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CRITICAL PHENOMENA IN THE FERMIONIC HIERARCHICAL MODEL

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The dynamics of the renormalization-group transformation in the coupling-constant space of the fermionic hierarchical model are discussed. The critical behavior of this model is described in terms of the complex behavior of the Grassmann-valued mean-spin distribution density with the proper normalization. Some critical indices are calculated.

1. Introduction

In [1–3], the renormalization-group (RG) study of the hierarchical fermionic model was started. In particular, the reduction of the Kadanoff–Wilson RG transformation to a rational transformation in the coupling-constant plane was shown, the RG transformation stable points and stable invariant curves passing through these points were described, and the global RG dynamics in some domains of the coupling-constant plane were investigated. In the present paper, we study the RG transformation dynamics in depth and describe critical phenomena of this model.

We recall the main definitions. A hierarchical lattice is the set of natural numbers N endowed with a hierarchical distance $d(i,j)$, $i,j \in N$, where $d(i,j) = n^{s(i,j)}$ if $i \neq j$. Here, $s(i,j) = \min\{s : \exists k : i \in V_{k,s}, j \in V_{k,s}\}$, $V_{k,s} = \{j : j \in N,\ (k-1)n^s < j \leq kn^s\}$, and n is a fixed natural number. Four-component spins $\psi^*(i) = (\bar\psi_1(i), \psi_1(i), \bar\psi_2(i), \psi_2(i))$, whose components are generators of the Grassmann algebra, are placed at the sites of this lattice.

We recall (see [1]) that we consider a fermionic field on the Grassmann subalgebra A_N, which is generated by $4 \cdot n^N$ generators $(\psi_1(i), \bar\psi_1(i), \psi_2(i), \bar\psi_2(i))$, $i \in \Lambda_N$; the field is determined in the volume $\Lambda_N \equiv V_{1,N}$ with the Gibbs state $\rho_N(r,g)$ (which also depends on the real parameter α). If $F(\psi^*) \in A_N$, then

$$\rho_N(r,g)\big(F(\psi^*)\big) = Z_N^{-1} \int F(\psi^*) e^{-H_N(\psi^*;r,g)} d\psi^*, \tag{1}$$

where

$$d\psi^* = \prod_{i\in\Lambda_N} d\psi_1(i)\, d\bar\psi_1(i)\, d\psi_2(i)\, d\bar\psi_2(i),$$

$$H_N(\psi^*;r,g) = H_{0,N}(\psi^*,\alpha) + \sum_{i\in\Lambda_N} L(\psi^*(i);r,g), \tag{2}$$

$$L(\psi^*(i);r,g) = r\big(\bar\psi_1(i)\psi_1(i) + \bar\psi_2(i)\psi_2(i)\big) + g\bar\psi_1(i)\psi_1(i)\bar\psi_2(i)\psi_2(i), \tag{3}$$

$$H_{0,N}(\psi^*;\alpha) = \sum_{i,j\in\Lambda_N} d_{0,N}(i,j)\big(\bar\psi_1(i)\psi_1(j) + \bar\psi_2(i)\psi_2(j)\big), \tag{4}$$

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$$d_{0,N}(i,j) = \frac{1-n^{\alpha-1}}{1-n^{-\alpha}} d^{-\alpha}(i,j) - \frac{(1-n^{\alpha-1})^2 n^{-\alpha(N+1)}}{(1-n^{-\alpha})(1-n^{-1})}, \quad i \neq j,$$

$$d_{0,N}(i,i) = \frac{1-n^{\alpha-2}}{1-n^{-1}} - \frac{(1-n^{\alpha-1})^2}{(1-n^{-\alpha})(1-n^{-1})} n^{-\alpha(N+1)}, \tag{5}$$

$$Z_N(r,g) = \int e^{-H_N(\psi^*;r,g)} d\psi^*.$$

The integration uses the superanalysis rules [4]. The RG transformation is

$$r(\alpha)\psi^*(i) = n^{-\alpha/2} \sum_{j \in V_{i,1}} \psi^*(j).$$

It was shown in [2] that

$$\rho_N(r,g)\big(F\big(r(\alpha)\psi^*\big)\big) = \rho_{N-1}(r',g')\big(F(\psi^*)\big).$$

In the coupling-constant space (r,g), the RG transformation acts as the rational mapping

$$r' = n^{\alpha-1}\left(\frac{(r+1)^2 - g}{(r+1)^2 - \frac{g}{n}}(r+1) - 1\right),$$

$$g' = n^{2\alpha-3}\left(\frac{(r+1)^2 - g}{(r+1)^2 - \frac{g}{n}}\right)^2 g, \tag{6}$$

which has nontrivial branches with the stable points $(r_\pm, g_\pm)$, where

$$r_\pm = \frac{\pm\sqrt{n} - n^{\alpha-1}}{1 \mp \sqrt{n}}, \qquad g_\pm = g_\pm(r_\pm), \tag{7}$$

$$g_\pm(r) = \frac{r(1+r)^2}{1+r \pm \frac{1}{\sqrt{n}}}. \tag{8}$$

It was shown in [3] that for $\alpha \geq 3/2$, the connection component of a stable RG-invariant curve that passes through the plus (minus) stable point (SP) is determined by a smooth, monotonically increasing (decreasing) function $g = h_+(r)$, $r > 0$ $\big(g = h_-(r)$, $r < -1\big)$. Then, $h_+(r) \to 0$ for $r \to 0$, $h_-(r) \to 0$ for $r \to -1$, $h_+(r) \to +\infty$ for $r \to +\infty$, and $h_-(r) \to +\infty$ for $r \to -\infty$. For $1 < \alpha \leq 3/2$, the curve $g = h_+(r)$ becomes a stable RG-invariant curve for the zero SP $(r = 0,\ g = 0)$.

The curve $g = h_+(r)$ lies in the domain

$$G_1 = \Big\{(r,g)\colon 0 \leq r \leq r_+,\ \max\{0, g_4(r),\ g_1^+(r)\} \leq g \leq g_+(r)\Big\} \cup$$
$$\cup \Big\{(r,g)\colon \max\{0, r_+\} \leq r,\ g_+(r) \leq g \leq g_1^+(r)\Big\},$$

and the curve $g = h_-(r)$ lies in the domain

$$G_2 = \big\{(r,g)\colon r \leq r_-,\ g_1^-(r) \leq g \leq g_-(r)\big\} \cup \big\{(r,g)\colon r_- < r < -1,\ g_-(r) < g < g_1^-(r)\big\},$$

where

$$g_1^{\pm}(r) = \frac{(r+1)(n \mp \sqrt{n}) - n + n^{2-\alpha}}{(r+1)(n \mp \sqrt{n}) - 1 + n^{1-\alpha}}(r+1)^2,$$
$$g_4(r) = \frac{r(1-n^{\alpha-2})}{r(1-n^{\alpha-2}) + 1 - n^{-1}}(r+1)^2. \tag{9}$$

Let γ_1 and γ_2 respectively denote the curves $g = h_+(r)$, $0 \le r < \infty$, and $g = h_-(r)$, $-\infty < r < -1$. In [2], it was shown that the upper half-plane domains to the right of γ_1 and to the left of γ_2 are RG invariant, and the asymptotic behavior of the RG-transformation iterations was found in these domains. These domains are denoted by Ω_1 and Ω_2 respectively.

