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VIRASORO AMPLITUDE FROM THE $S^N\mathbf{R}^{24}$-ORBIFOLD SIGMA MODEL

G. E. Arutyunov[1] and S. A. Frolov[2]

The four-tachyon scattering amplitude is derived from the $S^N\mathbf{R}^{24}$-orbifold sigma model in the large N limit. The closed string interaction is described by a vertex that is a bosonic analog of the supersymmetric vertex recently proposed by Dijkgraaf, H. Verlinde, and E. Verlinde.

1. Introduction

Compactification of M(atrix) theory [1] on a circle results in the $\mathcal{N}=8$ two-dimensional supersymmetric $SU(N)$ Yang–Mills model [2]. It was recently suggested in [3–5] that in the large N limit, the Yang–Mills theory describes the nonperturbative dynamics of type IIA string theory, while the string coupling constant was argued to be inversely proportional to the Yang–Mills coupling. This suggestion appears to be very natural since, in the IR limit, the gauge theory is strongly coupled and the IR fixed point may be described by the $\mathcal{N}=8$ supersymmetric conformal field theory on the orbifold target space $S^N\mathbf{R}^8$. In the large N limit, the Hilbert space of the orbifold model is known [6] to coincide with (to be precise, to contain) the Fock space of the free second-quantized type IIA string theory. Using these facts, Dijkgraaf, E. Verlinde, and H. Verlinde (DVV) [5] have suggested that perturbative string dynamics in the first order of the string coupling constant can be described by the $S^N\mathbf{R}^8$ supersymmetric orbifold conformal model perturbed by an irrelevant operator of conformal dimension $(3/2,3/2)$. An explicit form of this operator V was determined in [5] and it fits the conventional formalism of the light-cone string theory nicely.

The described approach does not appear to be limited to the supersymmetric case only. In particular, one can suggest [7] that the M(atrix) theory formulation for closed bosonic strings is provided by the large N limit of the two-dimensional Yang–Mills theory with 24 matter fields in the adjoint representation of the $U(N)$ gauge group. In this case, the IR limit of the gauge theory results in the $S^N\mathbf{R}^{24}$-orbifold conformal model. The closed bosonic string interactions are described via perturbation of the conformal field theory (CFT) action with a bosonic analogue of the DVV vertex [7].

An important problem posed by the above-described string interpretation of the S^N-orbifold sigma models is to obtain the usual string scattering amplitudes directly from the models. This problem seems to be nontrivial because the S^N-orbifold models are non-Abelian.

The aim of the present paper is to derive the four-tachyon scattering amplitude from the $S^N\mathbf{R}^{24}$-orbifold CFT perturbed by the bosonic analogue of the DVV interaction vertex.

Obviously, the first step in constructing the scattering amplitudes consists in defining the incoming and outgoing asymptotic states $|i\rangle$ and $|f\rangle$. The free string limit $g_s\to 0$ implies that the asymptotic states should be identified with some states in the Hilbert space of the orbifold CFT and, therefore, should be created by some conformal fields. Then, by conventional quantum field theory, the g_s^n-order scattering amplitude A can be extracted from the S-matrix element described as a correlation function of n conformal fields $V(z_i)$ with subsequent integration over the insertion points z_i,

$$\langle f|S|i\rangle \sim \int\prod_i d^2z_i\,\langle f|V(z_1)\dots V(z_n)|i\rangle.$$

[1]Steklov Mathematical Institute, Moscow, Russia; arut@genesis.mi.ras.ru.

[2]Steklov Mathematical Institute, Moscow, Russia; Munich University, Munich, Germany.

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Construction of the asymptotic states $|i\rangle$ and $|f\rangle$ that can be identified with incoming and outgoing tachyons and computation of the above-mentioned correlation functions in the $S^N\mathbf{R}^{24}$-orbifold CFT are the main questions we consider in order to obtain the four-tachyon scattering amplitude.

The paper is organized as follows. In Sec. 2, we review the description of the Hilbert space of the orbifold model. In Sec. 3, we introduce the twist fields that create the states of the Hilbert space and find their conformal dimensions. In Sec. 4, we calculate the scattering amplitude and show that it coincides with the Virasoro one. In the Conclusion, we discuss some unsolved problems.

2. $S^N\mathbf{R}^D$-orbifold sigma model

We consider two-dimensional field theory on a cylinder described by the action

$$S = \frac{1}{2\pi}\int d\tau\, d\sigma\, (\partial_\tau X_I^i \partial_\tau X_I^i - \partial_\sigma X_I^i \partial_\sigma X_I^i), \tag{1}$$

where $0 \le \sigma < 2\pi$, $i = 1, 2, \ldots, D$, $I = 1, 2, \ldots, N$, and the fields X take values in $S^N\mathbf{R}^D \equiv (\mathbf{R}^D)^N/S_N$.

As is usual in orbifold models [8, 9], the fields X^i can have twisted boundary conditions,

$$X^i(\sigma + 2\pi) = gX^i(\sigma), \tag{2}$$

where g belongs to the symmetric group S_N.

Multiplying (2) by some element $h \in S_N$ and taking into account that X^i and hX^i describe the same configuration, we observe that all possible boundary conditions are in a one-to-one correspondence with the conjugacy classes of the symmetric group. Therefore, the Hilbert space of the orbifold model is decomposed into the direct sum of the Hilbert spaces of the twisted sectors corresponding to the conjugacy classes $[g]$ of S_N [6],

$$\mathcal{H}(S^N\mathbf{R}^D) = \bigoplus_{[g]} \mathcal{H}_{[g]}.$$

It is well known that the conjugacy classes of S_N are described by partitions $\{N_n\}$ of N,

$$N = \sum_{n=1}^{s} nN_n,$$

and can be represented as

$$[g] = (1)^{N_1}(2)^{N_2}\cdots(s)^{N_s}. \tag{3}$$

Here, N_n is the multiplicity of the cyclic permutation (n) of n elements.

In any conjugacy class $[g]$, there exists a unique element g_c with the canonical block-diagonal form

$$g_c = \mathrm{diag}(\underbrace{\omega_1, \ldots, \omega_1}_{N_1 \text{times}}, \underbrace{\omega_2, \ldots, \omega_2}_{N_2 \text{times}}, \ldots, \underbrace{\omega_s, \ldots, \omega_s}_{N_s \text{times}}), \tag{4}$$

where ω_n is an $n \times n$ matrix that generates the cyclic permutations (n) of n elements

$$\omega_n = \sum_{i=1}^{n-1} E_{i,i+1} + E_{n1}$$

and E_{ij} are matrix units.

It is not difficult to show that ω_n generates the $\mathbf{Z}_n$ group since $\omega_n^n = 1$ and that only the matrices ω_n^k from $\mathbf{Z}_n$ commute with ω_n. Since the centralizer subgroup C_g of any element $g \in [g]$ is isomorphic to C_{g_c}, we conclude that

$$C_g = \prod_{n=1}^{s} S_{N_n} \times \mathbf{Z}_n^{N_n},$$

where the symmetric group S_{N_n} permutes the N_n cycles (n). It is obvious that the centralizer C_g contains $\prod_{n=1}^{s} N_n! n^{N_n}$ elements.

Due to factorization (3) of $[g]$, the Hilbert space $\mathcal{H}_{[g]} \equiv \mathcal{H}_{\{N_n\}}$ of each twisted sector can be decomposed into the N_n-fold symmetric tensor products of the Hilbert spaces $\mathcal{H}_{(n)}$, which correspond to cycles of length n,

$$\mathcal{H}_{\{N_n\}} = \bigotimes_{n=1}^{s} S^{N_n}\mathcal{H}_{(n)} = \bigotimes_{n=1}^{s} \Big(\underbrace{\mathcal{H}_{(n)} \otimes \cdots \otimes \mathcal{H}_{(n)}}_{N_n \text{times}}\Big)^{S_{N_n}}.$$

The space $\mathcal{H}_{(n)}$ is a $\mathbf{Z}_n$-invariant subspace of the Hilbert space for a sigma model of Dn fields X_I^i with the cyclic boundary condition

$$X_I^i(\sigma + 2\pi) = X_{I+1}^i(\sigma), \quad I = 1, 2, \ldots, n. \tag{5}$$

The fields $X_I(\sigma)$ can be glued together to make one field $X(\sigma)$ that is identified with a long string of length n. The states of the space $\mathcal{H}_{(n)}$ are obtained by the creation operators of the string acting on eigenvectors of the momentum operator. These eigenvectors have the standard normalization

$$\langle \mathbf{q}|\mathbf{k}\rangle = \delta^D(\mathbf{q}+\mathbf{k})$$

and can be regarded as states obtained by the operator $e^{i\mathbf{k}x}$ acting on the vacuum state (which is not normalizable): $|\mathbf{k}\rangle = e^{i\mathbf{k}x}|0\rangle$, $\langle \mathbf{q}| = \langle 0|e^{i\mathbf{q}x}$.

The $\mathbf{Z}_n$-invariant subspace is singled out by imposing the condition

$$(L_0 - \bar{L}_0)|\Psi\rangle = nm|\Psi\rangle,$$

where m is an integer and L_0 is the canonically normalized L_0 operator of the single string.

If $D = 24$, then the Fock space of the second-quantized closed bosonic string is recovered in the limit $N \to \infty$, $n_i/N \to p_i^+$ [6], where the finite ratio n_i/N is identified with the p_i^+ momentum of a long string. The $\mathbf{Z}_n$ projection reduces in this limit to the usual level-matching condition $L_0^{(i)} - \bar{L}_0^{(i)} = 0$. The individual p_i^- light-cone momentum is defined by means of the standard mass-shell condition $p_i^+ p_i^- = L_0^{(i)}$.

3. Twist fields

Let us consider the CFT of DN free scalar fields described by action (1). It is convenient to perform the Wick rotation $\tau \to -i\tau$ and to map the cylinder onto the sphere: $z = e^{\tau+i\sigma}$, $\bar{z} = e^{\tau-i\sigma}$.

The vacuum state $|0\rangle$ of the CFT is annihilated by momentum operators and by annihilation operators and must be normalizable. To identify this vacuum state with the vacuum state of the untwisted sector of the orbifold sigma model, we choose the normalization of $|0\rangle$ to be

$$\langle 0|0\rangle = R^{DN}.$$

Here, R should be regarded as a regularization parameter of the sigma model. We regularize the sigma model by compactifying the coordinates x_I^i on circles of radius R. Then the norm of the eigenvectors of the momentum operators in the untwisted sector is given by

$$\langle \mathbf{q}|\mathbf{k}\rangle = (2\pi)^{-DN} \int_0^{2\pi R} d^{DN}x\, e^{i(\mathbf{q}+\mathbf{k})x} = \prod_{I=1}^{N} \delta_R^D(\mathbf{q}_I + \mathbf{k}_I),$$

where $k_I^i = m_I^i/R$ and $q_I^i = n_I^i/R$ are momenta of the states, m_I^i and n_I^i are integers, since we compactified the coordinates, and

$$\delta_R^D(\mathbf{k}) = R^D \prod_{i=1}^{D} \delta_{m^i 0}$$

is the regularized δ-function. The usual normalization of the eigenvectors is recovered in the limit $R \to \infty$.

As usual, the field $X(z,\bar{z})$ can be decomposed into left- and right-moving components,

$$2X(z,\bar{z}) = X(z) + \overline{X}(\bar{z}). \tag{6}$$

In what follows, we concentrate our attention mainly on the left-moving sector.

Let $\sigma_g(z,\bar{z})$ be a primary field [10] that creates a vacuum of a twisted sector at the point z, i.e., the fields $X^i(z,\bar{z})$ satisfy the monodromy conditions

$$X^i(ze^{2\pi i}, \bar{z}e^{-2\pi i})\sigma_g(0,0) = gX^i(z,\bar{z})\sigma_g(0,0).$$

Note that the twist field $\sigma_g(z,\bar{z})$ cannot be represented as the tensor product of the twist fields $\sigma_g(z)$ and $\bar{\sigma}_g(\bar{z})$ creating the vacuum states of the left- and right-moving sectors, respectively.

It is obvious that the conformal dimension Δ_g depends only on $[g]$. To calculate Δ_g, let us assume that g has factorization (3). Then Δ_g is given by the equation

$$\Delta_g = \sum_{n=1}^{s} N_n \Delta_{(n)}, \tag{7}$$

where $\Delta_{(n)}$ denotes the conformal dimension of the twist field $\sigma_{(n)}$ that creates the vacuum state of the space $\mathcal{H}_{(n)}$ for the sigma model of Dn fields with the cyclic boundary condition (5). Let the twist field $\sigma_{(n)}$ be located at $z=0$ and let us denote the vacuum state[3] as $|(n)\rangle = \sigma_{(n)}(0)|0\rangle$. Since the twist field $\sigma_{(n)}$ creates one long string, we normalize the vacuum state $|(n)\rangle$ as follows:

$$\langle (n)|(n)\rangle = R^D. \tag{8}$$

The fields $X(z)$ have the following decomposition in the vicinity of $z=0$:

$$\partial X_I^i(z) = -i\frac{1}{n}\sum_m \alpha_m^i \exp\left\{-\frac{2\pi i}{n}Im\right\} z^{-\frac{m}{n}-1}, \tag{9}$$

where α_m^i $(m \neq 0)$ are the usual creation and annihilation operators with the commutation relation

$$[\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m+n,0} \tag{10}$$

and α_0^i is proportional to the momentum operator.[4]

The vacuum state $|(n)\rangle$ is annihilated by the operators α_m^i for $m \geq 0$.

