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THE REGIONAL STRATEGY IN THE ASYMPTOTIC EXPANSION OF TWO-LOOP VERTEX FEYNMAN DIAGRAMS

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General prescriptions for evaluating coefficients at arbitrary powers and logarithms in the asymptotic expansion of Feynman diagrams in the Sudakov limit are discussed and illustrated with two-loop examples. Peculiarities connected with the evaluation of individual terms in the expansion, in particular, the introduction of an auxiliary analytic regularization, are characterized.

1. The simplest explicit formulas [1–3] (see [4] for a review) for asymptotic expansions of Feynman diagrams in various off-mass-shell limits of momenta and masses in which the momenta are either large or small in the Euclidean sense have been generalized to some on-shell limits [5–7] typical for Minkowski space, in particular, to the Sudakov limit. The prescriptions for these limits were formulated by applying (pre)subtractions in a certain family of subgraphs of a given graph.

Explicit prescriptions for expanding Feynman integrals near threshold were recently presented using a standard physical strategy based on the analysis of regions in the space of loop momenta [8]. However, this regional strategy was usually used only to evaluate and sum the leading logarithms, in particular, in the Sudakov limit (see, e.g., [9]). We note that contributions to the leading logarithms come only from some specific regions and integrations in the other regions are usually not considered.

It was argued (and demonstrated for the threshold expansion) that this strategy can be used to evaluate coefficients at all powers and the logarithm in an arbitrary limit [8]. In such an extended form, the strategy reduces to the following prescriptions:

a. Consider all the regions of the loop momenta that are typical for the given limit. In each region, develop the integrand in a Taylor series with respect to the parameters that are considered small in the given region.
b. Integrate the developed integrand separately in each region and thus obtain the integral over the total integration domain of the loop momenta.
c. Set all scaleless integrals to zero (even if they are not regularized, e.g., using the dimensional regularization).

Step b is the most nontrivial [8]. We believe that this strategy succeeds for every limit of momenta and masses. For example, it leads to the well-known formulas for the asymptotic expansions in the standard Euclidean limits [1, 2] (the proof was in [3]), which indirectly confirms the assumption in this case. We note that for these limits as well as for the on-shell limit considered in [5, 6], the collection of relevant regions is determined by subdividing all the loop momenta into large (hard) and small (soft) momenta.

In the present paper, we check this heuristic procedure for evaluating the coefficients at arbitrary powers and logarithms in asymptotic expansions of Feynman diagrams in the Sudakov limit [10] with two-loop examples. We consider two commonly accepted variants of this limit for vertex diagrams with the external momenta p_1, p_2, and $q = p_1 - p_2$.

Limit 1 Two external momenta are off shell, $p_1^2 = p_2^2 = m^2 = -\mu^2$ and $Q^2 \equiv -q^2 \to \infty$; all internal masses are zero.

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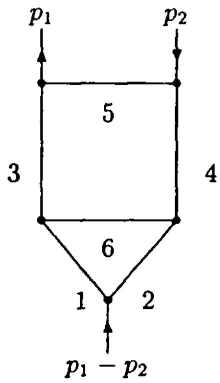

Fig. 1. Two-loop planar vertex diagram.

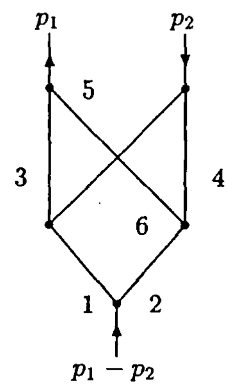

Fig. 2. Two-loop nonplanar vertex diagram.

Limit 2 Two external momenta are on shell, $p_1^2 = p_2^2 = 0$ and $Q^2 \to \infty$; some internal masses are nonzero.

We calculate the leading power behavior including all the logarithms, $\log^j(q^2/m^2)$, $j = 0, \ldots, 4$, of the massless planar diagram in Fig. 1 in the first limit and compare the obtained result with the well-known explicit expression [11]. After such confirmation, we apply the above heuristic prescription to the nonplanar diagram in Fig. 2 (for which no analytic results are known) in Limit 2 where $m_1 = \cdots = m_4 = 0$ and $m_5 = m_6 = m$. We also use the second example to describe techniques for evaluating individual terms of the expansion. A natural way to evaluate terms with the $1/(m^2)^{2\epsilon}$ dependence is to introduce an auxiliary analytic regularization. In contrast to the second limit of the planar diagram in which the poles of the first order arise in the analytic regularization parameter (and mutually cancel in the sum of two contributions) [7], the nonplanar diagram contains poles up to the second order, which occur in five contributions. These poles also cancel in the sum, and we obtain the result coinciding with the dimensional regularization.