In Sec. 2, the RG dynamics in the domain between the curves γ_1 and γ_2 are investigated for $2 > \alpha > 1$. We show that after a finite number of RG iterations with large enough g, the point (r,g) with $r \ge -(n^\alpha - n^{\alpha-1})(n^\alpha - 1)^{-1}$ enters the domain Ω_2 and with $r < -(n^\alpha - n^{\alpha-1})(n^\alpha - 1)^{-1}$ the domain Ω_1. Computer simulations showed that all other parts of stable invariant curves for the plus and minus SPs and the stable invariant curve for the infinitely distant SP are in the domain between γ_1 and γ_2. These curves slice the domain between γ_1 and γ_2 into an infinite number of connected domains, whose points enter either the domain Ω_1 or the domain Ω_2 under RG iteration; these domains are mixed in a fractal manner.

In Sec. 3, the global RG dynamics are used to describe critical phenomena in the model under consideration. The critical behavior results from the limiting behavior of the Grassmann-valued total-spin distribution density with the proper normalization,

$$q_N^{(a)}(x^*; r, g) = \rho(r,g)\delta\left(\frac{1}{n^{aN}} \sum_{i \in \Lambda_N} \psi^*(i) - x^*\right), \tag{10}$$

where $\rho(r,g)$ is the thermodynamic limit of the states $\rho_M(r,g)$ at $M \to \infty$, δ is the Grassmann delta function, and $x^* = (x_1, \bar{x}_1, x_2, \bar{x}_2)$ with x_1, $\bar{x}_1$, x_2, and $\bar{x}_2$ being Grassmann variables.

If $R^{N_0}(\alpha)(r,g)$ enters Ω_1 or Ω_2 for some $N_0 \ge 0$, where $R(\alpha)(r,g) = (r', g')$ is given by formulas (6), then

$$\lim_{N\to\infty} q_N^{1/2}(x^*; r, g) = \frac{1}{c_1^2(r,g)} e^{-c_1(r,g)(\bar{x}_1 x_1 + \bar{x}_2 x_2)}$$

in the normalization $a = 1/2$. Here, $c_1(r,g)$ is the limit

$$c_1^2(r,g) = \lim_{N\to\infty} r^{(N)} n^{-N(\alpha-1)},$$

where $r^{(N)}$ is determined by the relation $\left(r^{(N)}, g^{(N)}\right) = R^N(\alpha)(r,g)$. The existence of the constant $c_1(r,g)$ was proved in [2]. In other words, the limit is given by the "Gaussian" distribution; $c_1(r,g) > 0$ if (r,g) lies in the attraction zone of Ω_1, and $c_1(r,g) < 0$ if (r,g) lies in the attraction zone of Ω_2.

On a half-line (r,g), $g > 0$, a rearrangement of the limiting density occurs when g increases and $r > 0$ is fixed. At sufficiently small and sufficiently large g, the limiting density is Gaussian (with different signs of the coupling constant). A domain of values of g exists in which the limiting density becomes non-Gaussian infinitely many times (but on a set of zero measure), being Gaussian on a set of full measure on which the coupling-constant sign changes infinitely many times.

If (r,g) lies on the invariant curve γ_1 or γ_2, then we need the nonstandard normalization $a = \alpha/2$; the limiting behavior $q_N^{\alpha/2}(x^*; r, g)$ is then non-Gaussian and is determined by the limiting density that corresponds to the plus or minus SP respectively.

The singularity of the constant $c_1(r,g)$ at $g \to g_{\text{cr}}(r)$, where $\left(r, g_{\text{cr}}(r)\right)$ is the intersection point of the half-line (r,g), $g > 0$, with the invariant curve γ_1 (or γ_2), gives one more critical index of the model.

2. Renormalization-group dynamics in the domain between the curves γ_1 and γ_2

In what follows, together with the variables r and g, we use the notation

$$\beta = \beta(r,g) = \frac{g}{(r+1)^2},$$

$$\sigma = \sigma(r,g) = \frac{1-\beta(r,g)}{1-\frac{\beta(r,g)}{n}},$$

and $\lambda_1 = n^{\alpha-1}$, $\lambda_2 = n^{2\alpha-3}$. We note that λ_1 and λ_2 are the eigenvalues of the differential of the mapping $R(\alpha)$ at the zero point.

In [3] (Lemma 3), the following statement was proved: the parabola $g(r) = \beta(r+1)^2$, $\beta = \text{const}$, becomes the parabola $g'(r') = \beta(r'+\lambda_1)^2/n$ after one RG iteration; here, r' depends on r linearly, $r' = kr+b$, where $k > 0$ for $\beta < 1$ and $\beta > n$ and $k < 0$ for $1 < \beta < n$. Intersection points of the parabolas $g(r)$ and $g'(r)$ lie on the lines $r = r_\pm$.

Therefore, each point (r,g) such that $r > -1$ and $n > \beta(r,g) \geq 1$ passes to the domain $g < (r+\lambda_1)^2$, $r \leq -\lambda_1$, after one RG iteration. The invariant curve theorem implies that this point eventually enters the domain Ω_2.

We find where the points of the line $r = r_0$ go. The lemma follows.

Lemma 1. *The line $r = r_0$ passes to the curve*

$$g = \frac{r-\lambda_1 r_0}{r-\lambda_1 n r_0 - \lambda_1(n-1)}(r+\lambda_1)^2 \tag{11}$$

after one RG iteration. For $r_0 > -1$ or $r_0 < -1$, the points (r_0, g) such that $\beta(r_0,g) > n$ are mapped to that part of curve (11) for which $r > \lambda_1 n r_0 + \lambda_1(n-1)$ or $r < \lambda_1 n r_0 + \lambda_1(n-1)$ respectively. Furthermore, for $g \to +\infty$,

$$r'(r_0,g) \to \lambda_1 n r_0 + \lambda_1(n-1), \qquad g'(r_0,g) \to \infty.$$

In addition, the line $r = -1$ passes to the line $r = -\lambda_1$ while all other assertions of Lemma 1 remain valid.

Lemma 1 implies that for

$$r \geq -\frac{\lambda_1(n-1)}{\lambda_1 n - 1}, \tag{12}$$

the points (r,g), $\beta(r,g) > n$, move right in the r direction: $r'(r,g) > r$. The lemma follows.

Lemma 2. *If r satisfies condition (12) and $\beta(r,g) > n$, then the point (r,g) enters the domain Ω_2 after a finite number of RG iterations.*

Proof. We use the relation

$$\beta' = \beta(r',g') = \frac{\beta}{n}\left(1 - \frac{\lambda_1 - 1}{\lambda_1\sigma(r+1)}\right)^{-2}. \tag{13}$$

For $\beta > n$, we have $\sigma = (1-\beta)(1-\beta/n)^{-1} > n$, and inequality (12) implies

$$1 > \frac{\lambda_1 - 1}{\lambda_1\sigma(r+1)},$$

whence $\beta' > 1$. Furthermore, by virtue of Eq. (13), we obtain

$$\beta' < \frac{3\beta}{2n} \tag{14}$$

at sufficiently large r. Because we obtain

$$r' = \lambda_1 \left(\frac{1-\beta}{1-\frac{\beta}{n}}(r+1) - 1 \right) > \lambda_1\bigl(n(r+1)-1\bigr)$$

for $\beta > n$, the number k exists such that either the point $\bigl(r^{(k)}, g^{(k)}\bigr)$ enters the domain

$$(r+1)^2 < g < n(r+1)^2, \qquad r > -1, \tag{15}$$

consequently entering the domain Ω_2, or this point enters the domain where inequality (14) holds, therefore entering domain (15) after a few additional iterations.

