\Phi(\xi)\approx W(\xi)$. Hence Theorem 3.2 readily yields the theorem. □

In view of $\Phi(\xi)\approx W(\xi)$ and Proposition 3.8 we have the following well-known result ([1], [2] and [4]).

Corollary 4.5. *Let $\xi\in\partial D$. E is minimally thin at ξ if and only if $W(\xi)<\infty$.*

5. Good boundary layers

In this section we shall work with Liapunov or $C^{1,\alpha}$ domains again. So far we have considered boundary layers. There is also a *strong* type called good boundary layer defined by Volberg in [7, p. 155] for the case when D is the unit disk. The definition has a natural generalization. Let $D_n:=\{x\in D:\delta(x)>1/n\}$ and define Ω_n to be $\Omega\cup D_n$ and E_n to be $E\setminus D_n$. (We note that $\Omega_n=D\setminus E_n$.)

Definition 5.1. Ω is a good boundary layer if Ω_n is a $(1-\varepsilon_n)$-boundary layer with $\lim\varepsilon_n=0$.

The following proposition is a straightforward generalization of Theorem 1.4 in [7].

Proposition 5.2. *Ω is a good boundary layer if and only if $W(\xi)$ converges uniformly on the boundary ∂D.*

Proof. For simplicity we prove the theorem only for $d\geq 3$. The case when $d=2$ is similar. Since D is bounded, we may assume that Whitney cubes Q_k are enumerated as $Q_1,Q_2,\ldots$ so that Q_k approaches the boundary if and only if $k\to\infty$. We will prove the proposition in two steps.

Suppose that Ω is a good boundary layer. Take an arbitrary $\varepsilon>0$. We find $q=q(\varepsilon)>0$ so small that

$$M_7\frac{q}{1-q}\log\frac{2}{1-q}<\varepsilon,$$

where M_7 is the constant in Theorem 4.2. Since Ω is a good boundary layer, by choosing n large enough we see that Ω_n is a $(1-q)$-boundary layer. We have from Theorem 4.2(ii)

$$\sup_{\xi\in\partial D}W(\xi,E_n)\leq M_7\frac{q}{1-q}\log\frac{2}{1-q}<\varepsilon,$$

which means that

$$\sup_{\xi\in\partial D}\sum_{k>k_n}\frac{t_k^2}{\varrho_k(\xi)^d}\operatorname{cap}(E\cap Q_k)<\varepsilon,$$

with k_n being the least integer k_n such that $Q_k\subset\{x\in D:\delta(x)\leq 1/n\}$ for $k\geq k_n$. Thus $W(\xi)$ is uniformly convergent.

On the other hand, let us assume that $W(\xi)$ is uniformly convergent. Take an arbitrary $\varepsilon>0$. Then there is k_0 such that

$$\sup_{\xi\in\partial D}\sum_{k>k_0}\frac{t_k^2}{\varrho_k(\xi)^d}\operatorname{cap}(E\cap Q_k)\leq M_6\varepsilon, \tag{13}$$

where M_6 is the constant in Theorem 4.2. We find $n=n(k_0)$ such that

$$\left\{x\in D:\delta(x)\leq\frac{1}{n}\right\}\subset\bigcup_{k>k_0}Q_k. \tag{14}$$

Therefore,

$$\sup_{\xi\in\partial D}W(\xi,E_n)<M_6\varepsilon.$$

Theorem 4.2(i) gives us that $\Omega_n=D\setminus E_n$ is a $(1-\varepsilon)$-boundary layer. Thus, by definition, $\Omega=D\setminus E$ is a good boundary layer. □

Let us note that a good boundary layer is always a boundary layer. This property does not seem to follow from the definition directly. For the classical boundary layers this was proved by Essén [4, Theorem 3(b)]. Our proof heavily depends on Theorem 4.2.

