Theoretical and Mathematical Physics, Vol. 117, No. 2, 1998

GENERAL RELATIVISTIC ANALOGUE SOLUTIONS FOR THE YANG-MILLS THEORY

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We discuss several solutions to the Yang-Mills equations that can be found using the connection between general relativity and the Yang-Mills theory. Some comments about the possible physical meaning of these solutions are made. In particular, it is argued that some of these analogue solutions of the Yang-Mills theory may have some connection with the confinement phenomenon. To this end, we briefly look at the motion of test particles moving in the background potential of the Schwarzschild analogue solution.

Dedication

This article is dedicated to the memory of Professor Fedor Lunev.

1. Introduction

Yang-Mills theories [1] are non-Abelian gauge theories that have found a central role in particle physics for describing both electroweak and strong interactions. The non-Abelian nature of Yang-Mills theories make the field equations nonlinear and therefore much more difficult to handle compared with Abelian gauge theories such as pure electromagnetism. For example, at the classical level (and also approximately at the quantum level if the quantum corrections are not too large [2]), superposition can be used for Abelian gauge theories, while even at the classical level, superposition is not valid for Yang-Mills theories. This nonlinearity of the Yang-Mills field equations makes finding solutions difficult. There are some well-known solutions of the Yang-Mills field equations, such as the 't Hooft-Polyakov monopole [3], the Julia-Zee dyon [4], the Bogomolnyi-Prasad-Sommerfield (BPS) dyon [5, 6], and the instanton [7], but there is no systematic way to solve the Yang-Mills field equations.

General relativity can also be considered a non-Abelian gauge theory in some sense [8, 9], and a mathematical connection between the two theories can be made [10, 11]. One can ask if the solutions to the field equations of one theory could provide a starting point for using this connection to look for solutions in the other theory. This is in fact possible, and a host of solutions can thus be found.

In this paper, we review various solutions thus found and discuss some of their properties. All the solutions thus discovered have the apparent weak point that they have an infinite field energy, i.e., there are singularities in the fields of the solutions that make the field energy infinite. This is in contrast to the finite field-energy solutions in [3-6]. In addition to the purely mathematical interest in studying all types of solutions that occur in such nonlinear field theories, we present some ideas about the possible physical uses of such singular solutions. One speculation is that some of these may be connected with the confinement phenomenon of the strong interaction. Just as the various black-hole solutions (Schwarzschild or Kerr black holes) exhibit a type of confinement for any particle that crosses the event horizon, the Yang-Mills analogues of these solutions may exhibit a confining behavior.

In what follows, we discuss the spherically symmetrical solutions of the $SU(2)$ Yang-Mills equations coupled to a scalar field (these are usually called the Yang-Mills-Higgs equations). Then, we discuss solutions for gauge groups other than $SU(2)$. Finally, we examine the behavior of a test particle placed in

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Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 117, No. 2, pp. 308-324, November, 1998. Original article submitted April 14, 1998.

0040-5779/98/1172-1351$20.00 © 1999 Kluwer Academic/Plenum Publishers

the potential of the Schwarzschild analogue solution. We show that under certain conditions, this analogue solution can confine the test particle and that this system has a half-integer angular momentum, even though all the fields involved have an integer angular momentum.

2. $SU(2)$ Yang–Mills field equations for spherically symmetrical field configurations

In this section, we study an $SU(2)$ gauge theory coupled to a scalar field ϕ^a in the triplet representation. The scalar field is taken to have no mass or self-interaction. The Lagrangian for this system is

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + \frac{1}{2}(D_\mu\phi_a)(D^\mu\phi^a), \tag{1}$$

where $G^a_{\mu\nu}$ is the strength tensor defined by the gauge fields W^a_μ as

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon_{abc}W^b_\mu W^c_\nu \tag{2}$$

and D_μ is the covariant derivative of the scalar field given by

$$D_\mu\phi^a = \partial_\mu\phi^a + g\epsilon_{abc}W^b_\mu\phi^c. \tag{3}$$

The general equations of motion for this system are

$$\begin{aligned} \partial^\nu G^a_{\mu\nu} &= g\epsilon_{abc}\big[G_{\mu\nu b}W^\nu_c - (D_\mu\phi_b)\phi_c\big], \\ \partial^\mu D_\mu\phi^a &= g\epsilon_{abc}(D_\mu\phi_b)W^\mu_c. \end{aligned} \tag{4}$$

These field equations can be simplified using the generalized Wu–Yang ansatz [12] used by Witten [13] to study multi-instanton solutions,

$$\begin{aligned} W^a_i &= \epsilon_{aij}\frac{r^j}{gr^2}\left[1 - K(r)\right] + \left(\frac{r_i r_a}{r^2} - \delta_{ia}\right)\frac{G(r)}{gr}, \\ W^a_0 &= \frac{r^a}{gr^2}J(r), \\ \phi^a &= \frac{r^a}{gr^2}H(r). \end{aligned} \tag{5}$$

The ansatz functions $K(r)$, $G(r)$, $J(r)$, and $H(r)$ are to be determined by the equations of motion. Inserting these expressions into the field equations in Eq. (4), we find the set of coupled, non-linear equations

$$\begin{aligned} r^2K'' &= K(K^2 + G^2 + H^2 - J^2 - 1), \\ r^2G'' &= G(K^2 + G^2 + H^2 - J^2 - 1), \\ r^2J'' &= 2J(K^2 + G^2), \\ r^2H'' &= 2H(K^2 + G^2), \end{aligned} \tag{6}$$

where the primes denote differentiation with respect to r. The most well known solutions of these equations are those discovered by Prasad and Sommerfield [6] and independently by Bogomolnyi [5]. They are