We now consider the case $r < -\lambda_1(n-1)(\lambda_1 n - 1)^{-1}$. In the sequel, we need the family of curves $g = f(r;a)$, where

$$f(r;a) = \frac{r}{r-a}(r+1)^2. \tag{16}$$

In particular, $f(r;a) = g_+(r)$ for $a = -(1+n^{-1/2})$. Direct calculation shows that the RG image of the curve $g = f(r;a)$ is the curve $g = f'(r;a)$, where

$$f'(r;a) = \frac{r(r+\lambda_1)^2}{r - \lambda_1 n a - \lambda_1(n-1)}. \tag{17}$$

We prove the following assertion.

Lemma 3. *Let $r < -(1+n^{-1/2})$ and $g \geq g_+(r)$. Then the point (r,g) enters the domain Ω_1 after a finite number of iterations.*

Proof. First, we note that a point that satisfies the lemma conditions lies on a curve $g = f(r;a)$, $a < -(1+n^{-1/2})$. Let $r_1(a)$ denote the abscissa of the intersection point of the curve $g = f(r;a)$ with the parabola $g = n(r+1)^2$,

$$r_1(a) = \frac{an}{n-1}. \tag{18}$$

The relation

$$r' = -\lambda_1\bigl(1 + r_1(a)\bigr)\frac{r}{r - r_1(a)} \tag{19}$$

implies that the part of the curve $g = f(r;a)$ with $r < r_1(a)$ is mapped to the part of the curve $g = f'(r;a)$ with $r > r_2(a)$, where $r_2(a) = -\lambda_1\bigl(1 + r_1(a)\bigr)$, under the RG transformation.

For $r > 0$, the inequality $f'(r;a_1) > f'(r;a_2) > 0$ holds for $a_2 < a_1 < 1/n - 1$, and $r_2(a_2) > r_2(a_1)$ for $a_1 > a_2$. Therefore, if we prove that the part of the curve

$$g = f'\bigl(r; -(1+n^{-1/2})\bigr), \qquad r > r_2\bigl(-(1+n^{-1/2})\bigr),$$

lies to the right of the curve γ_1, then the same is true for the part $r > r_2(a)$ of the curve $g = f'(r;a)$ if $a < -(1+n^{-1/2})$.

Because $f\big(r;-(1+n^{-1/2})\big)=g_+(r)$, we have

$$f'\big(r;-(1+n^{-1/2})\big)=g'_+(r)=\frac{r}{r+\lambda_1(1+n^{1/2})}(r+\lambda_1)^2.$$

We thus obtain the decomposition

$$g_+(r)-g'_+(r)=\frac{(n^{1/2}-1)(\lambda_1+n^{-1/2})}{(r+1+n^{-1/2})\big(r+\lambda_1(1+n^{1/2})\big)}\times$$
$$\times\left(r+\frac{n^{1/2}-\lambda_1}{n^{1/2}-1}\right)\left(r+(1+n^{-1/2})\frac{\lambda_1}{\lambda_1+n^{-1/2}}\right). \tag{20}$$

Because

$$r_2\big(-(1+n^{-1/2})\big)=\lambda_1(n^{-1/2}-1)^{-1}>r_+(\alpha)=(\lambda_1-n^{1/2})(n^{1/2}-1)^{-1},$$

(20) implies that for $r>r_2\big(-(1+n^{-1/2})\big)$, we have $g_+(r)>g'_+(r)$, which means that $g'_+(r)$ lies to the right of γ_1.

Therefore, we have proved that any point from the domain

$$A_1=\big\{(r,g)\colon r<r_1\big(-(1+n^{-1/2})\big),\ f\big(r;-(1+n^{-1/2})\big)<g<n(r+1)^2\big\}$$

enters the domain Ω_1 after one RG iteration.

We now consider points lying in the domain

$$A_2=\big\{(r,g)\colon r<-(1+n^{-1/2}),\ g>\max\big(n(r+1)^2,\ f\big(r;-(1+n^{-1/2})\big)\big)\big\}.$$

The RG iteration maps the part $r_1(a)<r<a$ of the curve $g=f(r;a)$ to the part $r<\lambda_1(na+n-1)$ of the curve $g=f'(r;a)$, and $f'(r;a)>f(r;a)$ for $r<\lambda_1(na+n-1)$. This fact together with formula (13) implies that any point from the domain A_2 enters the domain A_1 after a finite number of RG iterations.

We note that the RG transformation formulas do not define this transformation for the points of the parabola $g=n(r+1)^2$. This restriction can be overcome if we can define the RG action in the space of expansion coefficients of the Grassmann-valued unit-spin distribution function. A nonnormalized spin density is

$$P(\bar\psi_1,\psi_1,\bar\psi_2,\psi_2)=c_0+c_1(\bar\psi_1\psi_1+\bar\psi_2\psi_2)+c_2(\bar\psi_1\psi_1\bar\psi_2\psi_2),$$

where $\bar\psi_1$, ψ_1, $\bar\psi_2$, and ψ_2 are the generators of the Grassmann algebra describing the spin at the fixed lattice site and (c_0,c_1,c_2) is a three-dimensional real-valued vector. In particular, the coefficient set $(1,-r,r^2-g)$ corresponds to the density

$$e^{-r(\bar\psi_1\psi_1+\bar\psi_2\psi_2)-g\bar\psi_1\psi_1\bar\psi_2\psi_2},$$

and the set $(0,0,1)$ corresponds to the δ-function $\delta(\psi)=\bar\psi_1\psi_1\bar\psi_2\psi_2$. Describing the density as a point in the two-dimensional projective space (c_0,c_1,c_2), we can prove that the RG action in the space of constants (c_0,c_1,c_2) is determined by

$$\begin{aligned}
c'_0&=(c_1-c_0)^2-\frac{1}{n}(c_1^2-c_0c_2),\\
c'_1&=\lambda_1\left((c_1-c_0)(c_2-c_1)-\frac{1}{n}(c_1^2-c_0c_2)\right),\\
c'_2&=\lambda_1^2\left((c_2-c_1)^2-\frac{1}{n}(c_1^2-c_0c_2)\right).
\end{aligned} \tag{21}$$