2. The Feynman integral for Fig. 1 can be written as

$$F_1(Q, m, \epsilon) = \iint \frac{d^d k\, d^d l}{(l^2 - 2p_1 l + m^2)(l^2 - 2p_2 l + m^2)} \frac{1}{(k^2 - 2p_1 k + m^2)(k^2 - 2p_2 k + m^2)k^2(k-l)^2}. \quad (1)$$

We use the dimensional regularization [12] with $d = 4 - 2\epsilon$. When presenting our results we omit $i\pi^{d/2}$ per loop, and when writing separate contributions through the expansion in ϵ, we also omit $\exp(-\gamma_E \epsilon)$ per loop (with γ_E the Euler constant).

For convenience, we choose the external momenta

$$\begin{aligned} p_1 &= \tilde{p}_1 + \frac{m^2}{Q^2}\tilde{p}_2, & p_2 &= \tilde{p}_2 + \frac{m^2}{Q^2}\tilde{p}_1, \\ \tilde{p}_1 &= \left(\frac{Q}{2}, -\frac{Q}{2}, 0, 0\right), & \tilde{p}_2 &= \left(\frac{Q}{2}, \frac{Q}{2}, 0, 0\right), \end{aligned} \quad (2)$$

which implies $p_i^2 = m^2$, $\tilde{p}_i^2 = 0$, and $2\tilde{p}_1\tilde{p}_2 = 2\tilde{p}_1 p_2 = Q^2$. The following regions are typical [9] in the given limit:

$$\begin{aligned} &\textit{hard} \ \text{(h)}: && k \sim Q, \\ &\textit{1-collinear} \ \text{(1c)}: && k_+ \sim Q, \quad k_- \sim \frac{m^2}{Q}, \quad \underline{k} \sim m, \\ &\textit{2-collinear} \ \text{(2c)}: && k_- \sim Q, \quad k_+ \sim \frac{m^2}{Q}, \quad \underline{k} \sim m, \\ &\textit{ultrasoft} \ \text{(us)}: && k \sim \frac{m^2}{Q}, \end{aligned} \quad (3)$$

where $k_\pm = k_0 \pm k_1$ and $\underline{k} = (k_2, k_3)$. The notation $k \sim Q$ means that any component of k_μ is of the order Q.

We assume that each loop momentum $k, l, \ldots$ belongs to one of the above types and consider various choices of the loop momenta (avoiding double counting). Other regions give zero contributions, in particular, when one of the loop momenta is *soft*, i.e., $k \sim m$. However, if some masses of the diagram are nonzero, then some soft regions would generate nonzero contributions (starting from a subleading order).

Integral (1) contains contributions of the leading order, $1/Q^4$, from the following nine regions: h-h, 1c-h, 2c-h, 1c-1c, 2c-2c, us-h, us-1c, us-2c, and us-us, where the region for the loop momentum k stands in the first place and for the loop momentum l in the second place. In the h-h region, the Taylor expansion of the integrand in the parameter m occurs. In the leading order, this is the massless planar diagram at $p_1^2 = p_2^2 = 0$ first evaluated in [13]. Although the result can be expressed through gamma functions for general ϵ using the method of integration by parts [14] (first performed in [15]), we here present it in the form of the ϵ expansion,

$$C^{(1)}_{(\text{h-h})} = \iint \frac{d^d k\, d^d l}{(l^2 - 2\tilde{p}_1 l)(l^2 - 2\tilde{p}_2 l)(k^2 - 2\tilde{p}_1 k)(k^2 - 2\tilde{p}_2 k) k^2 (k-l)^2} = \\ = \left(\frac{1}{4\epsilon^4} + \frac{5\pi^2}{24\epsilon^2} + \frac{29\zeta(3)}{6\epsilon} + \frac{3\pi^4}{32}\right) \frac{1}{(Q^2)^{2+2\epsilon}}. \quad (4)$$