$$\begin{aligned} K(r) &= \cos(\theta)Cr\,\mathrm{csch}(Cr), & G(r) &= \sin(\theta)Cr\,\mathrm{csch}(Cr), \\ J(r) &= \sinh(\gamma)\big[1 - Cr\coth(Cr)\big], & H(r) &= \cosh(\gamma)\big[1 - Cr\coth(Cr)\big], \end{aligned} \tag{7}$$

where C, θ, and γ are arbitrary constants. Solutions (7) have a finite field energy. Indeed, the energy density of the fields is

$$T^{00} = \frac{1}{g^2r^2}\left(K'^2 + G'^2 + \frac{(K^2+G^2-1)^2}{2r^2} + \frac{J^2(K^2+G^2)}{r^2} + \right. \\ \left. + \frac{(rJ'-J)^2}{2r^2} + \frac{H^2(K^2+G^2)}{r^2} + \frac{(rH'-H)^2}{2r^2}\right), \tag{8}$$

which yields a finite field energy $E = 4\pi C \cosh^2(\gamma)/g^2$ when integrated over all space. This finite-energy property of the BPS solution is a main reason for the interest in this classical solution.

We now examine the general relativistic analogue solutions of the Yang–Mills equations.

Solutions with spherical singularities. To find the general relativistic analogue solutions to the Yang–Mills field equations, we begin by examining the Schwarzschild solution of general relativity. The Schwarzschild solution written in Schwarzschild coordinates has two nonvanishing metric tensor components: g_{00} and g_{rr}. The nonvanishing spatial element has the form $g_{rr} = Kr/(1-Kr)$ and $g_{00} = -1/g_{rr}$, where $K = 1/(2GM)$. Trying this form of g_{rr} in Eq. (6), we find the solution

$$K(r) = \frac{\mp\cos\theta Cr}{1 \pm Cr}, \qquad G(r) = \frac{\mp\sin\theta Cr}{1\pm Cr}, \\ J(r) = \frac{\sinh\gamma}{1\pm Cr}, \qquad H(r) = \frac{\cosh\gamma}{1\pm Cr}, \tag{9}$$

where C, γ, and θ are arbitrary constants. The solution with the minus sign in the denominator (which we call the Schwarzschild-like solution) develops a singularity in the gauge fields W_μ^a and scalar fields ϕ^a on a spherical surface of radius $r = r_0 = 1/C$. Both the Schwarzschild-like solution and the solution with the plus sign in the denominator develop singularities in the fields at $r = 0$. These field singularities lead to the field energy of these solutions being infinite, as can be seen by inserting the ansatz functions from Eq. (9) in Eq. (8) and trying to integrate over all space. Such infinite-energy solutions of the Yang–Mills equations have been investigated by several authors [14–19], and the earliest discussion [14] actually predates the study of the finite-energy solutions such as the 't Hooft–Polyakov monopole or the BPS dyon.

The infinite field energy is not a serious drawback of solutions (9) as compared with the finite-energy solutions. A similar situation occurs in some classical field theories, whose solutions nevertheless have a physical meaning. An example in electromagnetism is provided by the Coulomb solution with a field singularity at $r = 0$ similar to the $r = 0$ singularities developed by solutions (9) when these are inserted in (5). It has been speculated that the Schwarzschild-like solution, with its singular spherical surface, has some connection with the quark confinement phenomenon [15, 17, 18, 20, 21]. The motion of a test particle that moves in the potentials given by the minus-sign solution in Eq. (9) shows that the spherical singularity in the fields represents a barrier that can trap the test particle inside the sphere. This is similar in spirit to bag models of hadron structure with test particles moving in some confining potential (such as an infinite spherical well). It is also interesting that this Schwarzschild-like solution was obtained from the general relativistic solution for a nonrotating black hole that exhibits its own type of "confinement:" any particle that passes within the event horizon becomes permanently trapped. One should be cautious about pushing this analogy too far, because the spherical singularities in general relativity and the Yang–Mills theory differ. The singularity at the event horizon of the general relativistic Schwarzschild solution is not a physical but a coordinate singularity, as can be seen by writing the Schwarzschild solution in Kruskal coordinates, where the only singularity is at $r = 0$. Both singularities in the Yang–Mills analogue of the Schwarzschild solution are true singularities in the fields.

The existence of singular solutions for certain field theories is not new (e.g., the singularities in the Coulomb solution of electromagnetism, the Wu–Yang monopole solution [12], or the meron solutions [22]). Even the appearance of a singularity in the gauge fields on a spherical surface, such as occurs in the

Schwarzschild-like solution of Eq. (9), which may at first seem unique, can be found in other infinite-energy solutions. These other solutions possess an infinite set of concentric spherical surfaces on which the fields develop a singularity. This solution behavior could be taken as evidence that such spherical surfaces with singularities are not uncommon features in classical solutions to the Yang–Mills field equations. The first of these solutions can be obtained by exchanging the hyperbolic functions of the BPS solution in Eq. (7) with their trigonometric counterparts,

$$\begin{aligned} K(r) &= \cos(\theta) Cr\csc(Cr), & G(r) &= \sin(\theta) Cr\csc(Cr), \\ J(r) &= \sinh(\gamma)\big[1 - Cr\cot(Cr)\big], & H(r) &= \cosh(\gamma)\big[1 - Cr\cot(Cr)\big]. \end{aligned} \tag{10}$$

This solution was briefly discussed in the derivation of the BPS solution in [23], where the authors start with a solution like that in Eq. (10) and apply the transformation $C \to iC$ to arrive at the BPS solution. Solution (10) exhibits a series of concentric spherical surfaces on which the gauge and scalar fields become singular. These singularities are located on the spherical surfaces $Cr = n\pi$ where $n = 1, 2, 3, 4, \ldots$. Inserting the ansatz functions of Eq. (10) in Eq. (8), we find the energy density

$$T^{00} = \frac{2\cosh^2(\gamma)}{r^2 g^2}\left[C^2\csc^2(Cr)\big(1 - Cr\cot(Cr)\big)^2 + \frac{\big(C^2 r^2\csc^2(Cr) - 1\big)^2}{2r^2}\right]. \tag{11}$$

The energy density becomes singular on the same spherical surfaces as the gauge and scalar fields. These spherical shells, on which the energy density becomes infinite, cause the total field energy of this solution to diverge.