Theorem 5.3. *If $\Omega=D\setminus E$ is a good boundary layer, then Ω is a boundary layer.*

Proof. For simplicity we prove the theorem only for $d\geq 3$. The case when $d=2$ is similar. Let us prove the theorem by contradiction. Let $\Omega=D\setminus E$ be a good boundary layer and suppose it is not a boundary layer. By Proposition 2.3 we find $\xi_i\in\partial D$ such that

$$\widehat{R}^E_{K_{\xi_i}}(x_0)\to 1\quad\text{as } i\to\infty. \tag{15}$$

Taking a subsequence, if necessary, we may assume that ξ_i converges to $\xi_0\in\partial D$. Since $W(\xi_0)<\infty$, it follows from Corollary 4.5 that E is minimally thin at ξ_0, and hence from Proposition 3.8 that $\widehat{R}^E_{K_{\xi_0}}(x_0)<1$. Let

$$\varepsilon=\frac{1-\widehat{R}^E_{K_{\xi_0}}(x_0)}{2+\widehat{R}^E_{K_{\xi_0}}(x_0)}>0. \tag{16}$$

By Proposition 5.2 $W(\xi)$ is uniformly convergent and we can find k_0 such that (13) holds. Let $n=n(k_0)$ be such that (14) holds. By Theorem 4.2 we have

$$\sup_{\xi\in\partial D} \widehat{R}^{E_n}_{K_\xi}(x_0)<\varepsilon. \tag{17}$$

By the Hölder continuity of the kernel functions [6, Theorem 7.1], we see that

$$K_{\xi_i}/K_{\xi_0}\to 1 \quad \text{uniformly on } F_n=\bigcup_{Q_k\cap\{x\in D:\delta(x)\ge 1/n\}\ne\emptyset} E\cap Q_k.$$

Hence we may assume that $K_{\xi_i}\le(1+\varepsilon)K_{\xi_0}$ on F_n. This implies

$$\widehat{R}^{F_n}_{K_{\xi_i}}\le(1+\varepsilon)\widehat{R}^{F_n}_{K_{\xi_0}}\le(1+\varepsilon)\widehat{R}^{E}_{K_{\xi_0}} \quad \text{on } D,$$

and in particular

$$\widehat{R}^{F_n}_{K_{\xi_i}}(x_0)\le(1+\varepsilon)\widehat{R}^{E}_{K_{\xi_0}}(x_0). \tag{18}$$

Now, (15), (16), (17) and (18) altogether and the subadditivity of reduced functions yield

$$\begin{aligned}1&=\lim_{i\to\infty}\widehat{R}^{E}_{K_{\xi_i}}(x_0)\le\limsup_{i\to\infty}\widehat{R}^{E_n}_{K_{\xi_i}}(x_0)+\limsup_{i\to\infty}\widehat{R}^{F_n}_{K_{\xi_i}}(x_0)\\&\le\varepsilon+(1+\varepsilon)\widehat{R}^{E}_{K_{\xi_0}}(x_0)=\frac{1+2\widehat{R}^{E}_{K_{\xi_0}}(x_0)}{2+\widehat{R}^{E}_{K_{\xi_0}}(x_0)}<1.\end{aligned}$$

Thus a contradiction arises. The theorem is proved. □

6. Weak boundary layers

In the original definition of boundary layers, we take the harmonic measure in the origin. In Definition 2.1 we put x_0 in that position. How important is the choice of reference point? We will in this section investigate that question.

Let D be an arbitrary NTA domain in $\mathbf{R}^d$, as in Section 2. In order to simplify the notation, we will introduce an auxiliary function. Let

$$H_\xi(x):=\frac{1}{K_\xi(x)}\widehat{R}^{E}_{K_\xi}(x).$$

From Proposition 2.3(ii) we see that Ω is a boundary layer at x_0 if and only if $H_\xi(x_0)\le q<1$ for all $\xi\in\partial D$. (Recall that $K_\xi(x_0)=1$.)

Let us now choose the "best" reference point for our purpose instead of x_0 to get a slightly weaker assumption on Ω, i.e. let

$$\inf_{x\in\Omega}\sup_{\xi\in\partial D} H_\xi(x)<1. \tag{19}$$

It turns out that this weakening does not make any essential difference.