Since $\sigma_{(n)}$ is a primary field, the conformal dimension $\Delta_{(n)}$ can be found from the equation

$$\langle (n)|T(z)|(n)\rangle = \frac{\Delta_{(n)}}{z^2}\langle (n)|(n)\rangle,$$

where $T(z)$ is the stress-energy tensor.

Using (9) and (10), we calculate the correlation function

$$\langle (n)|\partial X_I^i(z)\partial X_I^j(w)|(n)\rangle = -\delta^{ij}\frac{(zw)^{\frac{1}{n}-1}}{n^2\left(z^{\frac{1}{n}} - w^{\frac{1}{n}}\right)^2}\langle (n)|(n)\rangle.$$

[3] This vacuum state is a primary state of the CFT.

[4] $\alpha_0^i = p^i/2$ in the string units $\alpha' = 1/2$.

Taking into account that the stress-energy tensor is defined as

$$T(z) = -\frac{1}{2}\lim_{w\to z}\sum_{i=1}^{D}\sum_{I=1}^{n}\left(\partial X_I^i(z)\partial X_I^i(w) + \frac{1}{(z-w)^2}\right),$$

one obtains

$$\Delta_{(n)} = \frac{D}{24}\left(n - \frac{1}{n}\right). \tag{11}$$

The excited states of this sigma model are obtained by acting on $|(n)\rangle$ with some vertex operators. In particular, the state corresponding to a scalar particle with momentum $\mathbf{k}$ is given by

$$\sigma_{(n)}[\mathbf{k}](0,0)|0\rangle = :\exp\{ik_I^i X_I^i(0,0)\}:|(n)\rangle, \tag{12}$$

where summation over i and I is assumed, $k_I^i = m_I^i/R$ is the momentum carried by the field $X_I^i(z,\bar{z})$, and $k^i = \sum_{I=1}^{n} k_I^i$ is the total momentum of the long string.

Using the definition of the vacuum state $|(n)\rangle$, we can rewrite Eq. (12) in the form

$$\sigma_{(n)}[\mathbf{k}](0,0)|0\rangle = :\exp\left\{i\frac{k^i}{\sqrt{n}}Y^i(0,0)\right\}:|(n)\rangle, \tag{13}$$

where

$$Y^i(z,\bar{z}) = \frac{1}{\sqrt{n}}\sum_{I=1}^{n} X_I^i(z,\bar{z}). \tag{14}$$

The field $Y(z)$ has a trivial monodromy around $z=0$ and is canonically normalized, i.e., the part of the stress-energy tensor depending on Y is $-(1/2){:}\partial Y(z)\partial Y(z){:}$.

It is obvious from (13) that the conformal dimension of the primary field

$$\sigma_{(n)}[\mathbf{k}](z,\bar{z}) = :\exp\left\{i\frac{k^i}{\sqrt{n}}Y^i(z,\bar{z})\right\}:\sigma_{(n)}(z,\bar{z})$$

is equal to

$$\Delta_{(n)}[\mathbf{k}] = \Delta_{(n)} + \frac{\mathbf{k}^2}{8n} = \frac{D}{24}\left(n - \frac{1}{n}\right) + \frac{\mathbf{k}^2}{8n},$$

where decomposition (6) is taken into account.

According to Eqs. (7) and (11), the conformal dimension of σ_g is given by

$$\Delta_g = \sum_{n=1}^{s} N_n\frac{D}{24}\left(n - \frac{1}{n}\right) = \frac{D}{24}\left(N - \sum_{n=1}^{s}\frac{N_n}{n}\right).$$

We can also introduce a primary field that creates scalar particles with momenta k_α^i, $\alpha = 1,2,\ldots,$ $N_1 + N_2 + \cdots + N_s \equiv N_{\text{str}}$,

$$\sigma_g[\{\mathbf{k}_\alpha\}](z,\bar{z}) = :\exp\left\{i\frac{k_\alpha^i}{\sqrt{n_\alpha}}Y_\alpha^i(z,\bar{z})\right\}:\sigma_g(z,\bar{z}),$$

where $n_1 = n_2 = \cdots = n_{N_1} = 1$, $n_{N_1+1} = n_{N_1+2} = \cdots = n_{N_1+N_2} = 2$, and so on, Y_α^i corresponds to the cycle (n_α) and is defined by (14), and summation over i and α is assumed. The conformal dimension of the field $\sigma_g[\{\mathbf{k}_\alpha\}]$ is equal to

$$\Delta_g[\{\mathbf{k}_\alpha\}] = \frac{D}{24}\left(N - \sum_{n=1}^{s}\frac{N_n}{n}\right) + \sum_\alpha \frac{\mathbf{k}_\alpha^2}{8n_\alpha}. \tag{15}$$

It is obvious that the two-point correlation function of the twist fields σ_{g_1} and σ_{g_2} is not equal to zero if and only if $g_1 g_2 = 1$. Taking into account normalization (8), we find[5]

$$\langle \sigma_{g^{-1}}(\infty)\sigma_g(0)\rangle = R^{DN_{\rm str}}.$$

This means that the fields $\sigma_{g^{-1}}$ and σ_g have the OPE

$$\sigma_{g^{-1}}(z,\bar{z})\sigma_g(0,0) = \frac{R^{D(N_{\rm str}-N)}}{|z|^{4\Delta_g}} + \cdots .$$

The two-point correlation function of $\sigma_{g^{-1}}[\{\mathbf{q}_\alpha\}]$ and $\sigma_g[\{\mathbf{k}_\alpha\}]$, consequently, is equal to

$$\langle \sigma_{g^{-1}}[\{\mathbf{q}_\alpha\}](\infty)\sigma_g[\{\mathbf{k}_\alpha\}](0)\rangle = \prod_\alpha \delta_R^D(\mathbf{q}_\alpha + \mathbf{k}_\alpha). \tag{16}$$

A twist field σ_g does not create a twisted sector of the orbifold CFT since it is not invariant with respect to the action of the symmetric group. An invariant twist field can be defined by summing all of the twist fields from one conjugacy class,

$$\sigma_{[g]}(z,\bar{z}) = \frac{1}{N!}\sum_{h\in S_N} \sigma_{h^{-1}gh}(z,\bar{z}).$$

Using this definition, we can easily calculate the two-point correlation function

$$\langle \sigma_{[g]}(\infty)\sigma_{[g]}(0)\rangle = \frac{R^{DN_{\rm str}}}{N!}\prod_{n=1}^{s} N_n! n^{N_n},$$

where $\prod_{n=1}^s N_n! n^{N_n}$ is the number of elements of the centralizer C_g.

The definition of the twist field $\sigma_{[g]}[\{\mathbf{k}_\alpha\}]$ is not so straightforward. Let us consider the element $g_c \in [g]$ that has the canonical block-diagonal form (4). There are $N_1 + \cdots + N_s = N_{str}$ fields $Y_\alpha(z,\bar{z})$ that have a trivial monodromy in the vicinity of the twist field σ_{g_c}. According to (14), they are defined as

$$Y_\alpha(z,\bar{z}) = \frac{1}{\sqrt{n_\alpha}}\sum_{I\in(n_\alpha)} X_I(z,\bar{z}).$$

Now let us consider the fields X that have the monodromy

$$X(ze^{2\pi i}, \bar{z}e^{-2\pi i}) = h^{-1}g_c h X(z,\bar{z}). \tag{17}$$

We can see from (17) that the fields $Y_\alpha[h]$,

$$Y_\alpha[h](z,\bar{z}) = \frac{1}{\sqrt{n_\alpha}}\sum_{I\in(n_\alpha)} (hX)_I(z,\bar{z}),$$

have a trivial monodromy. Then an invariant twist field $\sigma_{[g]}[\{\mathbf{k}_\alpha\}]$ is defined as

$$\sigma_{[g]}[\{\mathbf{k}_\alpha\}](z,\bar{z}) = \frac{1}{N!}\sum_{h\in S_N} :\exp\left\{ i\frac{k_\alpha^i}{\sqrt{n_\alpha}} Y_\alpha^i[h](z,\bar{z})\right\}: \sigma_{h^{-1}g_c h}(z,\bar{z}). \tag{18}$$

[5]It is clear that $[g^{-1}] = [g]$ and, therefore, $\Delta_{g^{-1}} = \Delta_g$.

It is easy to verify that the twist field $\sigma_{[g]}[\{\mathbf{k}_\alpha\}]$ is invariant with respect to the permutation of momenta $\mathbf{k}_\alpha$, which correspond to cycles (n_α) of the same length.

The interaction vertex proposed by DVV [5] is defined with the help of the twist field σ_{IJ} that corresponds to the group element $g_{IJ} = 1 - E_{II} - E_{JJ} + E_{IJ} + E_{JI}$ transposing the fields X_I and X_J.

The twist field σ_g has the following OPE[6]:

$$\sigma_{g_1}(z, \bar{z})\sigma_{g_2}(0) = \frac{1}{|z|^{2\Delta_{g_1}+2\Delta_{g_2}-2\Delta_{g_1 g_2}}}\left(C^{g_1 g_2}_{g_1,g_2}\sigma_{g_1 g_2}(0) + C^{g_2 g_1}_{g_1,g_2}\sigma_{g_2 g_1}(0)\right) + \cdots. \tag{19}$$

Here, two leading terms appear because there are two different ways to go around the points z and 0. It is not difficult to see that $g_1 g_2$ and $g_2 g_1$ belong to the same conjugacy class and, hence, $\Delta_{g_1 g_2} = \Delta_{g_2 g_1}$.

Therefore, the twist field σ_{IJ} acting on the state $\sigma_g(0)|0\rangle$ creates the states $\sigma_{g_{IJ}g}(0)|0\rangle$ and $\sigma_{g g_{IJ}}(0)|0\rangle$. An arbitrary element g has a decomposition $(n_1)(n_2)\cdots(n_k)$ that describes a configuration with k strings. If the indices I and J belong, for instance, to the cycle (n_1) in the decomposition, then the element $g_{IJ}g$ has the decomposition $(n_1^{(1)})(n_1^{(2)})(n_2)\cdots(n_k)$ with $n_1^{(1)} + n_1^{(2)} = n_1$ and, hence, describes a configuration with $k+1$ strings. If the index I belongs to the cycle (n_1) and the index J belongs to (n_2), then the element $g_{IJ}g$ has the decomposition $(n_1 + n_2)(n_3)\cdots(n_k)$ and describes a configuration with $k-1$ strings. Thus, the twist field σ_{IJ} generates the elementary joining and splitting of strings.

To describe the DVV interaction vertex, it is useful to return to Minkowskian space–time. Then the interaction is described by the translation-invariant vertex

$$V_{\text{int}} = \frac{\lambda N}{2\pi}\sum_{I<J}\int d\tau\, d\sigma\, \sigma_{IJ}(\sigma_+, \sigma_-),$$

where λ is a coupling constant proportional to the string coupling and $\sigma_\pm = \tau \pm \sigma$ are light-cone coordinates.

If $D = 24$, then the twist field $\sigma_{IJ}(\sigma_+, \sigma_-)$ is a weight $(3/2, 3/2)$ conformal field and the coupling constant λ has dimension -1. Performing the Wick rotation and the conformal mapping onto the sphere, again, we obtain the following expression for V_{int} (and for $D = 24$):

$$V_{\text{int}} = -\frac{\lambda N}{2\pi}\sum_{I<J}\int d^2 z\, |z|\sigma_{IJ}(z, \bar{z}),$$

where the minus sign appears because σ_{IJ} has the conformal dimension $(3/2, 3/2)$.

Thus, the action of the interacting $S^N\mathbf{R}^{24}$-orbifold sigma model is given by the sum

$$S_{\text{int}} = S_0 + V_{\text{int}}.$$

In the next section, we calculate the S-matrix element corresponding to the scattering of four tachyons and show that the scattering amplitude coincides with the Virasoro amplitude.