Considered as a transformation of P^2 onto itself, this transformation is determined everywhere except the point (1,1,1). In the new coordinates, the parabola $g = n(r+1)^2$ is the curve $\left(1, -r, r^2 - n(r+1)^2\right)$, $r \neq -1$, which becomes the curve

$$\left(0, 1, \lambda_1(n-1)\left(r + \frac{n+1}{n-1}\right)\right) \tag{22}$$

after one iteration of transformation (21). This curve cannot be described in the coordinates (r, g). However, after one more iteration of transformation (21), curve (22) becomes the curve

$$\left(1, \lambda_1 \frac{t(r) - 1 - n^{-1}}{1 - n^{-1}}, \lambda_1^2 \frac{\left(t(r) - 1\right)^2 - n^{-1}}{1 - n^{-1}}\right),$$

where $t(r) = \lambda_1(n-1)\left(r + (n+1)(n-1)^{-1}\right)$. Returning to the coordinates (r, g), we therefore see that two iterations of the RG transformation transform the parabola $g = n(r+1)^2$ into the parabola

$$g^{(2)} = \frac{1}{n}\left(r^{(2)} + \lambda_1\right)^2,$$

$$r^{(2)} = -\lambda_1^2 nr - \lambda_1(\lambda_1 n - 1)\frac{n+1}{n-1}.$$

Under the RG transformation thus defined, the parabola part $g = n(r+1)^2$, $r < r_1\left(-(1 + n^{-1/2})\right)$, enters the domain Ω_1 after two RG iterations.

This completes the proof of Lemma 3.

As for the domain $-(1 + n^{-1/2}) < r < -\lambda_1(n-1)(\lambda_1 n - 1)^{-1}$, we formulate the following lemma.

Lemma 4. *If $-(1 + n^{-1/2}) \leq r < -\lambda_1(n-1)(\lambda_1 n - 1)^{-1}$ and g is large enough, then the point (r, g) enters the domain Ω_1 after a finite number of RG iterations.*

Proof. Let $s(t) = \left(r(t), g(t)\right)$ be a continuous curve parameterized by the parameter t, $0 < t < b$, with the vertical asymptote $r = r_0$: $r(t) \to r_0$, $g(t) \to \infty$ for $t \to 0$. Also let the curve $s(t)$ lie in the domain $g > n(r+1)^2$. Then the RG image of this curve, $f'(t) = \left(r'(t), g'(t)\right)$, is also a continuous curve in t with the asymptote $r = r_1(0)$: $r'(t) \to r_1$, $g'(t) \to \infty$ for $t \to 0$. Here, $r_1 = \lambda_1\left(n(r_0 + 1) - 1\right)$. We can choose $b_1 < b$ such that the curve $f'(t)$ lies in the domain $g > n(r+1)^2$ for $0 < t < b_1$. Let $r_k = \lambda_1\left(n(r_{k-1} + 1) - 1\right)$. If $r_0 < -\lambda_1(n-1)(\lambda_1 n - 1)^{-1}$, then after a finite number of steps k, the quantity r_k becomes less than $-(1 + n^{-1/2})$. Choosing the vertical line $r = r_0$ as the initial curve and applying the above reasoning several times, we obtain the assertion of the lemma.

We used a computer to investigate the RG dynamics in those upper half-plane domains where the assertions of the above lemmas are invalid. The results follow.

The point $(-1, 0)$ is the starting point of a curve that asymptotically tends to the line $r_0 = -\lambda_1(n-1)(\lambda_1 n - 1)^{-1}$, which, in turn, can be treated as the graph of a smooth, monotonically increasing function $g = h(r)$, $r_0 < -\lambda_1(n-1)(\lambda_1 n - 1)^{-1}$. This curve is a part of a stable invariant curve for the infinitely distant SP (given by the set $(0, 0, 1)$ in the projective coordinates). Let γ_∞ denote this curve. Let

$$\tilde{\gamma}_i = \bigcup_{n=0}^{\infty} R^{-n}\gamma_i, \quad i = 1, 2, \infty,$$

i.e., $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, and $\tilde{\gamma}_\infty$ are the whole stable curves for the plus, minus, and infinitely distant SPs respectively.

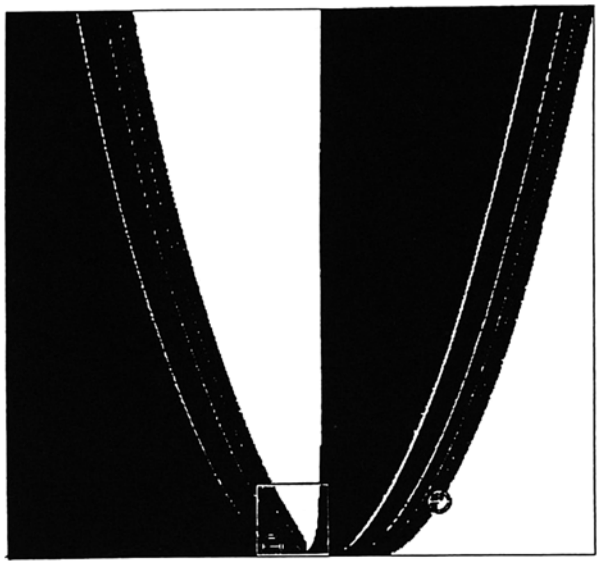

Fig. 1

In the projective coordinates, each curve γ_i is a smooth connected curve, which passes infinitely many times through the point $(-1,0)$ and its RG preimages $(-\lambda_1^{-k},0)$, $k=1,2,\dots$.

In the coordinates (r,g), these curves are not connected and are unifications of their connected parts, each of which passes through only one of the points $(-\lambda_1^{-k},0)$, $k=0,1,2,\dots$. We note that the RG transformation group is indeterminate at the point $(-1,0)$ and at its preimages, which must therefore be deleted from the curves $\tilde\gamma_i$, $i=1,2,\infty$.

The curves $\tilde\gamma_i$ are connected in the projective coordinates: the junction occurs on the set of points of the type $(0,c_1,c_2)$, which cannot be depicted in the (r,g) coordinates. Presumably, almost all upper half-plane points lying between the curves γ_1 and γ_2 and not belonging to the curves $\tilde\gamma_1$, $\tilde\gamma_2$, and $\tilde\gamma_\infty$ enter either the domain Ω_1 or the domain Ω_2 after a finite number of RG iterations. We consider the vertical line $r=r_0$, $g>0$. If $-\lambda_1(n-1)(\lambda_1 n-1)^{-1}<r_0<c_1$, $c_1<-\lambda_1^{-1}$, then the vertical line crosses the curve γ_∞ at a point (r_0,g_0) and is decomposed into the two intervals

$$I_1=\{r=r_0,0<g<g_0\} \qquad \text{and} \qquad I_2=\{r=r_0,g_0<g<\infty\}.$$

All the points from I_1 and I_2 enter the respective domains Ω_2 and Ω_1 after a finite number of RG iterations. For all other values of r_0, the half-line $r=r_0$, $g>0$, crosses each of the curves $\tilde\gamma_1$, $\tilde\gamma_2$, and $\tilde\gamma_\infty$ infinitely many times. If $r_0>0$, then the half-line $(r=r_0,\ g>0)$ is decomposed into the intervals $I_1=\{r=r_0,0<g<g_0(r_0)\}$, $I_2\{r=r_0,g_1(r_0)<g<\infty\}$, and infinitely many intervals into which the interval $(r=r_0,\ g_0(r_0)<g<g_1(r_0))$ is decomposed in a fractal manner. The points of I_1 belong to Ω_1, the points of I_2 enter the domain Ω_2 after a finite number of RG iterations, and the RG behavior alternates in the other intervals. For $r_0<-1$, we have an analogous picture with the only difference that the points of I_1 belong to Ω_2 and the points of I_2 enter the domain Ω_1 after a finite number of RG iterations. The same behavior occurs for $c_1<r_0<0$, but in this case, the points of both I_1 and I_2 enter the domain Ω_2 after a finite number of RG iterations.