Proposition 6.1. *Ω is a boundary layer at x_0 if and only if* (19) *holds.*

Proof. It suffices to show the 'if' part. Suppose that (19) holds. Then there exist q, $0<q<1$, and $x_1\in\Omega$ such that $\sup_{\xi\in\partial D} H_\xi(x_1)\le q$. Let $q<q'<1$. Since both K_ξ and $\widehat{R}^E_{K_\xi}$ are positive and harmonic in Ω, it follows from the Harnack principle that there is $\varepsilon>0$ such that $\overline{B}_\varepsilon\subset\Omega$ and

$$\sup_{\xi\in\partial D} H_\xi(x)\le q' \quad \text{for } x\in\overline{B}_\varepsilon,$$

where $B_\varepsilon=B(x_1,\varepsilon)$. In view of Proposition 2.3, we see that Ω is a $(1-q')$-boundary layer at $x_2\in\overline{B}_\varepsilon$, i.e.

$$\omega(x_2,I)\ge(1-q')\widetilde{\omega}(x_2,I) \tag{20}$$

for every Borel subset $I\subset\partial D$. By the minimum principle

$$\omega(x,I)\ge\omega(x,\partial B_\varepsilon,\Omega\setminus\overline{B}_\varepsilon)\min_{x_2\in\partial B_\varepsilon}\omega(x_2,I)$$

for $x\in\Omega\setminus\overline{B}_\varepsilon$. Using (20), we evaluate the above inequality at $x=x_0$ to obtain

$$\omega(x_0,I)\ge\omega(x_0,\partial B_\varepsilon,\Omega\setminus\overline{B}_\varepsilon)(1-q')\min_{x_2\in\partial B_\varepsilon}\widetilde{\omega}(x_2,I).$$

By the Harnack principle again

$$\widetilde{\omega}(x_2,I)\approx\widetilde{\omega}(x_0,I) \quad \text{for } x_2\in\partial B_\varepsilon,$$

where the constant of comparison is independent of I, and hence

$$\omega(x_0,I)\ge M_\varepsilon(1-q')M\widetilde{\omega}(x_0,I),$$

where

$$M_\varepsilon=\omega(x_0,\partial B_\varepsilon,\Omega\setminus\overline{B}_\varepsilon)>0.$$

Since I is an arbitrary Borel subset in ∂D, this implies that Ω is an $M_\varepsilon(1-q')M$-boundary layer at x_0. $\square$

The chain of inequalities

$$\sup_{\xi\in\partial D}\inf_{x\in\Omega} H_\xi(x)\le\inf_{x\in\Omega}\sup_{\xi\in\partial D} H_\xi(x)\le\sup_{\xi\in\partial D} H_\xi(x_0) \tag{21}$$

encourages us to define another variant of boundary layers.

Definition 6.2. We say that Ω is a weak boundary layer if

$$\sup_{\xi\in\partial D}\inf_{x\in\Omega} H_\xi(x)<1.$$

In view of Definition 3.1 we introduce

$$\Phi(\xi,x):=\sum_{j=1}^{\infty}\frac{1}{K_\xi(x)}\widehat{R}_{K_\xi}^{E_j(\xi)}(x),$$
$$\Phi_w(\xi):=\inf_{x\in\Omega}\Phi(\xi,x).$$

We have the following proposition (cf. Proposition 2.3).

Proposition 6.3. *The following statements are equivalent:*
(i) Ω *is a weak boundary layer.*
(ii) $\inf_x H_\xi(x)<1$ *for every* $\xi\in\partial D$.
(iii) $\inf_x H_\xi(x)=0$ *for every* $\xi\in\partial D$.
(iv) $\Phi_w(\xi)=0$ *for every* $\xi\in\partial D$.
(v) E *is minimally thin at every* $\xi\in\partial D$.
(vi) $\inf_x(1/h(x))\widehat{R}_h^E(x)<1$ *for every positive harmonic function* h.
(vii) $\inf_x(1/h(x))\widehat{R}_h^E(x)=0$ *for every positive harmonic function* h.

This proposition is an easy consequence of the following pointwise result, which can be shown by the well-known minimal fine limit theorem (e.g. [3, 1.XII.18]).