4. Scattering amplitude

In the second order in the coupling constant λ, the S-matrix element is given by the standard formula of quantum field theory,

$$\langle f|S|i\rangle = -\frac{1}{2}\left(\frac{\lambda N}{2\pi}\right)^2\left\langle f \,\middle|\, \int d^2 z_1\, d^2 z_2\, |z_1||z_2| T\left(\mathcal{L}_{\text{int}}(z_1, \bar{z}_1)\mathcal{L}_{\text{int}}(z_2, \bar{z}_2)\right) \,\middle|\, i\right\rangle, \tag{20}$$

[6]Let us stress that there are other primary fields on the r.h.s. of the OPE; however, these fields are nonessential in our consideration.

where the symbol T means the time-ordering, $|z_1| > |z_2|$, and

$$\mathcal{L}_{\text{int}}(z,\bar{z}) = \sum_{I<J} \sigma_{IJ}(z,\bar{z}).$$

The initial state $|i\rangle$ describes two tachyons with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ and is created by the twist field $\sigma_{[g_0]}[\mathbf{k}_1,\mathbf{k}_2]$,

$$|i\rangle = C_0\sigma_{[g_0]}[\mathbf{k}_1,\mathbf{k}_2](0,0)|0\rangle.$$

The element g_0 is taken in the canonical block-diagonal form

$$g_0 = (n_0)(N-n_0),$$

where $n_0 < N - n_0$.

The final state $\langle f|$ describes two tachyons with momenta $\mathbf{k}_3$ and $\mathbf{k}_4$ and is given by the formula (see [10])

$$\langle f| = C_\infty \lim_{z_\infty \to \infty} |z_\infty|^{4\Delta_\infty} \langle 0|\sigma_{[g_\infty]}[\mathbf{k}_3,\mathbf{k}_4](z_\infty,\bar{z}_\infty).$$

The element g_∞ has the canonical decomposition

$$g_\infty = (n_\infty)(N-n_\infty), \quad n_\infty < N - n_\infty.$$

The constants C_0 and C_∞ are chosen to be

$$C_0 = \sqrt{\frac{N!}{n_0(N-n_0)}}, \qquad C_\infty = \sqrt{\frac{N!}{n_\infty(N-n_\infty)}},$$

which guarantees the standard normalization of the initial and final states.

After the conformal transformation $z \to z/z_1$, Eq. (20) acquires the form

$$\langle f|S|i\rangle = -\frac{1}{2}\left(\frac{\lambda N}{2\pi}\right)^2 \int d^2z_1\, d^2z_2\, |z_1||z_2||z_1|^{2\Delta_\infty - 2\Delta_0 - 6} \left\langle f \,\middle|\, T\left(\mathcal{L}_{\text{int}}(1,1)\mathcal{L}_{\text{int}}\left(\frac{z_2}{z_1},\frac{\bar{z}_2}{\bar{z}_1}\right)\right) \,\middle|\, i\right\rangle, \tag{21}$$

where Δ_0 and Δ_∞ are conformal dimensions of the twist fields $\sigma_{[g_0]}[\mathbf{k}_1,\mathbf{k}_2]$ and $\sigma_{[g_\infty]}[\mathbf{k}_3,\mathbf{k}_4]$,

$$\begin{aligned} \Delta_0 &= N - \frac{1}{n_0} - \frac{1}{N-n_0} + \frac{\mathbf{k}_1^2}{8n_0} + \frac{\mathbf{k}_2^2}{8(N-n_0)}, \\ \Delta_\infty &= N - \frac{1}{n_\infty} - \frac{1}{N-n_\infty} + \frac{\mathbf{k}_3^2}{8n_\infty} + \frac{\mathbf{k}_4^2}{8(N-n_\infty)}. \end{aligned} \tag{22}$$

Let us introduce the light-cone momenta of the tachyons [5], taking into account the mass-shell condition for the tachyonic states,

$$\begin{aligned} k_1^+ &= \frac{n_0}{N}, & k_1^- k_1^+ - \mathbf{k}_1^2 &\equiv -k_1^2 = -8, \\ k_2^+ &= \frac{N-n_0}{N}, & k_2^- k_2^+ - \mathbf{k}_2^2 &\equiv -k_2^2 = -8, \\ k_3^+ &= -\frac{n_\infty}{N}, & k_3^- k_3^+ - \mathbf{k}_3^2 &\equiv -k_3^2 = -8, \\ k_4^+ &= -\frac{N-n_\infty}{N}, & k_4^- k_4^+ - \mathbf{k}_4^2 &\equiv -k_4^2 = -8. \end{aligned}$$

Using the light-cone momenta and the mass-shell condition, we rewrite (22) in the form

$$\Delta_0 = N + \frac{k_1^- + k_2^-}{8N},$$
$$\Delta_\infty = N - \frac{k_3^- + k_4^-}{8N}.$$

Performing the change of variables $z_2/z_1 = u$, we obtain

$$\langle f|S|i\rangle = -\frac{1}{2}\left(\frac{\lambda N}{2\pi}\right)^2 \int d^2 z_1\, |z_1|^{2\Delta_\infty - 2\Delta_0 - 2} \int d^2 u\, |u| \langle f|T(\mathcal{L}_{\text{int}}(1,1)\mathcal{L}_{\text{int}}(u,\bar{u}))|i\rangle.$$

The integral over z_1 is obviously divergent. To understand the meaning of this divergence, remember that we have made the Wick rotation. Returning to the σ, τ coordinates on the cylinder, we find for the integral over z_1 that

$$\int d^2 z_1\, |z_1|^{2\Delta_\infty - 2\Delta_0 - 2} \to i \int d\tau\, d\sigma\, e^{2i\tau(\Delta_\infty - \Delta_0)}.$$

Integration over σ and τ gives us the conservation law for the light-cone momenta k_i^-,

$$\int d\tau\, d\sigma\, e^{2i\tau(\Delta_\infty - \Delta_0)} = 4N(2\pi)^2 \delta(k_1^- + k_2^- + k_3^- + k_4^-).$$

Therefore, the S-matrix element is equal to

$$\langle f|S|i\rangle = -i2\lambda^2 N^3 \delta(k_1^- + k_2^- + k_3^- + k_4^-) \int d^2 u\, |u|\, \langle f|T\left(\mathcal{L}_{\text{int}}(1,1)\mathcal{L}_{\text{int}}(u,\bar{u})\right)|i\rangle. \quad (23)$$

To find the S-matrix element, we have to calculate the correlation function

$$F(u,\bar{u}) = \langle f|T(\mathcal{L}_{\text{int}}(1,1)\mathcal{L}_{\text{int}}(u,\bar{u}))|i\rangle =$$
$$= C_0 C_\infty \sum_{I<J;K<L} \langle \sigma_{[g_\infty]}[\mathbf{k}_3,\mathbf{k}_4](\infty) T(\sigma_{IJ}(1,1)\sigma_{KL}(u,\bar{u}))\sigma_{[g_0]}[\mathbf{k}_1,\mathbf{k}_2](0,0)\rangle. \quad (24)$$

We assume for definiteness that $n_0 < n_\infty$ and $|u| < 1$ in what follows.

Using definition (18) of $\sigma_{[g]}[\{\mathbf{k}_\alpha\}]$ and taking into account that the interaction vertex is S_N-invariant and that any correlation function of the twist fields is invariant with respect to the global action of the symmetric group

$$\langle \sigma_{g_1}\sigma_{g_2}\cdots\sigma_{g_n}\rangle = \langle \sigma_{h^{-1}g_1 h}\sigma_{h^{-1}g_2 h}\cdots\sigma_{h^{-1}g_n h}\rangle, \quad (25)$$

we rewrite the correlation function in the form

$$F(u,\bar{u}) = \frac{C_0 C_\infty}{N!} \sum_{h_\infty \in S_N} \sum_{I<J;K<L} \langle \sigma_{h_\infty^{-1} g_\infty h_\infty}[\mathbf{k}_3,\mathbf{k}_4](\infty)\sigma_{IJ}(1,1)\sigma_{KL}(u,\bar{u})\sigma_{g_0}[\mathbf{k}_1,\mathbf{k}_2](0,0)\rangle.$$

We note that the correlation function

$$\langle \sigma_{g_1}(\infty)\sigma_{g_2}(1,1)\sigma_{g_3}(u,\bar{u})\sigma_{g_4}(0,0)\rangle \quad (26)$$

does not vanish only if

$$g_1 g_2 g_3 g_4 = 1 \quad \text{or} \quad g_1 g_4 g_3 g_2 = 1. \quad (27)$$

This can be seen as follows. Due to the OPE (19) of σ_g, correlation function (26) reduces to the sum of the three-point correlation functions $\langle\sigma_{g_1}\sigma_{g_2}\sigma_{g_3g_4}\rangle$ and $\langle\sigma_{g_1}\sigma_{g_2}\sigma_{g_4g_3}\rangle$ in the limit $u\to 0$. This sum does not vanish if one of the following equations is fulfilled:

$$g_1g_2g_3g_4=1,\quad g_1g_3g_4g_2=1,\quad g_1g_2g_4g_3=1,\quad g_1g_4g_3g_2=1. \tag{28}$$

From the other side, in the limit $u\to 1$, we obtain the sum of the correlation functions $\langle\sigma_{g_1}\sigma_{g_2g_3}\sigma_{g_4}\rangle$ and $\langle\sigma_{g_1}\sigma_{g_3g_2}\sigma_{g_4}\rangle$. This sum does not vanish if

$$g_1g_2g_3g_4=1,\quad g_1g_4g_2g_3=1,\quad g_1g_3g_2g_4=1,\quad g_1g_4g_3g_2=1. \tag{29}$$

Comparing Eqs. (28) and (29), we obtain (27).

The contribution of the terms satisfying the equation $h_\infty^{-1}g_\infty h_\infty g_0g_{KL}g_{IJ}=1$, however, coincides with the contribution of the terms satisfying $h_\infty^{-1}g_\infty h_\infty g_{IJ}g_{KL}g_0=1$. To prove the statement, we note that the invariance of action (1) with respect to the world-sheet parity symmetry $z\to\bar z$ (or $\sigma\to-\sigma$ in Minkowskian space–time) leads to the equality

$$\langle\sigma_{g_1}\sigma_{g_2}\cdots\sigma_{g_n}\rangle=\langle\sigma_{g_1^{-1}}\sigma_{g_2^{-1}}\cdots\sigma_{g_n^{-1}}\rangle, \tag{30}$$

since the twist fields σ_g transform into $\sigma_{g^{-1}}$. Taking into account Eqs. (25) and (30) and the fact that the elements g and g^{-1} belong to the same conjugacy class, we obtain the desired equality,

$$\begin{aligned}\sum_{I<J;K<L}\langle\sigma_{g_{IJ}g_{KL}g_0^{-1}}\sigma_{IJ}\sigma_{KL}\sigma_{g_0}\rangle&=\sum_{I<J;K<L}\langle\sigma_{g_0g_{KL}g_{IJ}}\sigma_{IJ}\sigma_{KL}\sigma_{g_0^{-1}}\rangle=\\&=\sum_{I<J;K<L}\langle\sigma_{g_0^{-1}g_{KL}g_{IJ}}\sigma_{IJ}\sigma_{KL}\sigma_{g_0}\rangle.\end{aligned}$$

Therefore, the function $F(u,\bar u)$ is given by the sum of the correlation functions of twist fields which can be schematically represented as

$$\mathcal{S}=\sum_{h_\infty\in S_N}\sum_{I<J;K<L}\langle\sigma_{h_\infty^{-1}g_\infty h_\infty}\sigma_{IJ}\sigma_{KL}\sigma_{g_0}\rangle,$$

where the elements h_∞, g_{IJ}, and g_{KL} solve the equation $h_\infty^{-1}g_\infty h_\infty g_{IJ}g_{KL}g_0=1$. We can fix the values of the indices K and L using the action of the stabilizer of g_0 and invariance (25) of the correlation functions

$$\begin{aligned}\mathcal{S}=\sum_{h_\infty\in S_N}\sum_{I<J}\big(&n_0(N-n_0)\langle\sigma_{h_\infty^{-1}g_\infty h_\infty}\sigma_{IJ}\sigma_{n_0N}\sigma_{g_0}\rangle+\\&+(N-n_0)\langle\sigma_{h_\infty^{-1}g_\infty h_\infty}\sigma_{IJ}\sigma_{n_\infty N}\sigma_{g_0}\rangle+\\&+(N-n_0)\langle\sigma_{h_\infty^{-1}g_\infty h_\infty}\sigma_{IJ}\sigma_{n_0+n_\infty,N}\sigma_{g_0}\rangle\big).\end{aligned} \tag{31}$$