Figures 1 and 2 illustrate this situation. In Fig. 1, the upper half-plane $g>0$ is depicted. It is colored in black and white following the rule that black points move left, i.e., to the domain Ω_2, and white points move right, i.e., to the domain Ω_1, under RG iterations. The plus and minus SPs are circled in Fig. 1. We assume that almost all points lying at the boundary separating black and white domains belong to the invariant curves $\tilde\gamma_1$, $\tilde\gamma_2$, and $\tilde\gamma_\infty$. A sufficiently small neighborhood of an invariant-curve point is divided

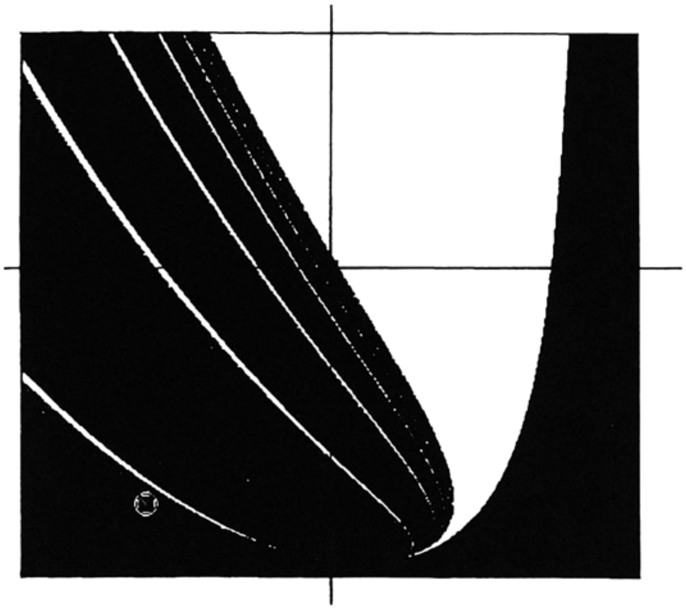

Fig. 2

by this curve into two parts. Then, the structures of these parts are different for different invariant curves. For the curve $\tilde{\gamma}_1$, one part of the neighborhood is entirely white (consists of white points), and the other part is mixed (contains both white and black points). For the curve $\tilde{\gamma}_\infty$, one part is white, and the other is black. For the curve $\tilde{\gamma}_2$, one part is black, and the other is mixed.

One can see the parts of the curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_\infty$ in Fig. 1. Parts of the curve $\tilde{\gamma}_2$ cannot be distinguished, but their existence becomes clear after amplifying the picture considerably. In Fig. 2, the amplified image of the square marked in Fig. 1 is given. The minus SP, which lies on the curve $\tilde{\gamma}_2$, is at the center of the small circle in Fig. 2. In the example depicted, the RG parameter $\alpha = 1.7$ and $n = 2$.

3. Critical phenomena

Critical phenomena in our model are related to the limiting behavior of the Grassmann-valued distribution "density" of the properly normalized total spin averaged w.r.t. the state ρ_N in the volume Λ_N.

Let $\tilde{q}_N^{(a)}(x^*; r, g)$ denote the distribution "density" of the normalized total spin

$$\psi^*_{N,a} = \frac{1}{n^{aN}} \sum_{i\in\Lambda_N} \psi^*(i)$$

for the field ψ^* in the state $\rho_N(\rho, g)$:

$$\tilde{q}_N^{(a)}(x^*; r, g) = \rho_N(r, g)\big(\delta(\psi^*_{N,a} - x^*)\big),$$

where $x^* = (x_1, \bar{x}_1, x_2, \bar{x}_2)$ (the δ-function for anticommuting variables is $\delta(\psi^*) = \psi_1\bar{\psi}_1\psi_2\bar{\psi}_2$).

The results of [1] imply

$$\tilde{q}_N^a(x^*; r, g) = \rho_N(r, g)\left(\delta\left(n^{\left(\frac{\alpha}{2}-a\right)N} n^{-\frac{\alpha}{2}N} \sum_{i\in\Lambda_N} \psi^*(i) - x^*\right)\right) =$$

$$= n^{4\left(\frac{\alpha}{2}-a\right)N} \tilde{q}_0^{(a)}\left(n^{\left(a-\frac{\alpha}{2}\right)N} x^*; r^{(N)}, g^{(N)}\right),$$

where $(r^{(N)}, g^{(N)}) = R^N(\alpha)(r,g)$ is the Nth iteration of the RG transformation $R(\alpha)$, $R(\alpha)(r,g) = (r', g')$, determined by relation (6).

Therefore, we can explicitly calculate $\tilde{q}_N^{(a)}(x^*; r, g)$ as

$$\tilde{q}_N^{(a)}(x^*; r, g) = \frac{n^{(2\alpha-4a)N}}{(r^N + c_0)^2 - g^N} \exp\{-L(x^*; n^{(2a-\alpha)N} r^{(N)}, n^{(4a-2\alpha)N} g^{(N)})\},$$

where (see formula (5))

$$c_0 = c_0(\alpha) \equiv d_{0,0}(i,i) = (1 - n^{\alpha-2})(1 - n^{-1})^{-1}.$$

Before stating the theorem, we recall that the existence of the constants $c_1(r,g), c_2(r,g), (r,g) \in \Omega_1 \cup \Omega_2$ such that

$$\lim_{N\to\infty} r^{(N)} n^{-N(\alpha-1)} = c_1(r,g), \qquad \lim_{N\to\infty} g^{(N)} n^{-N(2\alpha-3)} = c_2(r,g) \tag{23}$$

follows from Theorem 2 in [2]. Then, $c_1(r,g) > 0$ for $(r,g) \in \Omega_1$, $c_1(r,g) < 0$ for $(r,g) \in \Omega_2$, and $c_2(r,g) = 0$ only for $g = 0$. These results are valid for all $\alpha > 1$. Analogous relations hold for the lower half-plane as well.

If the spin distribution density admits the exponential representation

$$p(x^*; r, g) = \frac{1}{r^2 - g} e^{-L(x^*; r, g)},$$

then we can say

$$\lim_{m\to\infty} p(x^*; r_m, g_m) = p(x^*; r, g),$$

if $\lim_{m\to\infty}(r_m, g_m) = (r, g)$.