All the contributions connected with the ultrasoft regions are easily evaluated in gamma functions using alpha parameters. In the leading order, we have

$$C^{(1)}_{(\text{us-us})} = \iint \frac{d^d k\, d^d l}{(-2\tilde{p}_1 l + m^2)(-2\tilde{p}_2 l + m^2)(-2\tilde{p}_1 k + m^2)(-2\tilde{p}_2 k + m^2) k^2 (k-l)^2} = \\ = \frac{\Gamma(1-\epsilon)^2 \Gamma(2\epsilon)^2}{\epsilon^2 (-m^2)^{4\epsilon} (Q^2)^{2-2\epsilon}}, \quad (5)$$

$$C^{(1)}_{(\text{us-h})} = \iint \frac{d^d k\, d^d l}{(l^2 - 2\tilde{p}_1 l)(l^2 - 2\tilde{p}_2 l)(-2\tilde{p}_1 k + m^2)(-2\tilde{p}_2 k + m^2) k^2 l^2} = \\ = \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)\Gamma(\epsilon)^2\Gamma(-\epsilon)^2}{\Gamma(1-2\epsilon)(-m^2)^{2\epsilon}(Q^2)^2}, \quad (6)$$

$$C^{(1)}_{(\text{us-1c})} = \iint \frac{d^d k\, d^d l}{(-2\tilde{p}_1 l)(l^2 - 2p_2 l + m^2)(-2\tilde{p}_1 k + m^2)(-2\tilde{p}_2 k + m^2)} \frac{1}{k^2\left(l^2 - \frac{(2\tilde{p}_1 l)(2\tilde{p}_2 k)}{Q^2}\right)} = \\ = \frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)\Gamma(2\epsilon)\Gamma(-\epsilon)}{\epsilon\Gamma(1-2\epsilon)(-m^2)^{3\epsilon}(Q^2)^{2-\epsilon}} \equiv C^{(1)}_{(\text{us-2c})}. \quad (7)$$

For arbitrary ϵ, the remaining contributions can be represented through Mellin–Barnes integrals using the alpha parameters,

$$C^{(1)}_{(\text{1c-1c})} = \iint \frac{d^d k\, d^d l}{(-2\tilde{p}_1 l)(l^2 - 2p_2 l + m^2)(-2\tilde{p}_1 k)(k^2 - 2p_2 k + m^2) k^2 (k-l)^2} = \\ = \frac{\Gamma(\epsilon)\Gamma(-\epsilon)\Gamma(2\epsilon)}{\Gamma(1+\epsilon)(-m^2)^{2\epsilon}(Q^2)^2} \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} ds\, \frac{\Gamma(s-3\epsilon)\Gamma(s+1-2\epsilon)\Gamma(s+1-\epsilon)\Gamma(\epsilon-s)\Gamma(-s)}{\Gamma(s+1-3\epsilon)} \equiv \\ \equiv C^{(1)}_{(\text{2c-2c})}, \quad (8)$$

$$C^{(1)}_{(\text{1c-h})} = \iint \frac{d^d k\, d^d l}{(l^2 - 2\tilde{p}_1 l)(l^2 - 2\tilde{p}_2 l)(-2\tilde{p}_1 k)(k^2 - 2p_2 k + m^2)} \frac{1}{k^2\left(l^2 - \frac{(2\tilde{p}_1 k)(2\tilde{p}_2 l)}{Q^2}\right)} = \\ = \frac{\Gamma(\epsilon)\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)(-m^2)^{\epsilon}(Q^2)^{2+\epsilon}} \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} ds\, \frac{\Gamma(s+1)\Gamma(s-\epsilon)\Gamma(s+1+\epsilon)\Gamma(-\epsilon-s)\Gamma(-s)}{\Gamma(s+1-2\epsilon)} \equiv \\ \equiv C^{(1)}_{(\text{2c-h})}. \quad (9)$$

We assume the standard choice of the integration contours: the UV poles are to the right and the IR poles are to the left of these contours. The above Mellin–Barnes integrals are expanded in ϵ by shifting the contours and evaluating residua at points where UV and IR poles meet as $\epsilon \to 0$. As a result, we obtain

$$(Q^2)^2 [C^{(1)}_{(\text{1c-1c})} + C^{(1)}_{(\text{2c-2c})} + C^{(1)}_{(\text{1c-h})} + C^{(1)}_{(\text{2c-h})}] = -\frac{1}{2\epsilon^4} + \left(L^2 - \frac{\pi^2}{2}\right)\frac{1}{2\epsilon^2} + \left(\frac{1}{2}L^3 - \frac{\pi^2}{6}L - \frac{17\zeta(3)}{3}\right)\frac{1}{\epsilon} + \\ + \frac{7}{24}L^4 - 4\zeta(3)L - \frac{\pi^4}{144}, \tag{10}$$

where $L = \log(Q^2/\mu^2)$ and we set $\mu = 1$ for brevity. (We note that we have both $\log(Q^2/\mu^2)$ and $\log(\mu^2)$ in individual contributions.)