To obtain the next solution, we simply try the complementary trigonometric functions for the solution in Eq. (10). Doing this shows that the following is also a solution [24] of Eq. (6):

$$\begin{aligned} K(r) &= \cos(\theta) Cr\sec(Cr), & G(r) &= \sin(\theta) Cr\sec(Cr), \\ J(r) &= \sinh(\gamma)\big[1 + Cr\tan(Cr)\big], & H(r) &= \cosh(\gamma)\big[1 + Cr\tan(Cr)\big]. \end{aligned} \tag{12}$$

We note that because of the linear Cr term in each solution, solution (12) cannot be obtained from the other trigonometric solution in Eqs. (10) by simply letting $Cr \to Cr - \pi/2$. Although these two trigonometric solutions are in this sense distinct (i.e., they are not simply related by the transformation $Cr \to Cr - \pi/2$), they are physically similar, because most of the comments concerning the solution in Eqs. (10) apply here as well. Most obviously, the ansatz functions (and therefore the gauge and scalar fields) become singular when $Cr = n\pi/2$, where $n = 1, 3, 5, 7, \ldots$, and when $r = 0$. This solution thus exhibits a series of concentric spherical surfaces on which its fields become singular as well as a point singularity at the origin. These singularities also show up in the energy density of this solution as they did for the solution in Eqs. (10). The point singularity at $r = 0$ and the spherical singularities of the solutions in Eqs. (10) and (12) are similar to those of the solutions in Eqs. (9). However, the solutions in Eqs. (9) only possessed one spherical surface on which the fields and energy density diverged.

One conjectured application of the Schwarzschild-like solution is as a possible explanation of the confinement mechanism. When the Schwarzschild-like solution in [19] is treated as a background potential in which a test particle is placed, it is found that the spherical singularity can act as an impenetrable barrier that traps the test particle either in the interior or the exterior of the sphere [20], giving a classical type of confinement. Similar results have been found for other singular solutions [15, 17, 18]. In addition, it was pointed out in [15] that such a classical type of confinement is only possible with infinite-energy solutions. Treating the trigonometric solutions as a background potential would also trap test particles between any two of the concentric spherical singularities. These trigonometric solutions could possibly be used to solve the field equations in some limited range of r, and then the resulting solution could be patched to one of the other solutions that would solve the field equations for the remaining range of r. This is similar to attempts in general relativity to patch an exterior solution to some interior solution.

Finally, a third solution of Eqs. (6) can be obtained by applying the transformation $C \to iC$ to the solution [24] in Eqs. (12). This yields

$$K(r) = \mathrm{i}\cos(\theta) Cr\,\mathrm{sech}(Cr), \qquad G(r) = \mathrm{i}\sin(\theta) Cr\,\mathrm{sech}(Cr),$$
$$J(r) = \sinh(\gamma)\left[1 - Cr\tanh(Cr)\right], \qquad H(r) = \cosh(\gamma)\left[1 - Cr\tanh(Cr)\right], \tag{13}$$

Because the ansatz functions $K(r)$ and $G(r)$ are imaginary, the space components of the gauge fields are complex. Despite this, all the physical quantities associated with this complex solution, such as energy density, are real. Inserting the ansatz functions of Eqs. (13) in Eq. (8), we find that the field energy density is

$$T^{00} = \frac{2\cosh^2(\gamma)}{r^2 g^2}\left[-C^2\,\mathrm{sech}^2(Cr)\left(1 - Cr\tanh(Cr)\right)^2 + \frac{\left(C^2 r^2\,\mathrm{sech}^2(Cr) + 1\right)^2}{2r^2}\right]. \tag{14}$$

This energy density is real, but the total field energy is infinite due to the singularity at $r = 0$. This solution is thus more like a Wu–Yang monopole [12] or a charged point particle, as opposed to a finite-energy BPS dyon.

***SU*(2) solutions with increasing potentials.** In addition to the preceding infinite-energy solutions, whose gauge and scalar fields become singular on some spherical surface, there are other types of infinite-energy, general relativistic analogue solutions. In general relativity, if a nonzero cosmological constant Λ is allowed for, then the time–time component of the metric tensor for the Schwarzschild solution becomes [25]

$$g_{00} = 1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}. \tag{15}$$

The Newtonian potential for this solution is

$$\Phi = \frac{(g_{00} - 1)}{2} = \frac{-GM}{r} - \frac{\Lambda r^2}{6}. \tag{16}$$

Using Eq. (16) as a starting point, we find the simple solution [26] of Eq. (6)

$$K(r) = \cos\theta, \qquad G(r) = \sin\theta, \qquad J(r) = H(r) = \frac{B}{r} + Ar^2, \tag{17}$$

where a, B, and θ are arbitrary constants. If we set $A = 0$, then it can be seen that the Schwarzschild-like solutions in Eqs. (9) and those in Eqs. (17) become similar in the limit as $C \to \infty$ and $e^{\gamma}/C \to 2B$. Inserting the ansatz functions of Eqs. (17) into the gauge and scalar fields of Eqs. (5), we find that the time component of the gauge field (W_0^a) and the scalar field (ϕ^a) behave as $Ar + B/r^2$. The space part of the gauge fields (W_i^a) have a $1/r$ dependence. This classical solution exhibits a linear confining potential similar to those used in some phenomenological studies of hadronic spectra [27]. In addition, arguments from lattice gauge theory [28] seem to indicate that the confining potential between quarks should be linear. Classical solutions similar to those in Eqs. (17) were discussed in [29] in connection with the confinement problem.