Theorem 1. *Let $2 > \alpha > 1$. If $(r^{(N_0)}, g^{(N_0)}) \in \Omega_1 \cup \Omega_2$ for some $N_0 \geq 0$, then*

$$\lim_{N\to\infty} \tilde{q}_N^{(1/2)}(x^*; r, g) = p(x^*; c_1(r,g), 0).$$

If $(r,g) \in \tilde{\gamma}_2$, then

$$\lim_{N\to\infty} \tilde{q}_N^{(\alpha/2)}(x^*; r, g) = p(x^*; c_0(\alpha) + r_-(\alpha), g_-(\alpha)).$$

If $(r,g) \in \tilde{\gamma}_1$, then for $\alpha > 3/2$,

$$\lim_{N\to\infty} \tilde{q}_N^{(\alpha/2)}(x^*; r, g) = p(x^*; c_0(\alpha) + r_+(\alpha), g_+(\alpha)),$$

and for $3/2 \geq \alpha > 1$,

$$\lim_{N\to\infty} \tilde{q}_N^{(\alpha/2)}(x^*; r, g) = p(x^*; c_0(\alpha)).$$

Here, $r_\pm(\alpha)$ and $g_\pm(\alpha)$ are determined by formulas (7) and (8) respectively.

Proof. The proof of the first assertion of Theorem 1 follows from asymptotic behavior (23). The other statements follow because $\tilde{\gamma}_2$ is a stable invariant curve for the minus SP and $\tilde{\gamma}_1$ is the stable invariant curve for the plus SP at $\alpha > 3/2$ and for the trivial SP at $1 < \alpha \leq 3/2$.

We now demonstrate that analogous results can be obtained for the total-spin distribution density in the volume Λ_N, averaged w.r.t. the state that corresponds to the thermodynamic limit and belongs to an infinite-dimensional Grassmann algebra generated by all spins of the hierarchical lattice.

Let, for instance, $\big(r^{(N_0)}, g^{(N_0)}\big) \in \Omega_1 \cup \Omega_2$ for some $N_0 \geq 0$. The results in [2] imply that the thermodynamic limit exists in our model for $\alpha > 1$, which means that limits of all correlation functions $\rho_N\big(F(\psi^*)\big)$ exist as $N \to \infty$. The proof follows from the existence of limits of one-point correlation functions

$$u_N^1(r,g) = \rho_N(r,g)\big(\psi_1(i)\bar{\psi}_1(i) + \psi_2(i)\bar{\psi}_2(i)\big),$$

$$u_N^2(r,g) = \rho_N(r,g)\big(-\psi_1(i)\bar{\psi}_1(i)\psi_2(i)\bar{\psi}_2(i)\big), \quad i \in \Lambda_N.$$

Let

$$u_N(r,g) = \begin{pmatrix} u_N^1(r,g) \\ u_N^2(r,g) \end{pmatrix}.$$

It was proved in [2] that the limit

$$u(r,g) = \lim_{N\to\infty} u_N(r,g)$$

exists. Moreover, in the case where (r,g) enters Ω_1 after a number of iterations, the following representation is valid [2]:

$$u(r,g) = \sum_{i=0}^{\infty} \prod_{k=0}^{i-1} A\big(r^{(k)}, g^{(k)}\big) s\big(r^{(i)}, g^{(i)}\big), \tag{24}$$

where

$$A(r,g) = \frac{1}{n} \begin{pmatrix} \dfrac{\partial r'}{\partial r} & \dfrac{\partial g'}{\partial r} \\ \dfrac{\partial r'}{\partial g} & \dfrac{\partial g'}{\partial g} \end{pmatrix}, \qquad s(r,g) = \frac{1}{n} \begin{pmatrix} \dfrac{\partial \log c(r,g)}{\partial r} \\ \dfrac{\partial \log c(r,g)}{\partial g} \end{pmatrix}, \tag{25}$$

$$c(r,g) = \big((r+1)^2 - g\big)^{n-2} \left((r+1)^2 - \frac{g}{n}\right). \tag{26}$$

Let $\rho = \lim_{M\to\infty} \rho_M$, and let $q_N^{(a)}(x^*)$ denote the distribution density of the total spin $\psi^*_{N,a}$ calculated w.r.t. the state ρ,

$$q_N^{(a)} = \rho\big(\delta(\psi^*_{N,a} - x^*)\big).$$

We can easily obtain the representation

$$q_N^{(1/2)}(x^*) = -n^{2(\alpha-1)N} u_2\big(r^{(N)}, g^{(N)}\big) - \frac{1}{2} n^{(\alpha-1)N} u_1\big(r^{(N)}, g^{(N)}\big)(\bar{x}_1 x_1 + \bar{x}_2 x_2) + \bar{x}_1 x_1 \bar{x}_2 x_2.$$

Therefore, the limiting behavior of the density $q_N^{(1/2)}(x^*)$ is the same as the limiting behavior of the vector $D_N u\big(r^{(N)}, g^{(N)}\big)$, where

$$D_N = \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_1^{2N} \end{pmatrix}.$$

From formula (24), we obtain the representation

$$D_N u\big(r^{(N)}, g^{(N)}\big) = \sum_{i=0}^{\infty} \prod_{k=0}^{i-1} T_N\big(r^{(N+k)}, g^{(N+k)}\big) D_N s\big(r^{(N+i)}, g^{(N+i)}\big),$$

where

$$T_N\left(r^{(N+k)}, g^{(N+k)}\right) = D_N A\left(r^{(N+k)}, g^{(N+k)}\right) D_N^{-1}.$$

A convenient representation for the matrix A is

$$A(r,g) = \begin{pmatrix} \frac{2n^{\alpha-2}}{n-1}\left(\sigma^2 - \sigma\frac{n+3}{2} + n\right) & 4\frac{n^{2\alpha-3}}{n-1}(r+1)\sigma(1-\sigma)^2 \\ -\frac{n^{\alpha-1}}{n-1}\cdot\frac{1}{r+1}\cdot\left(1-\frac{\sigma}{n}\right)^2 & -\frac{2n^{2\alpha-4}}{n-1}\sigma\left(\sigma^2 - \sigma^3\frac{n+1}{2} + n\right) \end{pmatrix},$$

where

$$\sigma = \sigma(r,g) = \frac{(r+1)^2 - g}{(r+1)^2 - \frac{g}{n}}.$$

Using asymptotic behavior (23) (found in [2]) and the fact $1 - \sigma\left(r^{(N)}, g^{(N)}\right) = o(n^{-N/2})$, we obtain

$$\lim_{N\to\infty} T_N\left(r^{(N+k)}, g^{(N+k)}\right) = \begin{pmatrix} n^{\alpha-2} & 0 \\ -\left(1-\frac{1}{n}\right)\frac{n^{\alpha-2}}{c_1(r,g)}n^{-k(\alpha-1)} & n^{2(\alpha-2)} \end{pmatrix}, \tag{27}$$

$$\lim_{N\to\infty} D_N s\left(r^{(N+i)}, g^{(N+i)}\right) = \begin{pmatrix} \frac{2}{c_1(r,g)}\left(1-\frac{1}{n}\right)n^{-i(\alpha-1)} \\ -\left(\left(1-\frac{1}{n}\right)\frac{1}{c_1(r,g)}n^{-i(\alpha-1)}\right)^2 \end{pmatrix}. \tag{28}$$

By virtue of (27) and (28), we find

$$\lim_{N\to\infty} D_N u\left(r^{(N)}, g^{(N)}\right) = \begin{pmatrix} \dfrac{2}{c_1(r,g)} \\ -\dfrac{1}{c_1^2(r,g)} \end{pmatrix}.$$

Therefore, we have

$$\lim_{N\to\infty} q_N^{(1/2)}(x^*) = \frac{1}{c_1^2(r,g)} - \frac{1}{c_1(r,g)}(\bar{x}_1 x_1 + \bar{x}_2 x_2 + \bar{x}_1 x_1 \bar{x}_2 x_2) =$$
$$= \frac{1}{c_1^2(r,g)}\, e^{-c_1(r,g)(\bar{x}_1 x_1 + \bar{x}_2 x_2)},$$

i.e., the limiting case of the densities $q_N^{(1/2)}(x^*)$ is the Gaussian density with the "variance" $2c_1(r,g)$.