Collecting all nine contributions together, we observe that the poles in ϵ, which have very different (UV, IR, or collinear) natures, mutually cancel, and we obtain

$$(Q^2)^2 F_1(Q, m, 0) \overset{Q\to\infty}{\sim} \frac{1}{4}L^4 + \frac{\pi^2}{2}L^2 + \frac{7\pi^4}{60}, \tag{11}$$

which agrees with the leading order expansion of the well-known explicit result [11].

3. Setting $m_1 = \cdots = m_4 = 0$ and $m_5 = m_6 = m$, we obtain the expansion of the planar diagram in Fig. 1 in Limit 2 in arbitrary order using the subtraction operators strategy (see [7]). We note that the same expressions for all the contributions to this expansion can be obtained using the regional strategy. The list of nonzero contributions, in this language, consists of h-h, 1c-h, 2c-h, 1c-1c, and 2c-2c contributions plus a contribution that starts from the next-to-leading order and comes from the region where the momentum of the middle line is soft and the second loop momentum is hard.

We now consider the expansion of the nonplanar diagram (Fig. 2) in Limit 2. The Feynman integral is

$$F_2(Q, m, \epsilon) = \iint \frac{d^d k\, d^d l}{((k+l)^2 - 2p_1(k+l))((k+l)^2 - 2p_2(k+l))} \times \\ \times \frac{1}{(k^2 - 2p_1 k)(l^2 - 2p_2 l)(k^2 - m^2)(l^2 - m^2)}, \tag{12}$$

where p_1 and p_2 satisfy the relations for $\tilde{p}_1$ and $\tilde{p}_2$ in the previous section. We also use the second choice of the loop momenta, where k and l are the respective momenta of lines 3 and 4, which corresponds to permuting the masses, which results in Eq. (12) with $m_1 = m_2 = m_5 = m_6 = 0$ and $m_3 = m_4 = m$.

Nonzero contributions to the expansion in the leading order are generated by the following regions: h-h, h-2c, 2c-h, 1c-1c, 2c-2c, 2c-1c, (1c-1c)′, (2c-2c)′, and (us-us)′. As above, the regions for the loop momenta k and l stand in the first and second places. The primes label the regions for the second natural choice of the loop momenta. The h-h contribution is given by the massless nonplanar diagram. The result expanded in ϵ is [13]

$$C^{(2)}_{(\text{h-h})} = \left(\frac{1}{\epsilon^4} - \frac{\pi^2}{\epsilon^2} - \frac{83\zeta(3)}{3\epsilon} - \frac{59\pi^4}{120}\right)\frac{1}{(Q^2)^{2+2\epsilon}}. \tag{13}$$

The (us-us)′ contribution can be easily evaluated in gamma functions,

$$C^{(2)}_{(\text{us-us})'} = \iint \frac{d^d k\, d^d l}{(-2p_1(k+l))(-2p_2(k+l))(-2p_1 k + m^2)(-2p_2 l + m^2)k^2 l^2} = \\ = \frac{1}{(Q^2)^{2-2\epsilon}(m^2)^{4\epsilon}}\left[\Gamma(\epsilon)\Gamma(2\epsilon)\Gamma(1-2\epsilon)\right]^2. \tag{14}$$