The first term in (31) corresponds to the joining of two incoming strings and the factor $n_0(N-n_0)$ appears because, in this case, the index K takes n_0 values, $K=1,\dots n_0$, and the index L takes $N-n_0$ values, $L=n_0+1,\dots,N$. To fix $K=n_0$ and $L=N$, we need to use all of the elements of C_{g_0}. The second and the third terms correspond to splitting the string of length $N-n_0$ into two strings of lengths $n_\infty-n_0$ and $N-n_\infty$ and of lengths $N-n_0-n_\infty$ and n_∞, respectively. To fix the values of K and L in these cases, we use the $N-n_0$ elements, which compose the subgroup $\mathbf{Z}_{N-n_0}$ of C_{g_0}, that do not act on the cycle (n_0). Equation (31) can be further rewritten in the form

$$\begin{aligned}\mathcal{S}=n_0(N-n_0)n_\infty(N-n_\infty)\Bigg(&\sum_{I=1}^{n_\infty}\langle\sigma_{g_\infty(I)}\sigma_{I,I+N-n_\infty}\sigma_{n_0N}\sigma_{g_0}\rangle+\\&+\sum_{I=1}^{N-n_\infty}\langle\sigma_{g_\infty(I)}\sigma_{I,I+n_\infty}\sigma_{n_0N}\sigma_{g_0}\rangle+\sum_{J=n_0+1}^{n_\infty}\langle\sigma_{g_\infty(J)}\sigma_{n_0J}\sigma_{n_\infty N}\sigma_{g_0}\rangle+\\&+\sum_{J=n_0+n_\infty+1}^{N}\langle\sigma_{g_\infty(J)}\sigma_{n_0J}\sigma_{n_0+n_\infty,N}\sigma_{g_0}\rangle\Bigg),\end{aligned} \tag{32}$$

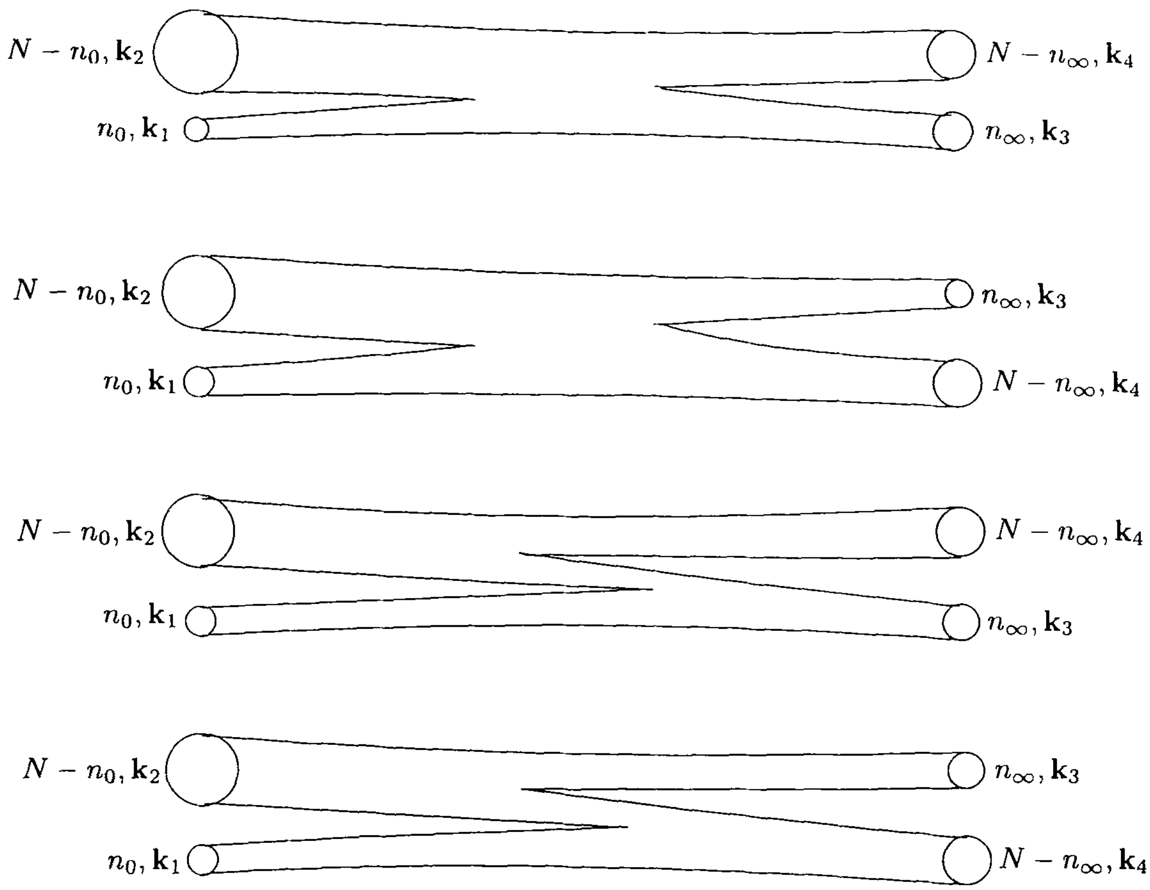

Fig. 1. The diagram representation of different correlation functions in Eq. (32).

where the elements g_∞ are found from the equation $g_\infty g_{IJ} g_{KL} g_0 = 1$.

Some comments are in order. The factor $n_\infty(N - n_\infty)$ is the volume of the stabilizer $\mathbf{Z}_{n_\infty} \times \mathbf{Z}_{N-n_\infty}$ of g_∞. The first two terms correspond to a splitting of the long string of length N into strings of lengths n_∞ and $N - n_\infty$. This can be achieved only if $J - I = N - n_\infty$ or $J - I = n_\infty$. In the third and fourth terms, we fixed the value of $I = n_0$ using the action of the subgroup $\mathbf{Z}_{n_0}$ of C_{g_0}. This gave the additional factor n_0. The third (fourth) term describes the joining of the strings of lengths n_0 and $n_\infty - n_0$ ($N - n_0 - n_\infty$) into one string of length n_∞ ($N - n_\infty$). Therefore, the total number of different correlation functions is equal to $2(N - n_0)$. The diagrams corresponding to these four terms are depicted in Fig. 1.

We need to compute the correlation functions (and the same correlation functions with the interchange $u \leftrightarrow 1$)

$$G(u, \bar{u}) = \langle \sigma_{g_\infty}[\mathbf{k}_3, \mathbf{k}_4](\infty) \sigma_{IJ}(1, 1) \sigma_{KL}(u, \bar{u}) \sigma_{g_0}[\mathbf{k}_1, \mathbf{k}_2](0, 0) \rangle, \tag{33}$$

where all possible elements $g_\infty, g_{IJ}, g_{KL}, g_0$ are listed in (32).

We employ the stress-energy tensor method [11] to calculate the correlation function (33). The idea of the method is as follows. Assume that we know the ratio

$$f(z, u) = \frac{\langle T(z) \phi_\infty(\infty) \phi_1(1) \phi_2(u) \phi_0(0) \rangle}{\langle \phi_\infty(\infty) \phi_1(1) \phi_2(u) \phi_0(0) \rangle}, \tag{34}$$

where $T(z)$ is the stress-energy tensor and each ϕ is a primary field. Taking into account that the OPE of $T(z)$ with any primary field has the form

$$T(z)\phi(0) = \frac{\Delta}{z^2}\phi(0) + \frac{1}{z}\partial\phi(0) + \cdots,$$

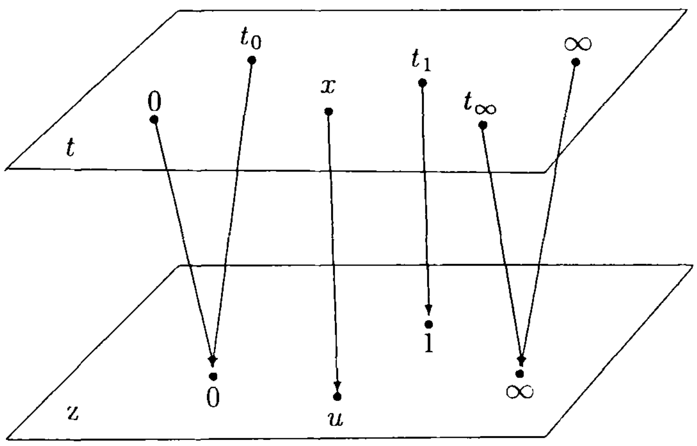

Fig. 2. The N-fold covering of the z-sphere by the t-sphere.

we obtain a differential equation for the correlation function $G(u,\bar{u}) = \langle \phi_\infty(\infty)\phi_1(1)\phi_2(u)\phi_0(0)\rangle$,

$$\partial_u \log G(u,\bar{u}) = H(u,\bar{u}),$$

where $H(u,\bar{u})$ is the second term in the decomposition of the function $f(z,u)$ in the vicinity of u,

$$f(z,u) = \frac{\Delta_2}{(z-u)^2} + \frac{1}{z-u} H(u,\bar{u}) + \cdots .$$

In the same way, we obtain the second equation for $G(u,\bar{u})$ using the stress-energy tensor $\overline{T}(\bar{z})$,

$$\partial_{\bar{u}} \log G(u,\bar{u}) = \overline{H}(u,\bar{u}).$$

A solution of these two equations determines the correlation function $G(u,\bar{u})$ up to a constant.

To calculate ratio (34), we first find the Green's functions[7] of the form

$$G^{ij}_{MS}(z,w) = \frac{\langle \partial X^i_M(z)\partial X^j_S(w)\sigma_{g_\infty}[\mathbf{k}_3,\mathbf{k}_4](\infty)\sigma_{IJ}(1,1)\sigma_{KL}(u,\bar{u})\sigma_{g_0}[\mathbf{k}_1,\mathbf{k}_2](0,0)\rangle}{\langle \sigma_{g_\infty}[\mathbf{k}_3,\mathbf{k}_4](\infty)\sigma_{IJ}(1,1)\sigma_{KL}(u,\bar{u})\sigma_{g_0}[\mathbf{k}_1,\mathbf{k}_2](0,0)\rangle} \equiv$$
$$\equiv \langle\langle \partial X^i_M(z)\partial X^j_S(w)\rangle\rangle.$$

These Green's functions have nontrivial monodromies around the points ∞, 1, u, and 0 and, in fact, are different branches of one multi-valued function. However, this function is single-valued on the sphere obtained by gluing the fields X^i_I at $z = 0$ and $z = \infty$. Therefore, to construct $G^{ij}_{MS}(z,w)$, we introduce the following map from this sphere onto the original sphere:

$$z = \frac{t^{n_0}(t-t_0)^{N-n_0}}{(t-t_\infty)^{N-n_\infty}} \frac{(t_1-t_\infty)^{N-n_\infty}}{t_1^{n_0}(t_1-t_0)^{N-n_0}} \equiv u(t). \tag{35}$$

Here the points $t = 0$ and $t = t_0$ are mapped to the point $z = 0$; $t = \infty$, $t = t_\infty \to z = \infty$; $t = t_1 \to z = 1$; and $t = x \to z = u$ (see Fig. 2). The map (35) may be viewed as the N-fold covering of the z-sphere by the t-sphere on which the Green's function is single-valued. A more detailed discussion of (35) is presented in the Appendix.

[7] We consider the correlation functions for general values of D, keeping in mind the application to the superstring case.

Due to the projective transformations, the positions of the points t_0, t_∞, and t_1 depend on x, and it is convenient to choose this dependence as follows:

$$
\begin{aligned}
t_0 &= x - 1, \\
t_\infty &= x - \frac{(N - n_\infty)x}{(N - n_0)x + n_0}, \\
t_1 &= \frac{N - n_0 - n_\infty}{n_\infty} + \frac{n_0 x}{n_\infty} - \frac{N(N - n_\infty)x}{n_\infty((N - n_0)x + n_0)}.
\end{aligned}
$$

This choice leads to the following expression for the rational function $u(x)$:

$$
\begin{aligned}
u = u(x) = (n_0 - n_\infty)^{n_0 - n_\infty} \frac{n_\infty^{n_\infty}}{n_0^{n_0}} \left(\frac{N - n_0}{N - n_\infty}\right)^{N - n_\infty} \left(\frac{x + \frac{n_0}{N - n_0}}{x - 1}\right)^{N} \times \\
\times \left(\frac{x - \frac{N - n_0 - n_\infty}{N - n_0}}{x}\right)^{N - n_0 - n_\infty} \left(x - \frac{n_0}{n_0 - n_\infty}\right)^{n_0 - n_\infty}.
\end{aligned} \tag{36}
$$

Since $n_0 < n_\infty$, the map $u(x)$ can be treated as the $2(N - n_0)$-fold covering of the u-sphere by the x-sphere, which means that the equation $u(x) = u$ has $2(N - n_0)$ different solutions. It is worthwhile to note that this number coincides with the number of nontrivial correlation functions in (32) and, therefore, different roots of Eq. (36) correspond to different correlation functions in (32). We see that the t-sphere can be represented as the union of $2(N - n_0)$ domains, and each domain V_{IJKL} contains the points x corresponding to the correlation function (33). If we take the appropriate system of cuts on the u-sphere, then every root of Eq. (36) realizes a one-to-one conformal mapping of the cut u-plain onto the corresponding domain V_{IJKL}.