This solution also has an infinite field energy. Inserting the ansatz functions of Eqs. (17) into the energy density of Eq. (8) and integrating to obtain the total field energy, we find

$$E = \int T^{00}\, d^3x = \frac{4\pi}{g^2}\int_{r_a}^{r_b} T^{00} r^2\, dr = \frac{4\pi A^2}{g^2}(r_b^3 - r_a^3) - \frac{8\pi B^2}{g^2}\left(\frac{1}{r_b^3} - \frac{1}{r_a^3}\right), \tag{18}$$

where we introduce the respective upper and lower cutoffs in the radial coordinate r_b and r_a. If $r_b \to \infty$, then the field energy becomes infinite because of the linear part of the gauge and scalar fields; if $r_a \to 0$, then the field energy becomes infinite because of the singularity at $r = 0$. Compared with the solutions in Eqs. (9), which had infinite field energy coming from local singularities (either at $r = 0$ or $r = 1/C$), the solution in Eqs. (17) can have a infinite field energy coming from the point singularity at $r = 0$ and from the linearly increasing gauge and scalar fields as $r \to \infty$. Again, although this classical solution has some undesirable characteristics, it also exhibits features found in some phenomenological studies of hadronic bound states.

$SU(3)$ solutions. Up to this point, we discussed classical solutions of the Yang–Mills field equations for $SU(2)$ fields. Because the quantum chromodynamics (QCD) involves the $SU(3)$ gauge group, it is natural to ask if there are any $SU(3)$ or even $SU(N)$ generalizations of the above solutions. One possibility is to embed the $SU(2)$ solutions above in an $SU(N)$ gauge theory [30]. Recently, a Schwarzschild-like classical solution was found that is not a simple embedding of the previous $SU(2)$ solutions in an $SU(N)$ gauge theory but a true $SU(3)$ solution [31]. To arrive at the $SU(3)$ solution, the Wu–Wu ansatz [32–34] is generalized as [31]

$$\begin{aligned} W_0 &= \frac{-\mathrm{i}\phi(r)}{gr^2}(\lambda^7 x - \lambda^5 y + \lambda^2 z) + \frac{1}{2}\lambda^a(\lambda^a_{ij} + \lambda^a_{ji})\frac{x^i x^j}{gr^3}w(r), \\ W^a_i &= (\lambda^a_{ij} - \lambda^a_{ji})\frac{\mathrm{i}x^j}{gr^2}\big(1 - f(r)\big) + \lambda^a_{jk}(\epsilon_{ilj}x^k + \epsilon_{ilk}x^j)\frac{x^l}{gr^3}v(r), \end{aligned} \tag{19}$$

where λ^a are the Gell-Mann matrices. Using this ansatz in the Yang-Mills field equations yields the following set of coupled differential equations for the functions $f(r)$, $v(r)$, $\phi(r)$, and $w(r)$:

$$\begin{aligned} r^2 f'' &= f^3 - f + 7fv^2 + 2vw\phi - f(w^2 + \phi^2), \\ r^2 v'' &= v^3 - v + 7vf^2 + 2fw\phi - v(w^2 + \phi^2), \\ r^2 w'' &= 6w(f^2 + v^2) - 12fv\phi, \\ r^2 \phi'' &= 2\phi(f^2 + v^2) - 4fvw, \end{aligned} \tag{20}$$

where the primes denote differentiation with respect to r. The nonlinear, coupled differential equations in Eqs. (20) are the $SU(3)$ equivalents of those in Eqs. (6). There were several simplifying assumptions to make the problem more tractable in [31]. First, taking $w = \phi = 0$ reduces Eqs. (20) to the system

$$\begin{aligned} r^2 f'' &= f(f^2 - 1 + 7v^2), \\ r^2 v'' &= v(v^2 - 1 + 7f^2). \end{aligned} \tag{21}$$

Then, the further simplification $f^2 = v^2 = q^2/8$ reduces Eqs. (21) to the Wu–Yang [12] equation for $q(r)$,

$$r^2 q'' = q(q^2 - 1). \tag{22}$$

This equation has been shown to have a solution that is singular at some radius $r = r_1$ [14, 18, 31]. In other words, the solution near $r = r_1$ has the form

$$q(r) \approx \frac{A}{r_1 - r}, \tag{23}$$

where A and r_1 are constant. Thus, even with no scalar field, solutions to the pure gauge field theory equations can be found that tend to trap test particles behind a spherical barrier in much the same way as the Schwarzschild-like solution in Eqs. (9). It is also possible to find closed-form solutions to a special case of system (20). With $v = w = 0$, Eqs. (20) become

$$\begin{aligned} r^2 f'' &= f(f^2 - \phi^2 - 1), \\ r^2 \phi'' &= 2\phi f^2, \end{aligned} \tag{24}$$

which has the simple closed-form solution

$$\begin{aligned} f(r) &= \mp\frac{Cr}{1 \pm Cr}, \\ \phi(r) &= \pm\frac{i}{1 \pm Cr}. \end{aligned} \tag{25}$$

Other, similar solutions can be found by making different simplifying assumptions such as $f = w = 0$. Thus, solutions with singular fields on a spherical surface are not unique to $SU(2)$ gauge theories but can also be found for $SU(3)$ [31] and, in general, for $SU(N)$ [30]. The interesting aspect of the solutions in [31] is that these solutions are true $SU(3)$ solutions and not simply embeddings of an $SU(2)$ solution in the $SU(N)$ gauge group as in [30]. Also, the $SU(3)$ solutions presented here are pure gauge field solutions, as opposed to the general $SU(2)$ solutions for the system given in Eq. (1) that involves scalar fields. In some sense, the role of the scalar field in the $SU(2)$ system is played by the time component of the gauge field in the $SU(3)$ system. This can be seen by comparing system (6) with system (20): the equations for $f(r)$ and $v(r)$ are similar to those for $K(r)$ and $G(r)$, and the equations for $w(r)$ and $\phi(r)$ are similar to those for $J(r)$ and $H(r)$.