Theorem C. *Let* $h=K\mu_h$ *be a positive harmonic function on* D *and let* u *be a Green potential. Then, for* μ_h *almost every boundary point* ξ, *there is a set* F_ξ *which is minimally thin at* ξ *such that*

$$\lim_{\substack{x\to\xi\\ x\in D\setminus F_\xi}}\frac{u(x)}{h(x)}=0.$$

Proposition 6.4. *Let* $\xi\in\partial D$. *Then the following statements are equivalent:*
(i) $\inf_x H_\xi(x)<1$.
(ii) $\inf_x H_\xi(x)=0$.
(iii) $\Phi_w(\xi)=0$.
(iv) E *is minimally thin at* ξ.

Proof. By the countable subadditivity of reduced functions and the definition of minimal thinness we readily have (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iv). Suppose (iv) holds.

By Proposition 3.8 we see that $\sum_{j=1}^{\infty} \widehat{R}_{K_\xi}^{E_j(\xi)}$ is a Green potential. By Theorem C there is a set F_ξ minimally thin at ξ such that

$$\lim_{\substack{x\to\xi \\ x\in D\setminus F_\xi}} \Phi(\xi,x)=0.$$

In particular (iii) holds.

Proof of Proposition 6.3. The equivalence (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) ⇔ (v) readily follows from Proposition 6.4. Obviously, (vii) ⇒ (vi). Since K_ξ is a positive harmonic function, it is obvious that (vi) ⇒ (ii). Let us show (v) ⇒ (vii). Suppose E is minimally thin at every $\xi\in\partial D$. Let $h=K\mu_h$ be a positive harmonic function. Since E is minimally thin at every $\xi\in\partial D$, it follows that $\widehat{R}_h^E$ is a Green potential (see e.g. [3, 1.XII.17 Example]). Hence Theorem C says that for μ_h-a.e. $\xi\in\partial D$, and hence at least one $\xi\in\partial D$, there is a set F_ξ minimally thin at ξ such that

$$\lim_{\substack{x\to\xi \\ x\in D\setminus F_\xi}} \frac{1}{h(x)}\widehat{R}_h^E(x)=0.$$

In particular, (vii) holds. □

7. Relationships between various boundary layers

We conclude with a list of implications between the different types of boundary layers. In this section we let D be a Liapunov or $C^{1,\alpha}$ domain. We have

(i) Ω is a good boundary layer ⇒ Ω is a boundary layer.

(ii) Ω is a boundary layer ⇒ Ω is a weak boundary layer.

(iii) For $\xi_0\in\partial D$ and $\alpha>0$ let $\Gamma(\xi_0)=\Gamma_\alpha(\xi_0)=\{x\in D:\delta(x)>\alpha|x-\xi_0|\}$ be a non-tangential cone or "Stoltz cone" with vertex at ξ_0. If $E\subset\Gamma(\xi_0)$, then the three types of boundary layers coincide.

In Theorem 5.3 we have observed (i); in view of (21) and Proposition 6.1, (ii) is obvious. These implications cannot be turned around as seen from examples [7, Ex. 5.1] and [4, Theorem 3(a)] combined with Proposition 6.3. The coincidence (iii) follows immediately from the following proposition.

Proposition 7.1. *Let $\xi_0\in\partial D$ and $\alpha>0$. Suppose $E\subset\Gamma(\xi_0)=\Gamma_\alpha(\xi_0)$. Then $\Omega=D\setminus E$ is a weak boundary layer if and only if Ω is a good boundary layer.*

Proof. Let us assume that Ω is a weak boundary layer. Then we have from Proposition 6.3 that E is minimally thin at ξ_0, or equivalently $W(\xi_0)<\infty$. For every

Whitney cube Q_k intersecting $\Gamma(\xi_0)$ we have $t_k \approx \varrho_k(\xi_0)$. Therefore we have that the convergence of $W(\xi_0)$ is equivalent to

$$\sum_k \left(\log \frac{4t_k}{\operatorname{cap}(E_k)}\right)^{-1} < \infty \quad \text{if } d=2,$$
$$\sum_k t_k^{2-d} \operatorname{cap}(E \cap Q_k) < \infty \quad \text{if } d \geq 3.$$

Since $t_k \leq \varrho_k(\xi)$ for every $\xi \in \partial D$, we conclude that $W(\xi)$ is uniformly convergent for $\xi \in \partial D$ in both cases. Hence, due to Proposition 5.2, Ω is a good boundary layer. The opposite implication is trivial.

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