The 2c-h contribution is

$$C^{(2)}_{(\text{2c-h})} = \iint \frac{d^d k\, d^d l}{\left(l^2 - 2p_1 l + \frac{(2p_2 k)(2p_1 l)}{Q^2}\right)\left(l^2 - 2p_2(k+l) + \frac{(2p_2 k)(2p_1 l)}{Q^2}\right)} \times \\ \times \frac{1}{(k^2 - 2p_1 k)(l^2 - 2p_2 l)(k^2 - m^2)l^2}, \tag{15}$$

and the same leading order h-2c contribution is obtained by permuting k and l. Using the alpha parameters and the Mellin–Barnes representation (twice), we obtain

$$C^{(2)}_{(\text{h-2c})} = C^{(2)}_{(\text{2c-h})} = \left(-\frac{3}{\epsilon^4} + \frac{\pi^2}{\epsilon^2} + \frac{22\zeta(3)}{\epsilon} + \frac{16\pi^4}{45}\right)\frac{1}{(Q^2)^{2+\epsilon}(m^2)^\epsilon}. \tag{16}$$

The 1c-1c contribution is

$$C^{(2)}_{(\text{1c-1c})} = \iint \frac{d^dk\, d^dl}{(-2p_1(k+l))((k+l)^2 - 2p_2(k+l))} \frac{1}{(-2p_1k)(l^2 - 2p_2l)(k^2 - m^2)(l^2 - m^2)}, \tag{17}$$

and the 2c-2c contribution is obtained by permuting k and l. We must also consider similar (1c-1c)′ and (2c-2c)′ contributions with the second choice of the loop momenta. The corresponding expressions are obtained by permuting the masses (see above). The fifth nonzero contribution of the collinear-collinear type originates from the 2c-1c region. These contributions are dimensionally regularized only in the sum. It is convenient to introduce an auxiliary analytic regularization into lines 3 and 4 as

$$\frac{1}{(k^2 - 2p_1k)^{1+x_1}(l^2 - 2p_2l)^{1+x_2}}.$$

In contrast to the planar two-loop diagram calculated in this limit [5], we obtain poles in x_i up to the second order. In particular, the 2c-1c contribution is evaluated in gamma functions for general ϵ as

$$\begin{aligned} C^{(1)}_{(\text{2c-1c})} &= \iint \frac{d^dk\, d^dl}{\left(-2p_1l + \frac{(2p_2k)(2p_1l)}{Q^2}\right)\left(-2p_2k + \frac{(2p_2k)(2p_1l)}{Q^2}\right)} \times \\ &\times \frac{1}{(k^2 - 2p_1k)(l^2 - 2p_2l)(k^2 - m^2)(l^2 - m^2)} = \\ &= \frac{\Gamma(x_1)\Gamma(x_2)\Gamma(-x_1-\epsilon)\Gamma(-x_2-\epsilon)\Gamma(x_1+\epsilon)\Gamma(x_2+\epsilon)}{\Gamma(1+x_1)\Gamma(1+x_2)\Gamma(-\epsilon)^2(-m^2)^{x_1+x_2+2\epsilon}(Q^2)^2}. \end{aligned} \tag{18}$$

Applying the technique of alpha parameters and the Mellin–Barnes representation to the other four e-e contributions, we obtain an expansion in x_i for each of them. We then switch off the analytic regularization (first $x_2 \to x_1$ and then $x_1 \to 0$), observe that the singular dependence in x_i drops out in the sum of all five contributions, and obtain the following result in the ϵ expansion:

$$\begin{aligned} (Q^2)^2\left[C^{(2)}_{(\text{1c-1c})} + C^{(2)}_{(\text{2c-2c})} + C^{(2)}_{(\text{1c-1c})'} + C^{(2)}_{(\text{2c-2c})'} + C^{(2)}_{(\text{2c-1c})}\right] &= \\ = \frac{19}{4\epsilon^4} - \frac{9}{2\epsilon^3}L + \left(L^2 - \frac{11\pi^2}{4}\right)\frac{1}{2\epsilon^2} - \left(\frac{3\pi^2}{4}L + \frac{97\zeta(3)}{6}\right)\frac{1}{\epsilon} + \frac{\pi^2}{12}L^2 + 9\zeta(3)L - \frac{23\pi^4}{32}&, \end{aligned} \tag{19}$$

where $L = \log(Q^2/m^2)$ and we set $m = 1$ for brevity.

Collecting all the leading-order contributions, we see that the poles in ϵ cancel, and we obtain

$$(Q^2)^2 F_2(Q, m, 0) \overset{Q\to\infty}{\sim} \frac{7}{12}L^4 - \frac{\pi^2}{2}L^2 + 20\zeta(3)L - \frac{31\pi^4}{180}. \tag{20}$$

It is possible to extend this result to any order in $1/Q^2$ at the expense of computer algebra.

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