3. Electromagnetic properties of the $SU(2)$ solutions

All the $SU(2)$ solutions to the Yang–Mills field equations have interesting "electromagnetic" features. To investigate these properties, we use 't Hooft's definition of a generalized, gauge-invariant, U(1) field strength tensor [3]

$$F_{\mu\nu} = \partial_\mu(\hat{\phi}^a W^a_\nu) - \partial_\nu(\hat{\phi}^a W^a_\mu) - \frac{1}{g}\epsilon_{abc}\hat{\phi}^a(\partial_\mu\hat{\phi}^b)(\partial_\nu\hat{\phi}^c), \tag{26}$$

where $\hat{\phi}^a = \phi^a(\phi^b\phi^b)^{-1/2}$. This generalized U(1) field strength tensor reduces to the usual expression for the field strength tensor if a gauge transformation to the Abelian gauge is performed, where the scalar field only points in one direction in isospin space (i.e., $\phi^a = \delta^{3a}v$) [35]. If this U(1) field is associated with the photon of electromagnetism, then the solutions in Eqs. (9), (10), (12), (13), and (17) carry magnetic and/or electric charges. In general, the electric and magnetic fields associated with these solutions are

$$\begin{aligned} E_i &= F_{i0} = \frac{r_i}{gr}\frac{d}{dr}\left(\frac{J(r)}{r}\right), \\ B_i &= \frac{1}{2}\epsilon_{ijk}F_{jk} = -\frac{r_i}{gr^3}. \end{aligned} \tag{27}$$

The magnetic field of all the solutions is that of a point monopole of strength $-4\pi/g$. The reason for this is discussed at the end of this section.

The electric field of the Schwarzschild-like solutions in Eqs. (9) is easily found by inserting the ansatz function $J(r)$ from Eqs. (9) in Eq. (27). This gives

$$E_i = \frac{-r_i \sinh\gamma(1 \pm 2Cr)}{gr^3(1 \pm Cr)^2}. \tag{28}$$

As $r \to \infty$, this electric field goes as $1/r^3$, which indicates that the net electric charge of this solution is zero, although there appears to be some kind of dipole charge distribution.

The electric fields of the two trigonometric solutions presented in Eqs. (10) and (12) are similar in that they indicate that these solutions carry an infinite electric charge. Inserting the ansatz function $J(r)$ from trigonometric solution (10) in Eq. (27) yields the electric field

$$E_i = \frac{-\sinh(\gamma)r_i}{gr^3}\left(1 - C^2r^2\csc^2(Cr)\right). \tag{29}$$

The electric field does not fall off for large r but behaves as $r_i\csc^2(Cr)/r$. This electric field also becomes singular on the spherical surfaces defined by $Cr = n\pi$, where $n = 1, 2, 3, 4, \ldots$. Trigonometric solution (12) exhibits the same type of electric field except that it becomes singular on spherical shells given by $Cr = n\pi/2$ with odd n and at $r = 0$. The electric charge of this solution is also infinite, because the electric field from Eq. (29) does not fall off as $r \to \infty$. For the special case where $\gamma = 0$, the solution carries no electric charge but only a magnetic charge. Even in this case, the energy density becomes singular on the concentric spherical surfaces and at the origin. Both the BPS solution and solutions (10) and (12) have the same finite magnetic strength $-4\pi/g$. Although this solution is a dyon in the sense that it carries both magnetic and electric charges, it is probably not correct to view it as a particle-like solution, because the electric field does not fall off, thus implying that these solutions have an infinite, spread-out electric charge.

The electric field associated with the complex solution in Eqs. (13) can be found in the same way as for the other solutions. Inserting the ansatz function $J(r)$ from Eqs. (13) in Eq. (27) yields

$$E_i = \frac{-\sinh(\gamma) r_i}{g r^3} \left(C^2 r^2 \operatorname{sech}^2(Cr) + 1 \right). \tag{30}$$

As with all the other solutions, the complex solution carries a magnetic charge of strength $-4\pi/g$. In addition, the behavior of the electric field in Eq. (30) as $r \to \infty$ shows that this complex solution carries the electric charge $-4\pi \sinh(\gamma)/g$, which is the same as that carried by the BPS solution. One interesting feature of solution (13) is that even though the space components of the gauge fields are complex, all the physical quantities (e.g., field energy, magnetic charge, and electric charge) calculated from it are real. Also, unlike solutions (10) and (12), this complex solution can be viewed as a pointlike dyon because it has a localized electric charge. The main difference between this solution and the BPS solution is the infinite field energy of the complex solution due to the field singularity at $r = 0$.

While many physical characteristics are substantially different in each of these various solutions, the magnetic charge of all the solutions is the same. This occurs because the magnetic charge of each solution is a topologic charge with the same value for each field configuration. A topologic current k_μ can be defined as [35]

$$k_\mu = \frac{1}{8\pi} \epsilon_{\mu\nu\alpha\beta} \epsilon_{abc} \partial^\nu \hat{\phi}^a \partial^\alpha \hat{\phi}^b \partial^\beta \hat{\phi}^c. \tag{31}$$

The topologic charge of this field configuration is then

$$\begin{aligned} q = \int k_0 \, d^3x = \frac{1}{8\pi} \int (\epsilon_{ijk} \epsilon_{abc} \partial^i \hat{\phi}^a \partial^j \hat{\phi}^b \partial^k \hat{\phi}^c) \, d^3x = \\ = \frac{1}{8\pi} \int \epsilon_{ijk} \epsilon_{abc} \partial^i (\hat{\phi}^a \partial^j \hat{\phi}^b \partial^k \hat{\phi}^c) \, d^3x. \end{aligned} \tag{32}$$

For all the solutions, we find that $\hat{\phi}^a = r^a/r$, which is the same regardless of the ansatz function $H(r)$. In all cases, we find that the topologic charge is $q = 1$. In the next section, where we examine the motion of a test particle in the background field of the Schwarzschild-like solution, we find that there is a field angular momentum due to the interaction of the test particle with the field configuration of the Schwarzschild-like solution. This field angular momentum can be seen to arise from the interaction of the topologic magnetic charge with the charge of the test particle, in much the same way as the configuration of a normal magnetic charge and an electric charge lead to a field angular momentum [36, 37, 2].