We now suppose that the point (r,g) lies on the invariant curve $\tilde{\gamma}_1$. The restrictions on α and n under which the thermodynamic limit of our model exists in the SP of the RG transformation were found in [1]. It was shown that if the spectrum of the matrix $A(r_+, g_+)$ or $A(r_-, g_-)$ lies inside the unit circle, then the respective thermodynamic limit in the plus or minus SP exists.

Under the same restrictions on α and n, the thermodynamic limit exists for the invariant curve points $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. The representation

$$u_N(r,g) = \sum_{i=0}^{N-1}\prod_{k=0}^{i-1} A\left(r^{(k)}, g^{(k)}\right) s\left(r^{(i)}, g^{(i)}\right) + \prod_{k=0}^{N-1} A\left(r^{(k)}, g^{(k)}\right) u_0\left(r^{(N)}, g^{(N)}\right),$$

where $A(r,g)$ and $s(r,g)$ are determined by Eqs. (25) and (26) respectively and

$$u_0(r,g) = \begin{pmatrix} \dfrac{\partial \log Z_0(r,g)}{\partial r} \\ \dfrac{\partial \log Z_0(r,g)}{\partial g} \end{pmatrix}, \qquad Z_0(r,g) = \left(\left(r + \frac{1-n^{\alpha-2}}{1+n^{-1}} \right)^2 - g \right),$$

was obtained in [2].

For definiteness, we consider the plus SP. If the eigenvalues of the matrix $A(r_+, g_+)$ are inside the unit circle, then the same is true for the matrix $A(r,g)$ given (r,g) sufficiently close to (r_+, g_+). If the point (r,g) belongs to the curve $\tilde{\gamma}_1$, then the spectrum of the matrix $A(r^{(k)}, g^{(k)})$ starting with some k also belongs to a circle of radius $\delta < 1$.

Because the vectors $s(r^{(i)}, g^{(i)})$, $i = 1, 2, \ldots$, and $u_0(r^{(N)}, g^{(N)})$, $N = 1, 2, \ldots$, are bounded in the norm by a constant, the limit $\lim_{N\to\infty} u_N(r,g) = u(r,g)$ exists and $u(r,g)$ also admits representation (24).

Considering the total-spin density normalized to $\alpha/2$, we obtain

$$q_N^{(\alpha/2)}(x^*) = -u_2(r^{(N)}, g^{(N)}) - \frac{1}{2} u_1(r^{(N)}, g^{(N)})(\bar{x}_1 x_1 + \bar{x}_2 x_2) + \bar{x}_1 x_1 \bar{x}_2 x_2.$$

The expansion

$$u(r^{(N)}, g^{(N)}) = \sum_{i=0}^{\infty} \prod_{k=0}^{i-1} A(r^{(N+k)}, g^{(N+k)}) s(r^{(N+i)}, g^{(N+i)})$$

and the limiting relations

$$A(r^{(N+k)}, g^{(N+k)}) \underset{N\to\infty}{\longrightarrow} A(r_+, g_+), \qquad s(r^{(N+i)}, g^{(N+i)}) \underset{N\to\infty}{\longrightarrow} s(r_+, g_+)$$

yield

$$u(r^{(N)}, g^{(N)}) \underset{N\to\infty}{\longrightarrow} u(r_+, g_+).$$

Therefore, we have proved that $\lim_{N\to\infty} q_N^{(\alpha/2)}(x^*)$ exists and is a non-Gaussian density determined by the coupling constants r_+ and g_+,

$$p_+(x^*; \alpha) = -u_2(r_+, g_+) - \frac{1}{2} u_1(r_+, g_+)(\bar{x}_1 x_1 + \bar{x}_2 x_2) + \bar{x}_1 x_1 \bar{x}_2 x_2,$$

where

$$u(r_+, g_+) = \sum_{i=0}^{\infty} (A(r_+, g_+))^i S(r_+, g_+) = (I - A(r_+, g_+))^{-1} S(r_+, g_+)$$

and I is the unit matrix. Analogous reasoning is valid when (r,g) belongs to the stable invariant curve $\tilde{\gamma}_2$. Then, the limiting density $p_-(x^*, \alpha)$ is determined by the vector $u(r_-, g_-)$, which can be found from the relation

$$u(r_-, g_-) = (I - A(r_-, g_-))^{-1} S(r_-, g_-).$$

The vectors $S(r_\pm, g_\pm)$ and matrices $A(r_\pm, g_\pm)$ can be calculated explicitly,

$$S(r_\pm, g_\pm) = \begin{pmatrix} \dfrac{2}{\sqrt{n} \pm 1} \dfrac{(n-2\pm\theta)\left(1 \mp \dfrac{\theta}{n}\right)}{\sqrt{n} - \theta} \\ \mp \dfrac{\sqrt{n} \mp 1}{\sqrt{n} \pm 1} \dfrac{\theta\left(n - 2 \pm \dfrac{\theta}{n}\right)\left(1 \mp \dfrac{\theta}{n}\right)}{(\theta - \sqrt{n})^2} \end{pmatrix},$$

$$A(r_\pm, g_\pm) = \begin{pmatrix} \dfrac{2}{(n-1)\sqrt{n}\theta}\left(\theta^2 \mp \theta\dfrac{n+3}{2} + n\right) & \pm\dfrac{4(\sqrt{n}-\theta)(1\mp\theta)^2}{\theta^2(n-1)(\pm\sqrt{n}-1)} \\ \mp\dfrac{\sqrt{n}(\sqrt{n}\mp 1)}{(n-1)(\sqrt{n}-\theta)}\left(1 \mp \dfrac{\theta}{n}\right)^2 & \mp\dfrac{2}{\theta n(n-1)}\left(\theta^2 \mp \theta\dfrac{3n+1}{2} + n\right) \end{pmatrix}. \tag{29}$$

The following theorem holds.