Now let us choose some root of Eq. (36). We can always cut the z-sphere and numerate the roots $t_R(z)$ of Eq. (35) in such a way that they have the same monodromies as the fields X. Then the Green's functions are obviously not equal to zero only if $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4 = 0$ and are given by

$$
\begin{aligned}
G^{ij}_{MS}(z, w) = -\delta^{ij} \frac{t'_M(z) t'_S(w)}{(t_M(z) - t_S(w))^2} - \frac{k_1^i k_1^j t'_M(z) t'_S(w)}{4 t_M(z) t_S(w)} - \\
- \frac{k_1^i k_2^j t'_M(z) t'_S(w)}{4 t_M(z)(t_S(w) - t_0)} - \frac{k_2^i k_1^j t'_M(z) t'_S(w)}{4(t_M(z) - t_0) t_S(w)} - \\
- \frac{k_2^i k_2^j t'_M(z) t'_S(w)}{4(t_M(z) - t_0)(t_S(w) - t_0)} - \frac{k_1^i k_4^j t'_M(z) t'_S(w)}{4 t_M(z)(t_S(w) - t_\infty)} - \\
- \frac{k_2^i k_4^j t'_M(z) t'_S(w)}{4(t_M(z) - t_0)(t_S(w) - t_\infty)} - \frac{k_4^i k_1^j t'_M(z) t'_S(w)}{4(t_M(z) - t_\infty) t_S(w)} - \\
- \frac{k_4^i k_2^j t'_M(z) t'_S(w)}{4(t_M(z) - t_\infty)(t_S(w) - t_0)} - \frac{k_4^i k_4^j t'_M(z) t'_S(w)}{4(t_M(z) - t_\infty)(t_S(w) - t_\infty)}.
\end{aligned} \tag{37}
$$

It is easy to verify that these functions have the singularity $-\delta^{ij}\delta_{MS}/(z - w)^2$ in the vicinity of $z - w = 0$ and proper monodromies at the points $z = \infty$, 1, u, and 0.

Recall that the stress-energy tensor is defined as

$$
T(z) = -\frac{1}{2} \lim_{w \to z} \sum_{i=1}^{D} \sum_{I=1}^{N} \left(\partial X^i_I(z) \partial X^i_I(w) + \frac{1}{(z - w)^2}\right).
$$

Using this definition and (37), we obtain[8]

$$\langle\langle T(z)\rangle\rangle = \sum_M \left(\frac{D}{12}\left(\left(\frac{t''_M(z)}{t'_M(z)}\right)' - \frac{1}{2}\left(\frac{t''_M(z)}{t'_M(z)}\right)^2 \right) + \frac{\mathbf{k}_1^2 (t'_M(z))^2}{8(t_M(z))^2} + \right.$$
$$+ \frac{\mathbf{k}_1\mathbf{k}_2 (t'_M(z))^2}{4t_M(z)(t_M(z)-t_0)} + \frac{\mathbf{k}_2^2 (t'_M(z))^2}{8(t_M(z)-t_0)^2} +$$
$$+ \frac{\mathbf{k}_1\mathbf{k}_4 (t'_M(z))^2}{4t_M(z)(t_M(z)-t_\infty)} + \frac{\mathbf{k}_2\mathbf{k}_4 (t'_M(z))^2}{4(t_M(z)-t_0)(t_M(z)-t_\infty)} +$$
$$\left. + \frac{\mathbf{k}_4^2 (t'_M(z))^2}{8(t_M(z)-t_\infty)^2} \right).$$

The term

$$\left(\frac{t''}{t'}\right)' - \frac{1}{2}\left(\frac{t''}{t'}\right)^2 = \frac{t'''}{t'} - \frac{3}{2}\left(\frac{t''}{t'}\right)^2$$

is the Schwartzian derivative, as could be expected from the very beginning.

To obtain the differential equation for correlation function (33), we expand $\langle\langle T(z)\rangle\rangle$ in the vicinity of $z = u$. This expansion is given by

$$\langle\langle T(z)\rangle\rangle = \frac{D}{16(z-u)^2} - \frac{D}{16(z-u)u}\left(1 + \frac{2a_2}{a_0^2} - \frac{3a_1^2}{2a_0^3}\right) +$$
$$+ \frac{1}{4a_0(z-u)u}\left(\frac{\mathbf{k}_1^2}{x^2} + \frac{\mathbf{k}_2^2}{(x-t_0)^2} + \frac{2\mathbf{k}_1\mathbf{k}_2}{x(x-t_0)} + \frac{\mathbf{k}_4^2}{(x-t_\infty)^2} + \right.$$
$$\left. + \frac{2\mathbf{k}_1\mathbf{k}_4}{x(x-t_\infty)} + \frac{2\mathbf{k}_2\mathbf{k}_4}{(x-t_0)(x-t_\infty)}\right) + \cdots, \tag{38}$$

where the coefficients a_k are defined as

$$a_k = \frac{(-1)^{k-1}}{k+2}\left(\frac{n_0}{x^{k+2}} + \frac{N-n_0}{(x-t_0)^{k+2}} - \frac{N-n_\infty}{(x-t_\infty)^{k+2}}\right).$$

The first term shows that the conformal dimension of the twist field σ_{KL} is equal to $D/16$, as it should be, and the other terms lead to the differential equation for $G(u,\bar{u})$,

$$u\partial_u \log G(u,\bar{u}) = -\frac{D}{16}\left(1 + \frac{2a_2}{a_0^2} - \frac{3a_1^2}{2a_0^3}\right) +$$
$$+ \frac{1}{4a_0}\left(\frac{\mathbf{k}_1^2}{x^2} + \frac{\mathbf{k}_2^2}{(x-t_0)^2} + \frac{2\mathbf{k}_1\mathbf{k}_2}{x(x-t_0)} + \frac{\mathbf{k}_4^2}{(x-t_\infty)^2} + \right.$$
$$\left. + \frac{2\mathbf{k}_1\mathbf{k}_4}{x(x-t_\infty)} + \frac{2\mathbf{k}_2\mathbf{k}_4}{(x-t_0)(x-t_\infty)}\right). \tag{39}$$

It is useful to make the change of variables $u \to u(x)$. Then, performing the simple but tedious calculations outlined in the Appendix, we obtain the following differential equation for $G(u,\bar{u})$:

$$\partial_x \log G(u(x),\bar{u}(\bar{x})) = -\frac{D}{16}\frac{d}{dx}\log u + \frac{d_0}{x} + \frac{d_1}{x-1} + \frac{d_2}{x+\frac{n_0}{N-n_0}} +$$
$$+ \frac{d_3}{x - \frac{N-n_0-n_\infty}{N-n_0}} + \frac{d_4}{x - \frac{n_0}{n_0-n_\infty}} - \frac{D}{24}\left(\frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2}\right), \tag{40}$$

[8] If all $\mathbf{k}_i = 0$, the expectation value of $T(z)$ in the presence of twist fields can be equivalently found by using $t_M(z)$ to map the stress-energy tensor on the t-sphere onto the z-sphere with the subsequent summation over M (see e.g. [11]).

where

$$\alpha_i = \frac{n_0}{n_0 - n_\infty} + (-1)^i \sqrt{\frac{n_0 n_\infty (N - n_\infty)}{(n_0 - n_\infty)^2 (N - n_0)}}$$

are roots of the equation $x^2 a_0 = 0$ and the coefficients d_i are given by the following formulas:

$$\begin{aligned} d_0 &= \frac{D}{24} + \frac{n_0}{8(N - n_\infty)}\left(\mathbf{k}_4^2 - \frac{D}{3}\right) + \frac{N - n_\infty}{8n_0}\left(\mathbf{k}_1^2 - \frac{D}{3}\right) + \frac{1}{4}\mathbf{k}_1\mathbf{k}_4, \\ d_1 &= \frac{D}{24} - \frac{n_\infty}{8(N - n_\infty)}\left(\mathbf{k}_4^2 - \frac{D}{3}\right) - \frac{N - n_\infty}{8n_\infty}\left(\mathbf{k}_3^2 - \frac{D}{3}\right) + \frac{1}{4}\mathbf{k}_3\mathbf{k}_4, \\ d_2 &= \frac{D}{24} - \frac{n_0}{8(N - n_0)}\left(\mathbf{k}_2^2 - \frac{D}{3}\right) - \frac{N - n_0}{8n_0}\left(\mathbf{k}_1^2 - \frac{D}{3}\right) + \frac{1}{4}\mathbf{k}_1\mathbf{k}_2, \\ d_3 &= \frac{D}{24} + \frac{n_\infty}{8(N - n_0)}\left(\mathbf{k}_2^2 - \frac{D}{3}\right) + \frac{N - n_0}{8n_\infty}\left(\mathbf{k}_3^2 - \frac{D}{3}\right) + \frac{1}{4}\mathbf{k}_2\mathbf{k}_3, \\ d_4 &= \frac{D}{24} + \frac{n_0}{8n_\infty}\left(\mathbf{k}_3^2 - \frac{D}{3}\right) + \frac{n_\infty}{8n_0}\left(\mathbf{k}_1^2 - \frac{D}{3}\right) + \frac{1}{4}\mathbf{k}_1\mathbf{k}_3. \end{aligned} \tag{41}$$

Taking into account that the second equation on $G(u, \bar{u})$ has the same form with the obvious substitution $u \to \bar{u}, x \to \bar{x}$, we obtain the solution of Eq. (40),

$$\begin{aligned} G(u, \bar{u}) = C(g_0, g_\infty)\delta_R^D(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \times \\ \times |u|^{-\frac{D}{8}}|x - \alpha_1|^{-\frac{D}{12}}|x - \alpha_2|^{-\frac{D}{12}}|x|^{2d_0}|x - 1|^{2d_1} \times \\ \times \left|x + \frac{n_0}{N - n_0}\right|^{2d_2}\left|x - \frac{N - n_0 - n_\infty}{N - n_0}\right|^{2d_3}\left|x - \frac{n_0}{n_0 - n_\infty}\right|^{2d_4}. \end{aligned} \tag{42}$$

Here, $x = x(u)$ is the root of the equation $u = u(x)$ that corresponds to given values of the indices I, J, K, and L, and $C(g_0, g_\infty)$ is a normalization constant that does not depend on u and $\bar{u}$.

To determine this constant, let us consider an auxiliary correlation function

$$G_0(u, \bar{u}) = \langle \sigma_{g_0^{-1}}[-\mathbf{k}_1, -\mathbf{k}_2](\infty)\sigma_{IJ}(1, 1)\sigma_{IJ}(u, \bar{u})\sigma_{g_0}[\mathbf{k}_1, \mathbf{k}_2](0, 0)\rangle, \tag{43}$$

where $I = 1, \ldots, n_0$, $J = n_0 + 1, \ldots, N$. We can fix $I = n_0$, $J = N$ using the action of C_{g_0}. This correlation function corresponds to the case $n_\infty = n_0$ and the rational function $u(x)$ is

$$u(x) = \left(1 + \frac{2n_0 - N}{N - n_0}\frac{1}{x}\right)^{N - 2n_0}\left(\frac{1 + \frac{n_0}{N - n_0}\frac{1}{x}}{1 - \frac{1}{x}}\right)^N. \tag{44}$$

The root of Eq. (44) that corresponds to correlation function (43) behaves as

$$\frac{1}{x} = \frac{1}{4n_0}(u - 1) + o(u - 1) \qquad \text{for} \quad u \to 1. \tag{45}$$

The expression for the correlation function $G_0(u, \bar{u})$ derived from (42) in the limit $n_\infty \to n_0$ is

$$\begin{aligned} G_0(u, \bar{u}) = C(g_0)R^D|u|^{-\frac{D}{8}}\left|x - \frac{N - 2n_0}{2(N - n_0)}\right|^{-\frac{D}{12}} \times \\ \times |x|^{2d_0}|x - 1|^{2d_1}\left|x + \frac{n_0}{N - n_0}\right|^{2d_2}\left|x - \frac{N - 2n_0}{N - n_0}\right|^{2d_3}, \end{aligned} \tag{46}$$

where the coefficient d_i is given by Eq. (41) with the obvious substitution $n_\infty \to n_0$, $\mathbf{k}_3 = -\mathbf{k}_1$, and $\mathbf{k}_4 = -\mathbf{k}_2$.