4. Motion of test particles in a Schwarzschild-like potential

We now study the motion of a test particle in the background potential of the Schwarzschild-like solution in Eqs. (9). We make several assumptions in doing this.

First, we take our test particle to be scalar as in [18, 20]. One reason for this choice is to illustrate the spin-from-isospin effect [38] that occurs with these solutions. As discussed in Sec. 3, all these solutions carry a magnetic charge. Many researchers have noted that the composite system of a magnetic charge and an electric charge has an angular momentum due to the configuration of electric and magnetic fields [36, 37]. Even when the magnetic charge is topologic, as with 't Hooft–Polyakov monopoles, a similar effect whereby the composite system of a topologic magnetic charge and a particle with an isotopic charge has an angular momentum in the gauge fields is found [38]. This has an interesting consequence: to construct fermionic objects from the singular solutions, scalar particles in the fundamental representation of the gauge group ($SU(2)$ for the solutions considered here) should be used. (Fermionic test particles in the adjoint representation would also give a net fermionic bound state [21]).

Second, we assume that the test particle is coupled to the scalar-field part of the solution of Eqs. (9) via the substitution $m^2 \to (m + \lambda\sigma^a\phi^a/2)^2$, where λ is an arbitrary coupling constant. Finally, we assume. in order to ignore the ansatz function $G(r)$, that $\theta = 0$ in Eqs. (9).

Thus, the scalar particle Φ^A moving in the background field of the Schwarzschild-like solution given in Eqs. (9) becomes

$$\left(\partial_0 - \frac{ig}{2}\sigma^a W_0^a\right)\left(\partial_0 - \frac{ig}{2}\sigma^a W_0^a\right)^A_B \Phi^B(x,t) - \\ - \left(\partial_i - \frac{ig}{2}\sigma^a W_i^a\right)\left(\partial_i - \frac{ig}{2}\sigma^a W_i^a\right)^A_B \Phi^B(x,t) = -\left(m + \frac{\lambda}{2}\sigma^a\phi^a\right)^2 \Phi^A(x,t), \tag{33}$$

where σ^a are the standard Pauli matrices and A and B are $SU(2)$ group indices with the values 1 or 2. Taking $\Phi^A(x,t) = \Phi^A(x)e^{-iEt}$, using $(\sigma^a v^a)^2 = v^a v^a$, and expanding, we find that Eq. (33) becomes

$$\left[-E^2 - g\sigma^a W_0^a E - \frac{g^2}{4}(W_0^a)^2 - \nabla^2 + ig\sigma^a W_i^a \partial_i + \frac{g^2}{4}(W_i^a)^2\right]^A_B \Phi^B(x) = \\ = -\left[m^2 + \lambda m \sigma^a\phi^a + \frac{\lambda^2}{4}(\phi^a)^2\right]^A_B \Phi^B(x). \tag{34}$$

Inserting the ansatz form of the gauge and scalar fields from Eq. (5) in Eq. (34) then yields

$$-\left[\nabla^2 - \frac{(1-K(r))}{r^2}\sigma^a l^a - \frac{(1-K(r))^2}{2r^2} + \frac{\sigma^a r^a}{r^2}EJ(r) + \frac{J(r)^2}{4r^2}\right]^A_B \Phi^B(x) = \\ = \left(E^2 - m^2 - \frac{\lambda m}{gr^2}\sigma^a r^a H(r) - \frac{\lambda^2}{4g^2r^2}H(r)^2\right)^A_B \Phi^B(x), \tag{35}$$

where we use

$$ig\sigma^a W_i^a \partial_i = -i(1-K(r))\sigma^a \epsilon_{aji}\frac{r^j\partial^i}{r^2} = (1-K(r))\sigma^a l^a/r^2$$

(with l^a being the standard orbital angular momentum operator) and

$$(W_i^a)^2 = \epsilon_{aij}\epsilon_{aik}r^j r^k \frac{(1-K(r))^2}{g^2r^4} = 2\frac{(1-K(r))^2}{g^2r^2}.$$

Because $\sigma^a l^a$ does not commute with $\sigma^a r^a$, Eq. (35) is difficult to handle. By taking advantage of the free parameter γ in the ansatz functions $J(r)$ and $H(r)$, we can chose γ such that $E\sinh\gamma = \lambda m\cosh\gamma/g$. With this choice, the two $\sigma^a r^a$ terms in Eq. (35) cancel each other. To handle the $\sigma^a l^a$ term, the total angular momentum operator must be defined as

$$J^a = l^a + S^a = l^a + \frac{1}{2}\sigma^a. \tag{36}$$

Thus, the total angular momentum comes not only from the orbital angular momentum; it has a contribution that looks like a spin angular momentum. The σ matrices in the last term of Eq. (36) are, however, connected with the isospin of the system rather than with the spin. This is just the spin-from-isospin effect [38] and is connected with the fact that the Schwarzschild-like solution of Eq. (9) carries a topologic magnetic charge. Thus, even though our system involves only integer-spin fields (i.e., W_μ^a, ϕ^a, Φ^A), the combined system is a spin-1/2 object. Using Eq. (36), we can expand the $\sigma^a l^a$ term in the usual way as $\sigma^a l^a = 2S^a l^a = J_{op}^2 - l_{op}^2 - S_{op}^2$, except that S_{op} is now the isospin, not spin, operator.