Theorem 2. *Let $2 > \alpha > 1$. If $\left(r^{(N_0)}, g^{(N_0)}\right) \in \Omega_1 \cup \Omega_2$ for some $N_0 \geq 0$, then*

$$\lim_{N\to\infty} q_N^{(1/2)}(x^*; r, g) = p\big(x^*; c_1(r,g), 0\big).$$

If $(r,g) \in \tilde{\gamma}_2$, then

$$\lim_{N\to\infty} q_N^{(\alpha/2)}(x^*; r, g) = p_-(x^*; \alpha),$$

and if $(r,g) \in \tilde{\gamma}_1$, then for $\alpha > 3/2$,

$$\lim_{N\to\infty} q_N^{(\alpha/2)}(x^*; r, g) = p_+(x^*; \alpha),$$

and for $1 < \alpha \leq 3/2$,

$$\lim_{N\to\infty} q_N^{(\alpha/2)}(x^*; r, g) = p\big(x^*; c_0(\alpha), 0\big),$$

where the densities $p_\pm(x^*;\alpha)$ are determined by the formulas

$$p_\pm(x^*;\alpha) = -u_2(r_\pm, g_\pm) - \frac{1}{2}u_1(r_\pm, g_\pm)(\bar{x}_1 x_1 + \bar{x}_2 x_2) + \bar{x}_1 x_1 \bar{x}_2 x_2$$

with

$$u_1(r_\pm, g_\pm) = \frac{1}{\Delta}\frac{2(n-1)\left(1\mp\frac{\theta}{n}\right)^2}{(\sqrt{n}\pm 1)(\sqrt{n}-\theta)},$$

$$u_2(r_\pm, g_\pm) = \mp\frac{1}{\Delta}\frac{(\sqrt{n}\mp 1)^2\left(1\mp\frac{\theta}{n}\right)\theta}{\sqrt{n}(\sqrt{n}\pm 1)(\sqrt{n}-\theta)^2}\left(n \pm 2\sqrt{n} - \theta\frac{2\sqrt{n}\pm 1}{n}\right), \quad \theta = n^{3/2-\alpha},$$

$$\Delta = 1 \pm n^{-3/2} - \frac{2n^{-3/2}}{1 \pm n^{-1/2})\theta}\left(\theta^2 \mp \theta\frac{n}{2}(1\mp n^{-1/2})^2 + n\right).$$

Therefore, in the case where (r,g) belongs to either the invariant curve $\tilde{\gamma}_1$ or the invariant curve $\tilde{\gamma}_2$, the limiting behavior of the total normalized spin $\psi^*_{N,\alpha/2}$ is non-Gaussian, only slightly different from Theorem 1.

Theorem 2 gives the following picture of the critical behavior. We fix the value of r_0. For definiteness, we set $r_0 > 0$ and $\alpha > 3/2$. We consider the half-line (r_0, g), $g > 0$, and let (r_0, g_{cr}) denote the point where this half-line crosses the curve γ_1. Then for $0 < g < g_{\text{cr}}$, the total normalized spin in the volume Λ_N has the "Gaussian" distribution with the "variance"

$$\chi(r_0, g) = \int (x_1 \bar{x}_1 + \bar{x}_2 x_2) p\big(x^*; c_1(r_0, g), 0\big)\, dx^* = \frac{2}{c_1(r, g)}$$

in the limit $N \to \infty$. We note that $\chi(r, g)$ can be also written as

$$\chi(r, g) = \lim_{N \to \infty} \rho(r, g) \frac{1}{n^N} \Big[\sum_{i \in \Lambda_N} \psi_1(i) \sum_{j \in \Lambda_N} \bar{\psi}_1(j) + \sum_{i \in \Lambda_N} \psi_2(i) \sum_{j \in \Lambda_N} \bar{\psi}_2(j) \Big]$$

and this quantity is therefore an anticommuting analogue of the susceptibility.

For $g = g_{\text{cr}}$, the total spin density has a nontrivial limit for the normalization factor $a = \alpha/2$, and this limit is determined by the plus "non-Gaussian" SP of the RG transformation $p_+(x^*; \alpha)$. For large enough g, for instance, for $g > (r_0 + 1)^2$ (as follows from the results in Sec. 2), the total spin with the normalization factor $a = 1/2$ has the limiting "Gaussian" distribution with the "variance" $\chi(r_0, g) = 2c_1(r_0, g)^{-1}$, but in contrast to the case $g < g_{\text{cr}}$, this "variance" is negative.

In the interval from g_{cr} to $(r_0 + 1)^2$, computer simulations show that we jump from the attraction domain of the invariant set Ω_1 to the attraction domain of the invariant set Ω_2 and back infinitely many times while crossing the stable invariant curves for nontrivial SPs infinitely many times. In this respect, the critical behavior differs from the critical behavior in "bosonic" models where only one critical temperature exists (in our model, g is an analogue of the reciprocal temperature).

Computer experiments show that if g approaches $\tilde{\gamma}_1$ from below, the variance $\chi(r_0, g)$ tends to $+\infty$, and if g approaches $\tilde{\gamma}_2$ from above, the variance $\chi(r_0, g)$ tends to $-\infty$; if g crosses $\tilde{\gamma}_\infty$, then $\chi(r_0, g) = 0$. The graph of the function $\chi(r_0, g)$ has a complex form with infinitely many poles.

We consider the behavior of $\chi(r_0, g)$ when $g \uparrow g_{\text{cr}}$, where g_{cr} is the ordinate of intersection of the half-line (r_0, g) with γ_1. We write g in the form $g = g_{\text{cr}} - b(g)\mu_+^{k(g)}$, where μ_+ is the largest eigenvalue of the RG differential in the plus SP, $k(g)$ is a natural number, and $1 \le b(g) < \mu_+$. If $g \to g_{\text{cr}}$, then $k(g) \to \infty$. We recall that the explicit formula $D(r_\pm, g_\pm) = nA(r_\pm, g_\pm)^{\text{tr}}$, where $A(r_\pm, g_\pm)$ is given by formula (29), holds for the RG differential in the plus (minus) SP. If $\alpha > 1$, then $\mu_+ > 1$.

Because $\chi(r, g) = 2c_1(r, g)^{-1}$ and $c_1(r, g) = \lim_{N \to \infty} r^{(N)} \lambda_1^{-N}$, we obtain the recurrent relations

$$c_1\big(r^{(k)}, g^{(k)}\big) = \lambda_1^k c_1(r, g),$$

$$\chi(r, g) = \lambda_1^k \chi\big(r^{(k)}, g^{(k)}\big).$$

We consider the limit

$$\lim_{g \to g_{\text{cr}}} \frac{\log \chi(r_0, g)}{\log |g - g_{\text{cr}}|} = \lim_{k(g) \to \infty} \frac{k(g) \log \lambda_1 + \log \chi\big(r_0^{(k(g))}, g^{(k(g))}\big)}{-k(g) \log \mu_+ + \log b(g)}.$$

Using methods in [5], we can show that $c_1\big(r_0^{(k(g))}, g^{(k(g))}\big) < \text{const}$. Hence,

$$\lim_{g \to g_{\text{cr}}} \frac{\log \chi(r_0, g)}{\log |g - g_{\text{cr}}|} = -\frac{\log \lambda_1}{\log \mu_+},$$

which determines the critical index in our model (in the bosonic models, it is denoted by γ). If $r_0 < -1$, then the half-line (r_0, g) first crosses the curve γ_2 at some point (r_0, g_{cr}). Analogous reasoning shows that

$$\lim_{g \to g_{\text{cr}}} \frac{\log |\chi(r_0, g)|}{\log |g - g_{\text{cr}}|} = -\frac{\log \lambda_1}{\log \mu_-},$$

where μ_- is the largest eigenvalue of the RG differential in the minus SP.

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