Taking into account the OPE

$$\sigma_{IJ}(1,1)\sigma_{IJ}(u,\bar{u}) = \frac{R^{-D}}{|u-1|^{\frac{D}{4}}} + \cdots$$

and normalization (16) of the two-point correlation functions, we obtain

$$G_0(u,\bar{u}) \to \frac{R^D}{|u-1|^{\frac{D}{4}}}. \tag{47}$$

On the other side, using (45) and (46), we derive

$$G_0(u,\bar{u}) \to C(g_0)R^D\left(\frac{1}{4n_0}|u-1|\right)^{-2\left(d_0+d_1+d_2+d_3-\frac{D}{24}\right)} = \frac{R^D}{|u-1|^{\frac{D}{4}}}C(g_0)(4n_0)^{\frac{D}{4}} \tag{48}$$

in the limit $u \to 1$. Comparing (47) and (48), we see that the normalization constant is

$$C(g_0) = (4n_0)^{-\frac{D}{4}}. \tag{49}$$

Next, let us consider the limit $u \to 0$. Taking into account the OPE

$$\begin{aligned}\sigma_{n_0N}(u,\bar{u})\sigma_{g_0}[\mathbf{k}_1,\mathbf{k}_2](0) = {}& \frac{C^{g_{n_0N}g_0}_{n_0N,g_0}(\mathbf{k}_1,\mathbf{k}_2)}{|u|^{\frac{D}{8}+2\Delta_{g_0}[\mathbf{k}_1,\mathbf{k}_2]-2\Delta_{g_{n_0N}g_0}[\mathbf{k}_1+\mathbf{k}_2]}}\sigma_{g_{n_0N}g_0}[\mathbf{k}_1+\mathbf{k}_2](0)+\\ &+\frac{C^{g_0g_{n_0N}}_{n_0N,g_0}(\mathbf{k}_1,\mathbf{k}_2)}{|u|^{\frac{D}{8}+2\Delta_{g_0}[\mathbf{k}_1,\mathbf{k}_2]-2\Delta_{g_{n_0N}g_0}[\mathbf{k}_1+\mathbf{k}_2]}}\sigma_{g_0g_{n_0N}}[\mathbf{k}_1+\mathbf{k}_2](0)+\cdots,\end{aligned} \tag{50}$$

we obtain

$$\begin{aligned}G_0(u,\bar{u}) \to {}& \frac{C^{g_{n_0N}g_0}_{n_0N,g_0}(\mathbf{k}_1,\mathbf{k}_2)}{|u|^{\frac{D}{8}+2\Delta_{g_0}[\mathbf{k}_1,\mathbf{k}_2]-2\Delta_{g_{n_0N}g_0}[\mathbf{k}_1+\mathbf{k}_2]}}\times\\ &\times\langle\sigma_{g_0^{-1}}[-\mathbf{k}_1,-\mathbf{k}_2](\infty)\sigma_{n_0N}(1)\sigma_{g_{n_0N}g_0}[\mathbf{k}_1+\mathbf{k}_2](0)\rangle+\\ &+\frac{C^{g_0g_{n_0N}}_{n_0N,g_0}(\mathbf{k}_1,\mathbf{k}_2)}{|u|^{\frac{D}{8}+2\Delta_{g_0}[\mathbf{k}_1,\mathbf{k}_2]-2\Delta_{g_{n_0N}g_0}[\mathbf{k}_1+\mathbf{k}_2]}}\times\\ &\times\langle\sigma_{g_0^{-1}}[-\mathbf{k}_1,-\mathbf{k}_2](\infty)\sigma_{n_0N}(1)\sigma_{g_0g_{n_0N}}[\mathbf{k}_1+\mathbf{k}_2](0)\rangle.\end{aligned} \tag{51}$$

It is not difficult to show that the correlation functions $\langle\sigma_{g_0^{-1}}\sigma_{n_0N}\sigma_{g_{n_0N}g_0}\rangle$ and $\langle\sigma_{g_0^{-1}}\sigma_{n_0N}\sigma_{g_0g_{n_0N}}\rangle$ are equal to $C^{g_0g_{n_0N}}_{n_0N,g_0}$ and $C^{g_{n_0N}g_0}_{n_0N,g_0}$, respectively, and, moreover, are equal to each other. This follows from (25) and (30) and from the obvious symmetry property of the structure constant $C^{g_{n_0N}g_0}_{n_0N,g_0}(-\mathbf{k}_1,-\mathbf{k}_2) = C^{g_{n_0N}g_0}_{n_0N,g_0}(\mathbf{k}_1,\mathbf{k}_2)$,

$$\begin{aligned}\langle\sigma_{g_0^{-1}}\sigma_{n_0N}\sigma_{g_{n_0N}g_0}\rangle &= \langle\sigma_{g_{n_0N}g_0}\sigma_{n_0N}\sigma_{g_0^{-1}}\rangle =\\ &= \begin{cases}\langle\sigma_{g_{n_0N}g_0^{-1}}\sigma_{n_0N}\sigma_{g_0}\rangle = R^DC^{g_0g_{n_0N}}_{n_0N,g_0},\\ \langle\sigma_{g_0^{-1}g_{n_0N}}\sigma_{n_0N}\sigma_{g_0}\rangle = R^DC^{g_{n_0N}g_0}_{n_0N,g_0}.\end{cases}\end{aligned} \tag{52}$$

Therefore, the correlation function $G_0(u,\bar{u})$ in the limit $u \to 0$, using the structure constant

$$C(n_0,\mathbf{k}_1;N-n_0,\mathbf{k}_2) \equiv C^{g_{n_0N}g_0}_{n_0N,g_0}(\mathbf{k}_1,\mathbf{k}_2),$$

becomes

$$G_0(u,\bar u) \to \frac{2R^D C^2(n_0,\mathbf{k}_1;N-n_0,\mathbf{k}_2)}{|u|^{\frac{D}{8}+2\Delta_{g_0}[\mathbf{k}_1,\mathbf{k}_2]-2\Delta_{g_{n_0N}g_0}[\mathbf{k}_1+\mathbf{k}_2]}}. \tag{53}$$

On the other hand, taking into account that the root $x(u)$ behaves as

$$\left|x+\frac{n_0}{N-n_0}\right| \to N n_0^{\frac{N-2n_0}{N}}(N-n_0)^{\frac{2n_0-2N}{N}}|u|^{\frac{1}{N}},$$

in the limit $u\to 0$, we obtain

$$G_0(u,\bar u) \to \frac{2^{\frac{D}{12}}R^D C(g_0)}{|u|^{\frac{D}{8}-\frac{2}{N}d_2}} N^{-\frac{D}{12}+4d_2}(N-n_0)^{-\frac{D}{12}+4\frac{n_0-N}{N}d_2} n_0^{\frac{D}{6}-4\frac{n_0}{N}d_2} \tag{54}$$

from (46). Comparing (53) and (54), we obtain the structure constant

$$C(n_0,\mathbf{k}_1;N-n_0,\mathbf{k}_2) = 2^{-\frac{5D+12}{24}} N^{-\frac{D}{24}+2d_2}(N-n_0)^{-\frac{D}{24}-2\frac{N-n_0}{N}d_2} n_0^{-\frac{D}{24}-2\frac{n_0}{N}d_2}, \tag{55}$$

where

$$\begin{aligned} d_2 &\equiv d_2(n_0,\mathbf{k}_1;N-n_0,\mathbf{k}_2) = \\ &= \frac{D}{24} - \frac{n_0}{8(N-n_0)}\left(\mathbf{k}_2^2-\frac{D}{3}\right) - \frac{N-n_0}{8n_0}\left(\mathbf{k}_1^2-\frac{D}{3}\right) + \frac{1}{4}\mathbf{k}_1\mathbf{k}_2. \end{aligned} \tag{56}$$

Now it is not difficult to express any three-point correlation function of the form $\langle\sigma_{g^{-1}g_{IJ}}\sigma_{IJ}\sigma_g\rangle$ through the structure constant $C(n,\mathbf{k};m,\mathbf{q})$. First, we note that any twist field $\sigma_g[\{\mathbf{k}_\alpha\}]$ has the following decomposition into the tensor product of the twist fields $\sigma_{(n)}[\mathbf{k}]$:

$$\sigma_g[\{\mathbf{k}_\alpha\}] = \bigotimes_{\alpha=1}^{N_{str}} \sigma_{(n_\alpha)}[\mathbf{k}_\alpha], \tag{57}$$

where the element g has the decomposition $(n_1)(n_2)\cdots(n_{N_{str}})$.[9] With the help of (52), we obtain the structure constant $C(n,\mathbf{k};m,\mathbf{q})$ with arbitrary n and m,

$$C(n,\mathbf{k};m,\mathbf{q}) = R^{-D}\langle\sigma_{(-n-m)}[-\mathbf{k}-\mathbf{q}](\infty)\sigma_{IJ}(1)\sigma_{(n)}[\mathbf{k}]\otimes\sigma_{(m)}[\mathbf{q}](0)\rangle, \tag{58}$$

where $I\in(n)$ and $J\in(m)$.

Using (57) and (58), we easily obtain the three-point correlation function

$$\begin{aligned} &\langle\sigma_{g^{-1}g_{IJ}}[\{\mathbf{q}_\alpha\}](\infty)\sigma_{IJ}(1)\sigma_g[\{\mathbf{k}_\alpha\}](0)\rangle = \langle\sigma_g[\{\mathbf{k}_\alpha\}](\infty)\sigma_{IJ}(1)\sigma_{g^{-1}g_{IJ}}[\{\mathbf{q}_\alpha\}](0)\rangle = \\ &= \langle\sigma_{(-n_1-n_2)}[\mathbf{q}]\bigotimes_{\alpha=3}^{N_{str}}\sigma_{(-n_\alpha)}[\mathbf{q}_\alpha](\infty)\sigma_{IJ}(1)\sigma_{(n_1)}[\mathbf{k}_1]\otimes\sigma_{(n_2)}[\mathbf{k}_2]\bigotimes_{\alpha=3}^{N_{str}}\sigma_{(n_\alpha)}[\mathbf{k}_\alpha](0)\rangle = \\ &= \prod_{\alpha=3}^{N_{str}}\delta_R^D(\mathbf{q}_\alpha+\mathbf{k}_\alpha)\langle\sigma_{(-n_1-n_2)}[\mathbf{q}](\infty)\sigma_{IJ}(1)\sigma_{(n_1)}[\mathbf{k}_1]\otimes\sigma_{(n_2)}[\mathbf{k}_2](0)\rangle = \\ &= C(n_1,\mathbf{k}_1;n_2,\mathbf{k}_2)\delta_R^D(\mathbf{q}+\mathbf{k}_1+\mathbf{k}_2)\prod_{\alpha=3}^{N_{str}}\delta_R^D(\mathbf{q}_\alpha+\mathbf{k}_\alpha), \end{aligned} \tag{59}$$

[9] We use the notation $(-n_1)(-n_2)\cdots(-n_{N_{str}})$ for the decomposition of the element g^{-1}.

where $I \in (n_1)$ and $J \in (n_2)$.

It is now clear that the structure constant $C^{g_{IJ}g}_{IJ,g}$ in the OPE of σ_{IJ} and σ_g is equal to $C(n_1, \mathbf{k}_1; n_2, \mathbf{k}_2)$ and that the structure constant $C^{g^{-1}}_{IJ,g^{-1}g_{IJ}}$ (which coincides with $C^{g_{IJ}g^{-1}g_{IJ}}_{IJ,g^{-1}g_{IJ}}$ due to (52)) in the OPE

$$\sigma_{IJ}(u,\bar{u})\sigma_{g^{-1}g_{IJ}}[\{\mathbf{q}_\alpha\}](0) = \sum_{\mathbf{q}_1,\mathbf{q}_2} \frac{\delta_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q},0}}{|u|^{\frac{D}{8}+2\Delta_{g^{-1}g_{IJ}}[\{\mathbf{q}_\alpha\}]-2\Delta_g[\{\mathbf{q}_\alpha\}]}} \times$$
$$\times \left(C^{g^{-1}}_{IJ,g^{-1}g_{IJ}}(\mathbf{q}_1,\mathbf{q}_2)\sigma_{g^{-1}}[\{\mathbf{q}_\alpha\}](0) + \right.$$
$$\left. + C^{g^{-1}}_{IJ,g^{-1}g_{IJ}}(\mathbf{q}_1,\mathbf{q}_2)\sigma_{g_{IJ}g^{-1}g_{IJ}}[\{\mathbf{q}_\alpha\}](0)\right) + \cdots \tag{60}$$

is

$$C^{g^{-1}}_{IJ,g^{-1}g_{IJ}}(\mathbf{q}_1,\mathbf{q}_2) = R^{-D}C(n_1,\mathbf{q}_1;n_2,\mathbf{q}_2).$$

In particular, the structure constants $C^{g_{n_\infty}Ng_0}_{n_\infty N,g_0}$ and $C^{g_{n_0+n_\infty},Ng_0}_{n_0+n_\infty,N;g_0}$, which are used to find the normalization constant $C(g_0, g_\infty)$, are given by

$$\begin{aligned} C^{g_{n_\infty}Ng_0}_{n_\infty N,g_0}(\mathbf{k}_1,\mathbf{k}_2) &= R^{-D}C(n_\infty - n_0, \mathbf{k}_1; N - n_\infty, \mathbf{k}_2), \\ C^{g_{n_0+n_\infty},Ng_0}_{n_0+n_\infty,N;g_0}(\mathbf{k}_1,\mathbf{k}_2) &= R^{-D}C(N - n_\infty - n_0, \mathbf{k}_1; n_\infty, \mathbf{k}_2). \end{aligned} \tag{61}$$

We are now ready to determine the normalization constant $C(g_0, g_\infty)$ by the factor $G(u,\bar{u})$ in the limit $u \to 0$ on three-point functions. According to (36), $u \to 0$ in the three cases

$$x \to -\frac{n_0}{N-n_0},$$
$$x \to \infty,$$
$$x \to \frac{N-n_0-n_\infty}{N-n_0},$$

and, conversely, any root $x_M = x_M(u)$ of Eq. (36) tends to one of these values when $u \to 0$. Evidently, these three possible asymptotic behaviors correspond to three different choices of the indices K and L in Eq. (32).