Finally, we assume for simplicity that $\lambda = g$, which allows the $J(r)^2$ and $H(r)^2$ terms to be combined more easily. At first glance, there seems to be nothing special in this choice, but we can see that according to the arguments in [18] and [39], the barrier at $r = 1/C$ absolutely confines a particle only if $\lambda \geq g$, while for $\lambda < g$ there is some probability for the test particle to tunnel through the barrier. Combining all the preceding assumptions, we find

$$-\left[\nabla^2 - \frac{(1-K(r))}{r^2}(J_{op}^2 - l_{op}^2 - S_{op}^2) - \frac{(1-K(r))^2}{2r^2} + \frac{1}{4r^2}(J(r)^2 - H(r)^2)\right]_B^A \Phi^B(x) = \\ = (E^2 - m^2)_B^A \Phi^B(x). \quad (37)$$

In order to separate the radial equation from Eq. (37), we take

$$\Phi^A(x) = \frac{1}{r} f_{Jl}(r) Y_{JlM}^A(\theta, \phi), \quad (38)$$

where Y_{JlM}^A are the standard spinor spherical harmonics that can be obtained by adding the orbital angular momenta l^a to a spin 1/2. Here, spin is replaced by isospin, but the mathematics and the spinor spherical harmonics are exactly the same. Now, inserting Eq. (38) in Eq. (37) yields

$$-\left[\frac{d^2}{dr^2} - \frac{D}{r^2} - \frac{F(1-K(r))}{r^2} - \frac{(1-K(r))^2}{2r^2} + \frac{1}{4r^2}(J(r)^2 - H(r)^2)\right] f_{Jl}(r) = (E^2 - m^2) f_{Jl}(r) \quad (39)$$

with the constants $D = l(l+1)$ and $F = J(J+1) - l(l+1) - 3/4$. If we then set $x = Cr$ and insert the ansatz functions $K(r)$, $J(r)$, and $H(r)$ from Eqs. (9) in Eq. (39), the problem becomes the effective one-dimensional Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + \frac{D}{x^2} + \frac{F(1-2x)}{x^2(1-x)} + \frac{(1-2x)^2}{2x^2(1-x)^2} + \frac{1}{4x^2(1-x)^2}\right] f_{Jl}(x) = \frac{(E^2 - m^2)}{C^2} f_{Jl}(x), \quad (40)$$

where all the nonderivative terms in the left-hand side are viewed as the effective potential. The key feature of this effective potential is the singularities at $x = 0$ and $x = 1$. As $x \to 1$, the leading term in the effective potential goes as

$$V_{\text{eff}}(x) = \frac{D}{x^2} + \frac{F(1-2x)}{x^2(1-x)} + \frac{(1-2x)^2}{2x^2(1-x)^2} + \frac{1}{4x^2(1-x)^2} \to \frac{3}{4(1-x)^2}. \quad (41)$$

It was argued in [39, 18] that such a singularity would only present a true barrier to the test particle (i.e., the probability of the test particle tunneling through the barrier would be zero) if the coefficient in Eq. (41) were greater than or equal to 3/4. Thus, the effective potential of Eq. (40) just confines the test particle to the range $0 \leq x \leq 1$. That the effective potential is just able to confine the test particle stems from our choice of $\lambda = g$ for the coupling of the scalar potential ϕ^a to the test particle Φ^A. If we had taken $\lambda < g$, the coefficient in the limiting form of the effective potential from Eq. (41) would have been less than 3/4, and the test particle would no longer be confined (e.g., if we take $\lambda = 0$, it is straightforward to show starting from Eq. (35) that we obtain the coefficient 1/2). Conversely, when $\lambda > g$, the coefficient in Eq. (41) becomes greater than 3/4, and the test particle is confined. This has the interesting implication that the scalar potential plays an important role in this confinement mechanism. Although confinement is generally thought to be just the result of the gauge interaction, there are phenomenological studies [40–42] that indicate that an effective scalar potential is involved in the confinement mechanism.

To consider the solution of Eq. (40) in more detail, we must pick particular values of J and l (which determine the constants D and F in Eq. (40)) and solve for the eigenfunctions $f_{Jl}(x)$ and the eigenvalues $(E^2 - m^2)/C^2$. In general, this must be done numerically [21, 20], but the key features of the effective one-dimensional potential of Eq. (41) (i.e., the singularities in the potential at $x = 0$ and $x = 1$) make this potential similar to the Pöschl–Teller potential [43]

$$V(x) = \frac{1}{2} V_0 \left[\frac{\alpha(\alpha - 1)}{\sin^2(\pi x/2)} + \frac{\beta(\beta - 1)}{\cos^2(\pi x/2)} \right], \tag{42}$$

where α, β, and V_0 are constants. With α, β, and V_0 chosen correctly, the Pöschl–Teller potential can be made similar to the effective potential from Eq. (41). The known eigenfunctions and eigenvalues of the Pöschl–Teller potential should then give a good approximation to the eigenvalues and eigenfunctions of the potential from Eq. (41). The eigenfunctions for the Pöschl–Teller potential are given by [43]

$$f_n(x) = K \sin^\alpha(\pi x/2) \cos^\beta(\pi x/2), \qquad {}_2F_1\left(-n, \alpha + \beta + n, \alpha + \frac{1}{2} \sin^2(\pi x/2)\right), \tag{43}$$

where K is a constant fixed by normalization, n is the radial quantum number with the values $n = 0, 1, 2, 3, \ldots$, and ${}_2F_1(a, b, c; x)$ is the hypergeometric function. The energy eigenvalues for the Pöschl–Teller potential are [43]