We begin with the case where $K = n_0$ and $L = N$. Using OPE (50) and normalization (16) of the two-point correlation functions, we obtain, in the limit $u \to 0$,

$$G(u,\bar{u}) \to \delta^D_R(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\frac{C(n_0,\mathbf{k}_1;N-n_0,\mathbf{k}_2)C(n_\infty,\mathbf{k}_3;N-n_\infty,\mathbf{k}_4)}{|u|^{\frac{D}{8}+2\Delta_{g_0}[\mathbf{k}_1,\mathbf{k}_2]-2\Delta_{g_{n_0}Ng_0}[\mathbf{k}_1+\mathbf{k}_2]}}. \tag{62}$$

In this case, the root $x(u)$ has the following behavior in the vicinity of $u = 0$:

$$\left|x + \frac{n_0}{N-n_0}\right| \to Nn_0^{\frac{N-n_0}{N}} n_\infty^{-\frac{n_\infty}{N}} (N-n_0)^{\frac{n_0-2N}{N}} (N-n_\infty)^{\frac{n_\infty}{N}} |u|^{\frac{1}{N}}. \tag{63}$$

Using (42) and (63), we easily find

$$G(u,\bar{u}) \to \delta^D_R(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\frac{C(g_0,g_\infty)}{|u|^{\frac{D}{8}-\frac{2}{N}d_2}}\left(\frac{n_0N(N-n_\infty)}{(N-n_0)^2(n_0-n_\infty)}\right)^{-\frac{D}{12}} \times$$
$$\times \left(\frac{n_0}{N-n_0}\right)^{2d_0}\left(\frac{N}{N-n_0}\right)^{2d_1}\left(\frac{n_0(N-n_\infty)}{(N-n_0)(n_0-n_\infty)}\right)^{2d_4}\left(\frac{N-n_\infty}{N-n_0}\right)^{2d_3} \times$$
$$\times \left(Nn_0^{\frac{N-n_0}{N}} n_\infty^{-\frac{n_\infty}{N}} (N-n_0)^{\frac{n_0-2N}{N}} (N-n_\infty)^{\frac{n_\infty}{N}}\right)^{2d_2}. \tag{64}$$

It is not difficult to verify that

$$\Delta_{g_0}[\mathbf{k}_1, \mathbf{k}_2] - \Delta_{g_{n_0}N g_0}[\mathbf{k}_1 + \mathbf{k}_2] = -\frac{1}{N}d_2,$$

as should be expected. Comparing (62) and (64), we obtain the normalization constant. However, for general values of D, the corresponding expression is rather complicated and is not written here. For $D = 24$, the coefficient d_i is given by the following simple formulas:

$$\begin{aligned} d_0 &= 1 + \frac{1}{4}k_1k_4, \qquad d_1 = 1 + \frac{1}{4}k_3k_4, \qquad d_2 = 1 + \frac{1}{4}k_1k_2, \\ d_3 &= 1 + \frac{1}{4}k_2k_3, \qquad d_4 = 1 + \frac{1}{4}k_1k_3, \end{aligned} \tag{65}$$

where $k_ik_j \equiv \mathbf{k}_i\mathbf{k}_j - (1/2)k_i^+k_j^- - (1/2)k_i^-k_j^+$.

Using (65), we easily obtain

$$C(g_0, g_\infty) = \frac{2^{-11}}{n_0(N-n_0)n_\infty(N-n_\infty)(n_\infty - n_0)^2}\left(\frac{N-n_0}{n_\infty - n_0}\right)^{2+\frac{1}{2}(k_1+k_3)k_4}. \tag{66}$$

Thus, we have found the normalization constant for the N correlation functions presented in the first and second terms of Eq. (32).

Now let us determine the normalization constant for $n_\infty - n_0$ correlation functions of the form $\langle\sigma_{g_\infty(J)}\sigma_{n_0J}\sigma_{n_\infty N}\sigma_{g_0}\rangle$. Using (60) and (61), we find

$$\begin{aligned} G(u,\bar{u}) \to\ & \delta_R^D(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)\times \\ & \times \frac{C(n_\infty - n_0, \mathbf{k}_2 + \mathbf{k}_4; N - n_\infty, -\mathbf{k}_4)C(n_\infty - n_0, \mathbf{k}_2 + \mathbf{k}_4; n_0, \mathbf{k}_1)}{|u|^{\frac{D}{8}+2\Delta_{g_0}[\mathbf{k}_1,\mathbf{k}_2]-2\Delta_{g_{n_\infty}N g_0}[\mathbf{k}_1,\mathbf{k}_2+\mathbf{k}_4,\mathbf{k}_4]}} \end{aligned} \tag{67}$$

in the limit $u \to 0$. Taking into account the behavior of the root $x(u)$ in the vicinity of $u = 0$,

$$|x| \to \left((n_\infty - n_0)^{n_0 - n_\infty}\frac{n_\infty^{n_\infty}}{n_0^{n_0}}\left(\frac{N-n_0}{N-n_\infty}\right)^{N-n_\infty}\right)^{\frac{1}{n_\infty - n_0}}|u|^{\frac{1}{n_0 - n_\infty}},$$

we obtain

$$\begin{aligned} G(u,\bar{u}) \to\ & \frac{\delta_R^D(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)C(g_0, g_\infty)}{|u|^{\frac{D}{8}+\frac{2(d_0+d_1+d_2+d_3+d_4-\frac{D}{12})}{n_\infty - n_0}}}\times \\ & \times \left((n_\infty - n_0)^{n_0-n_\infty}\frac{n_\infty^{n_\infty}}{n_0^{n_0}}\left(\frac{N-n_0}{N-n_\infty}\right)^{N-n_\infty}\right)^{\frac{2(d_0+d_1+d_2+d_3+d_4-\frac{D}{12})}{n_\infty - n_0}} \end{aligned} \tag{68}$$

from (42). A simple calculation shows that

$$\Delta_{g_0}[\mathbf{k}_1, \mathbf{k}_2] - \Delta_{g_{n_\infty}N g_0}[\mathbf{k}_1, \mathbf{k}_2 + \mathbf{k}_4, \mathbf{k}_4] = \frac{d_0 + d_1 + d_2 + d_3 + d_4 - \frac{D}{12}}{n_\infty - n_0}.$$

The normalization constant $C(g_0, g_\infty)$ can be found from (67) and (68). For $D = 24$, the computation is drastically simplified if we note that

$$\begin{aligned} & d_0 + d_1 + d_2 + d_3 + d_4 - 2 = -1 - \frac{1}{4}k_1k_3, \\ & d_2(n_\infty - n_0, \mathbf{k}_2 + \mathbf{k}_4; N - n_\infty, -\mathbf{k}_4) = -\frac{N-n_0}{n_\infty - n_0}\left(1 + \frac{1}{4}k_1k_3\right), \\ & d_2(n_\infty - n_0, \mathbf{k}_2 + \mathbf{k}_4; n_0, \mathbf{k}_1) = -\frac{n_\infty}{n_\infty - n_0}\left(1 + \frac{1}{4}k_1k_3\right). \end{aligned}$$

Then, it can be easily shown that $C(g_0, g_\infty)$ is, again, given by Eq. (66).

We can find the normalization constant for the remaining $N - n_0 - n_\infty$ correlation functions of the form $\langle \sigma_{g_\infty(J)} \sigma_{n_0 J} \sigma_{n_0+n_\infty, N} \sigma_{g_0} \rangle$ in the same manner; once again, it is defined by (66).

Up to now, we have considered the correlation functions

$$G_{IJKL}(u, \bar{u}) = \langle \sigma_{g_\infty}(\infty) \sigma_{IJ}(1) \sigma_{KL}(u, \bar{u}) \sigma_{g_0}(0) \rangle$$

with $|u| < 1$. The correlation functions $G_{IJKL}(u, \bar{u})$ with $|u| > 1$ can be calculated in the same way and their dependence on u is given by Eq. (42) as well. The normalization constant, in this case, is derived by studying the limit $u \to \infty$ and it coincides with the previously found constant (66). Therefore, the time ordering can be omitted. To complete the computation of the S-matrix element, we need to integrate the correlation function $F(u, \bar{u})$ given by (24) over the complex plane. With the help of the momentum conservation law, the mass-shell condition, (65), and the equality

$$\frac{1}{u} \frac{du}{dx} = \frac{(n_0 - n_\infty)(x - \alpha_1)^2 (x - \alpha_2)^2}{x(x-1)\left(x - \frac{N - n_0 - n_\infty}{N - n_0}\right)\left(x - \frac{n_0}{n_0 - n_\infty}\right)\left(x + \frac{n_0}{N - n_0}\right)},$$

we can rewrite Eq. (42) as

$$\begin{aligned} G_{IJKL}(u, \bar{u}) = \delta_R^D(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) C(g_0, g_\infty)(n_\infty - n_0)^2 \times \\ \times \left|\frac{du}{dx}\right|^{-2} |u|^{-1} \frac{|x - \alpha_1|^2 |x - \alpha_2|^2}{|x - \frac{n_0}{n_0 - n_\infty}|^4} \times \\ \times \left|\frac{x(x - \frac{N - n_0 - n_\infty}{N - n_0})}{x - \frac{n_0}{n_0 - n_\infty}}\right|^{\frac{1}{2} k_1 k_4} \left|\frac{(x-1)(x + \frac{n_0}{N - n_0})}{x - \frac{n_0}{n_0 - n_\infty}}\right|^{\frac{1}{2} k_3 k_4}. \end{aligned}$$

Now the integral $\int d^2 u |u| G_{IJKL}(u, \bar{u})$ can be easily calculated by changing the variables $u \to x$,

$$\begin{aligned} \int d^2 u \, |u| G_{IJKL}(u, \bar{u}) = \delta_R^D(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) C(g_0, g_\infty)(n_\infty - n_0)^2 \times \\ \times \int_{V_{IJKL}} d^2 x \, \frac{|x - \alpha_1|^2 |x - \alpha_2|^2}{|x - \frac{n_0}{n_0 - n_\infty}|^4} \left|\frac{x(x - \frac{N - n_0 - n_\infty}{N - n_0})}{x - \frac{n_0}{n_0 - n_\infty}}\right|^{\frac{1}{2} k_1 k_4} \left|\frac{(x-1)(x + \frac{n_0}{N - n_0})}{x - \frac{n_0}{n_0 - n_\infty}}\right|^{\frac{1}{2} k_3 k_4}, \end{aligned} \tag{69}$$

where we have taken into account that under this change of variables, the u-sphere is mapped onto the domain V_{IJKL}. Since the correlation function

$$F(u, \bar{u}) = \frac{C_0 C_\infty}{N!} 2 n_0 (N - n_0) n_\infty (N - n_\infty) \sum_{IJKL} G_{IJKL}(u, \bar{u}),$$

where the summation goes over the set of indices listed in (32), we have the integral

$$\begin{aligned} \int d^2 u \, |u| F(u, \bar{u}) = \frac{2^{-10} \delta_R^D(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)}{\sqrt{n_0 (N - n_0) n_\infty (N - n_\infty)}} \left(\frac{N - n_0}{n_\infty - n_0}\right)^{2 + \frac{1}{2}(k_1 + k_3) k_4} \times \\ \times \int d^2 x \, \frac{|x - \alpha_1|^2 |x - \alpha_2|^2}{|x - \frac{n_0}{n_0 - n_\infty}|^4} \left|\frac{x(x - \frac{N - n_0 - n_\infty}{N - n_0})}{x - \frac{n_0}{n_0 - n_\infty}}\right|^{\frac{1}{2} k_1 k_4} \left|\frac{(x-1)(x + \frac{n_0}{N - n_0})}{x - \frac{n_0}{n_0 - n_\infty}}\right|^{\frac{1}{2} k_3 k_4}. \end{aligned} \tag{70}$$

Finally, performing the change of variables

$$\frac{n_\infty - n_0}{N - n_0} z = \frac{x(x - \frac{N - n_0 - n_\infty}{N - n_0})}{x - \frac{n_0}{n_0 - n_\infty}},$$

we obtain

$$\int d^2u\,|u|F(u,\bar u) = \frac{2^{-9}\delta^D_R(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)}{\sqrt{n_0(N-n_0)n_\infty(N-n_\infty)}} \int d^2z\,|z|^{\frac{1}{2}k_1k_4}|1-z|^{\frac{1}{2}k_3k_4}. \tag{71}$$

The S-matrix element can be found using (23) and taking the limit as $R\to\infty$,

$$\langle f|S|i\rangle = -i\frac{\lambda^2 2^{-8}N^3\delta(k_1^-+k_2^-+k_3^-+k_4^-)\delta^D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)}{\sqrt{n_0(N-n_0)n_\infty(N-n_\infty)}} \int d^2z\,|z|^{\frac{1}{2}k_1k_4}|1-z|^{\frac{1}{2}k_3k_4} =$$

$$= -i\frac{\lambda^2 2^{-8}N\delta(\sum_i k_i^-)\delta^D(\sum_i \mathbf{k}_i)}{\sqrt{k_1^+k_2^+k_3^+k_4^+}} \int d^2z\,|z|^{\frac{1}{2}k_1k_4}|1-z|^{\frac{1}{2}k_3k_4}. \tag{72}$$

We represent the light-cone momenta k_i^+ as $k_i^+ = m_i/N$ and rewrite (72) as

$$\langle f|S|i\rangle = -i\frac{\lambda^2 2^{-8}N\delta_{m_1+m_2+m_3+m_4,0}(\sum_i k_i^-)\delta^D(\sum_i \mathbf{k}_i)}{\sqrt{k_1^+k_2^+k_3^+k_4^+}} \int d^2z\,|z|^{\frac{1}{2}k_1k_4}|1-z|^{\frac{1}{2}k_3k_4}. \tag{73}$$

In the limit $N\to\infty$, the combination $N\delta_{m_1+m_2+m_3+m_4,0}$ goes to $\delta(\sum_i k_i^+)$ and Eq. (73) becomes

$$\langle f|S|i\rangle = -i\frac{\lambda^2 2^{-9}\delta^{D+2}(\sum_i k_i^\mu)}{\sqrt{k_1^+k_2^+k_3^+k_4^+}} \int d^2z\,|z|^{\frac{1}{2}k_1k_4}|1-z|^{\frac{1}{2}k_3k_4}. \tag{74}$$

Taking into account that the scattering amplitude A is related to the S-matrix by (see, e.g., [12])

$$\langle f|S|i\rangle = -i\frac{\delta^{D+2}(\sum_i k_i^\mu)}{\sqrt{k_1^+k_2^+k_3^+k_4^+}}A(1,2,3,4),$$

we finally have

$$A(1,2,3,4) = \lambda^2 2^{-9}\int d^2z\,|z|^{\frac{1}{2}k_1k_4}|1-z|^{\frac{1}{2}k_3k_4},$$

which is the well-known Virasoro amplitude.

5. Conclusion

In this paper, we have developed the technique for calculating the scattering amplitudes of bosonic string states, using the interacting $S^N\mathbf{R}^{24}$-orbifold sigma model. The scattering amplitude turned out to be automatically Lorentz-invariant. This gives strong evidence that the corresponding two-dimensional Yang–Mills model should possess the same invariance.

It would be interesting to trace the appearance of loop amplitudes in the framework of the $S^N\mathbf{R}^{24}$-orbifold sigma model. Obviously, the one-loop amplitude requires the computation of the correlation function for four Z_2-twist fields sandwiched between the asymptotic states; technically, this results in the construction of noncommutative Green's functions in the presence of six twist fields. We note that cancellation of possible divergences in the amplitude may require further perturbation of the CFT action by higher-order contact terms.

The next important problem to be solved is to consider the $S^N\mathbf{R}^8$ supersymmetric orbifold sigma model and to prove the DVV conjecture. It is not difficult to introduce twist fields for fermionic variables and calculate their conformal dimensions. However, calculation of the four-point correlation functions of the twist fields is a more complicated problem and is now under consideration. We do not exclude the possibilty that simplest way to solve the problem is to bosonize the fermion fields.

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Appendix

In this appendix, we consider some properties of map (35) and outline the derivation of differential equation (40) for the four-point correlation functions (33). Let us consider map (35),

$$z = \frac{t^{n_0}(t-t_0)^{N-n_0}}{(t-t_\infty)^{N-n_\infty}} \frac{(t_1-t_\infty)^{N-n_\infty}}{t_1^{n_0}(t_1-t_0)^{N-n_0}} \equiv u(t). \tag{A.1}$$

This map is the N-fold covering of the z-sphere by the t-sphere. Obviously, it branches at the points $t=0$, t_0, t_∞, and ∞. To find the other branch points, we have to solve the equation

$$\begin{aligned}\frac{d\log z}{dt} &= \frac{n_0}{t} + \frac{N-n_0}{t-t_0} - \frac{N-n_\infty}{t-t_\infty} = \\ &= \frac{n_\infty t^2 + ((N-n_0-n_\infty)t_0 - Nt_\infty)\,t + n_0 t_0 t_\infty}{t(t-t_0)(t-t_\infty)}.\end{aligned} \tag{A.2}$$

In general, there are two different solutions of Eq. (A.2), t_1 and t_2, and map (A.1) has the form

$$z - z_i \sim (t-t_i)^2, \quad z_1 = 1 = u(t_1), \quad z_2 = u = u(t_2)$$

in the vicinity of these points. Due to the projective transformations, we can impose three relations on the positions of the branch points. However, we have already chosen the points 0 and ∞ as two branch points; therefore, only one relation remains to be imposed. Since the differential equation on the four-point correlation function is written with respect to the point u, it is convenient not to fix the position of the point $t_2 \equiv x$. Then, the remaining relation that leads to the rational dependence of points t_0, t_∞ and t_1 on x is

$$t_0 = x - 1. \tag{A.3}$$

The point x is assumed to be a solution of Eq. (A.2). Therefore, from (A.2) and (A.3), we can immediately derive that t_∞ is expressed through the point x as follows:

$$t_\infty = x - \frac{(N-n_\infty)x}{(N-n_0)x+n_0}. \tag{A.4}$$

The second solution of Eq. (A.2) can now be easily found. It is

$$\begin{aligned}t_1 &= \frac{N-n_0-n_\infty}{n_\infty} + \frac{n_0 x}{n_\infty} - \frac{N(N-n_\infty)x}{n_\infty((N-n_0)x+n_0)} = \\ &= \frac{n_0(x-1)\,((N-n_0)x+n_0+n_\infty-N)}{n_\infty\,((N-n_0)x+n_0)}.\end{aligned} \tag{A.5}$$

The rational function $u(x)$ is defined by the equation

$$u(x) = \frac{x^{n_0}(x-t_0)^{N-n_0}(t_1-t_\infty)^{N-n_\infty}}{(x-t_\infty)^{N-n_\infty}t_1^{n_0}(t_1-t_0)^{N-n_0}}. \tag{A.6}$$

Using (A.3)-(A.5), we derive the relations

$$\begin{aligned}t_1 - t_0 &= \frac{(N-n_0)(x-1)\big((n_0-n_\infty)x-n_0\big)}{n_\infty\big((N-n_0)x+n_0\big)}, \\ t_1 - t_\infty &= \frac{\big((n_0-n_\infty)x-n_0\big)\big((N-n_0)x+n_0+n_\infty-N\big)}{n_\infty\big((N-n_0)x+n_0\big)}.\end{aligned}$$

Then, the rational function $u(x)$ becomes

$$u = u(x) = (n_0 - n_\infty)^{n_0 - n_\infty} \frac{n_\infty^{n_\infty}}{n_0^{n_0}} \left(\frac{N - n_0}{N - n_\infty} \right)^{N - n_\infty} \left(\frac{x + \frac{n_0}{N - n_0}}{x - 1} \right)^N \times$$
$$\times \left(\frac{x - \frac{N - n_0 - n_\infty}{N - n_0}}{x} \right)^{N - n_0 - n_\infty} \left(x - \frac{n_0}{n_0 - n_\infty} \right)^{n_0 - n_\infty}. \qquad \text{(A.7)}$$

To obtain differential equation (40), we need to know the decomposition of the roots $t_K(z)$ and $t_L(z)$ in the vicinity of $z = u$. Let us take the logarithm of both sides of Eq. (A.1),

$$\log \frac{z}{u} = n_0 \log \frac{t}{x} + (N - n_0) \log \frac{t - t_0}{x - t_0} - (N - n_\infty) \log \frac{t - t_\infty}{x - t_\infty}. \qquad \text{(A.8)}$$

Decomposition of the l.h.s. of (A.8) around $z = u$ and the r.h.s. of (A.8) around $t = x$ gives

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{z - u}{u} \right)^k = (t - x)^2 \sum_{k=0}^{\infty} a_k (t - x)^k, \qquad \text{(A.9)}$$

where

$$a_k = \frac{(-1)^{k-1}}{k+2} \left(\frac{n_0}{x^{k+2}} + \frac{N - n_0}{(x - t_0)^{k+2}} - \frac{N - n_\infty}{(x - t_\infty)^{k+2}} \right). \qquad \text{(A.10)}$$

It is clear from (A.9) that $t(z)$ has the decomposition

$$t - x = \sum_{k=1}^{\infty} c_k (z - u)^{\frac{k}{2}}. \qquad \text{(A.11)}$$

Substituting (A.11) into (A.9), we find

$$c_1^2 = \frac{1}{u a_0}, \qquad c_2 = -\frac{a_1}{2 u a_0},$$
$$2 a_0 c_1 c_3 = -\frac{1}{2u^2} + \frac{5 a_1^2}{4 u^2 a_0^3} - \frac{a_2}{u^2 a_0^2}. \qquad \text{(A.12)}$$

The other coefficients of this expansion are not important for us.

Then, using decomposition (A.11) and Eq. (A.12), we obtain

$$\left(\frac{t''}{t'} \right)' = \frac{1}{2(z - u)^2} + O(1),$$
$$\left(\frac{t''}{t'} \right)^2 = \frac{1}{4(z - u)^2} + \frac{3}{z - u} \left(\frac{c_2^2}{c_1^2} - \frac{c_3}{c_1} \right) + O(1),$$
$$\frac{c_2^2}{c_1^2} - \frac{c_3}{c_1} = \frac{1}{4u} \left(1 + \frac{2 a_2}{a_0^2} - \frac{3 a_1^2}{2 a_0^3} \right).$$

Finally, taking into account that only the two roots $t_K(z)$ and $t_L(z)$ in the set of N roots $t_M(z)$ have decomposition (A.11), we obtain (38) and (39). The coefficients a_k can be rewritten as the following functions of x:

$$
\begin{aligned}
a_0 &= \frac{n_0(n_0+n_\infty-N)}{2(N-n_\infty)x^2} + \frac{n_0(N-n_0)}{(N-n_\infty)x} + \frac{(N-n_0)(n_\infty-n_0)}{2(N-n_\infty)} = \\
&= \frac{(N-n_0)(n_\infty-n_0)}{2(N-n_\infty)x^2}(x-\alpha_1)(x-\alpha_2), \\
a_1 &= \frac{n_0((N-n_\infty)^2-n_0^2)}{3(N-n_\infty)^2x^3} - \frac{n_0^2(N-n_0)}{(N-n_\infty)^2x^2} - \\
&\quad - \frac{n_0(N-n_0)^2}{(N-n_\infty)^2x} + \frac{(N-n_0)((N-n_\infty)^2-(N-n_0)^2)}{3(N-n_\infty)^2}, \\
a_2 &= -\frac{n_0((N-n_\infty)^3-n_0^3)}{4(N-n_\infty)^3x^4} + \frac{n_0^3(N-n_0)}{(N-n_\infty)^3x^3} + \frac{3n_0^2(N-n_0)^2}{2(N-n_\infty)^3x^2} + \\
&\quad + \frac{n_0(N-n_0)^3}{(N-n_\infty)^3x} - \frac{(N-n_0)((N-n_\infty)^3-(N-n_0)^3)}{4(N-n_\infty)^3}.
\end{aligned}
\tag{A.13}
$$

To obtain differential equation (40), we need to use the following important equalities for $d\log u/dx$, which can be derived using (A.7) and (A.13):

$$
\begin{aligned}
\frac{1}{u}\frac{du}{dx} &= \frac{n_0+n_\infty-N}{x} - \frac{N}{x-1} + \frac{N}{x+\frac{n_0}{N-n_0}} + \\
&\quad + \frac{N-n_0-n_\infty}{x-\frac{N-n_0-n_\infty}{N-n_0}} + \frac{n_0-n_\infty}{x-\frac{n_0}{n_0-n_\infty}}, \\
\frac{1}{u}\frac{du}{dx} &= \frac{4(N-n_\infty)^2x^4a_0^2}{(N-n_0)^2(n_0-n_\infty)x(x-1)(x-\frac{N-n_0-n_\infty}{N-n_0})(x-\frac{n_0}{n_0-n_\infty})(x+\frac{n_0}{N-n_0})} = \\
&= \frac{(n_0-n_\infty)(x-\alpha_1)^2(x-\alpha_2)^2}{x(x-1)(x-\frac{N-n_0-n_\infty}{N-n_0})(x-\frac{n_0}{n_0-n_\infty})(x+\frac{n_0}{N-n_0})}.
\end{aligned}
$$

Finally, to obtain (40), we use the Lagrange interpolation formula for the ratio of two polynomials,

$$
\frac{P(x)}{Q(x)} = \sum_i \frac{P(x_i)}{Q'(x_i)}\frac{1}{x-x_i},
$$

where each x_i is a simple root of $Q(x)$ and $\deg P < \deg Q$. These equalities drastically simplify the derivation of Eq. (40).

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