$$E_n = \frac{1}{2} V_0 (\alpha + \beta + 2n)^2. \tag{44}$$

We see from the shape of both the Pöschl–Teller potential and the effective potential in Eq. (41) that this is exactly the kind of dependence to be expected for the energy eigenvalues. For small energies (i.e., $\alpha + \beta > 2n$), both potentials behave as a harmonic oscillator potential, and it would therefore be expected that the leading term in E_n would go as $2V_0(\alpha + \beta)n \propto n$. For large energies (i.e., $2n > \alpha + \beta$), both potentials behave as infinite spherical wells, and it would therefore be expected that the leading term in E_n would go as $2V_0 n^2 \propto n^2$. As a simple example, we consider the $l = 0$ case for the potential in Eq. (41). For $l = 0$, we find $J = 1/2$, $D = 0$, and $F = 0$; the potential in Eq. (41) becomes

$$V_{\text{eff}}(x) = \frac{3 - 8x + 8x^2}{4x^2(1 - x)^2}. \tag{45}$$

This potential approaches $3/\big(4(1 - x)^2\big)$ as $x \to 1$, and the test particle is therefore just confined to the range $0 < x < 1$. In this range, $V_{\text{eff}}(x)$ of Eq. (45) reaches its minimum value of 4 at $x = 1/2$, and the potential is symmetrical about this point. For the Pöschl–Teller potential to also be symmetrical about $x = 1/2$ and to also take a value of 4 at this point, we can choose $V_0 = 1$ and $\alpha = \beta = 2$. Now, inserting these into Eq. (44) and recalling that our eigenvalue from Eq. (40) is $(E^2 - m^2)/C^2$, we find that the approximate energy of the bound states for this case with $l = 0$ is

$$E_n^2 = m^2 + C^2(2 + n)^2. \tag{46}$$

We note that this energy depends on the arbitrary constant C that sets the radius of the confining sphere ($r = 1/C$). The radius of the spherical shell decreases as C increases, and we can see from Eq. (46) that the energy of the state increases as would be expected. Although it was particularly easy to determine V_0, α, and β in this $l = 0$ case, the form of the bound-state energy given by Eq. (46) is similar even when $l \neq 0$.

5. Discussion and Conclusions

In this article, we have presented a variety of solutions to the field equations of the Yang-Mills theory. Although finding exact solutions of nonlinear field theories is difficult in general, many of the solutions given above were found using the mathematical connection that exists between the Yang–Mills theory and general relativity. Because general relativity has been studied longer than the Yang–Mills theory, there exists a body of known solutions that can serve as guides for finding solutions to the Yang–Mills or Yang–Mills–Higgs field equations. The Schwarzschild solution of general relativity, both without and with a cosmological term, gave rise to the solution with a spherical singularity in Eqs. (9) and the linearly increasing solution in Eqs. (17). Although both of these solutions suffered from the apparent drawback of an infinite field energy, they also exhibited some possible connection with the confinement phenomenon. The linear solution in Eqs. (17) has the form of phenomenological potentials [27] that are often used in studies of heavy-quark bound states. In addition, arguments from lattice gauge theory [28] favor a linear type of confining potential. The Schwarzschild-like solution in Eqs. (9) has some similarities to bag models for quark bound states. Spherical singularities, similar to those of the Schwarzschild-like solution, were also found in several other solutions as given in Eqs. (10) and (12). Actually, the solutions given in Eqs. (10) and (12) possess an infinite set of concentric spheres on which the gauge and scalar fields became infinite. Thus, such spherical singularities may not be uncommon features of Yang–Mills field theories. The $SU(2)$ Schwarzschild-like solution can be easily generalized to $SU(N)$ by simply embedding the $SU(2)$ solutions in an $SU(N)$ gauge theory [30]. It has also recently been found that true $SU(3)$ solutions (not simple embeddings of the $SU(2)$ solutions) can be given [31].

In Sec. 4, we examined the behavior of a scalar test particle placed in the background potential presented by the Schwarzschild-like solution. In order for the Schwarzschild-like potential to confine the test particle Φ^A, it was necessary to couple Φ^A to the scalar part of the Schwarzschild-like solution ϕ^a via the coupling $m^2 \to (m+\lambda\sigma^a\phi^a/2)^2$, where λ is the strength of the coupling between the Φ^A and ϕ^a particles. Even with this coupling, it was found that confinement occurred only for $\lambda \geq g$ and there would be some finite probability for Φ^A to tunnel out of the spherical well for $\lambda < g$. Although it is normally thought that the confinement phenomenon is the result of only gauge interactions, there has been some work [40–42] that indicates that an effective scalar interaction may be needed to completely explain confinement.

Another interesting aspect of the bound-state system studied in the previous section is that the total system was a fermion even though only integer-spin fields were involved. The spin-1/2 nature of the bound-state system resulted from the fact that the isospin 1/2 of the test particle Φ^A was converted into spin 1/2 when it was placed inside the Schwarzschild-like solution. Another way of obtaining this result is to note that almost all the solutions presented here could be shown to carry a topologic magnetic charge. Thus, in the same way that a standard magnetic–electric charge system carries a field angular momentum of 1/2 in the combined electromagnetic fields, the combined charges of the Schwarzschild-like solution and Φ^A carry a field angular momentum of 1/2 in their combined non-Abelian fields. If a realistic model of hadronic bound states can be constructed from these classical field-theory solutions, then the fact that the net angular momentum of these states does not come entirely from the constituent particles may offer a possible explanation of the European Muon Collaboration effect [44], which shows that a large part of the net spin of the proton does not come from the valence quarks.

In addition to the Schwarzschild-like solutions presented here, it is also possible to take more complex solutions from general relativity to find other Yang–Mills solutions. In [45], the general relativistic Kerr solution was used to construct a new Yang–Mills solution. Although the final form of this solution was not as simple as the Schwarzschild-like solutions, it did share the common feature of having confining surfaces on which the fields become singular. Finally, it is also possible to use this method to find solutions of nonlinear field equations in reverse: starting from known solutions to the Yang–Mills equations, solutions to the general relativistic field equations can be obtained [46].

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