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On Leray's self-similar solutions of the Navier–Stokes equations

by

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1. Introduction

In the 1934 paper [Le] Leray raised the question of the existence of self-similar solutions of the Navier–Stokes equations

$$\left.\begin{aligned} u_t-\nu\Delta u+(u\cdot\nabla)u+\nabla p=0\\ \operatorname{div} u=0\end{aligned}\right\}\quad \text{in } \mathbf{R}^3\times(t_1,t_2), \tag{1.1}$$

where, as usual, $\nu>0$. These are the solutions of the form

$$u(x,t)=\frac{1}{\sqrt{2a(T-t)}}\,U\biggl(\frac{x}{\sqrt{2a(T-t)}}\biggr), \tag{1.2}$$

where $T\in\mathbf{R}$, $a>0$, and $U=(U_1,U_2,U_3)$ is defined in $\mathbf{R}^3$. (Hence u is defined in $\mathbf{R}^3\times(-\infty,T)$.) One also requires that certain natural energy norms of u are finite. If $U\not\equiv 0$, then u given by (1.2) develops a singularity at time $t=T$. The Navier–Stokes equations for u give the system

$$\left.\begin{aligned} -\nu\Delta U+aU+a(y\cdot\nabla)U+(U\cdot\nabla)U+\nabla P=0\\ \operatorname{div} U=0\end{aligned}\right\}\quad \text{in } \mathbf{R}^3 \tag{1.3}$$

for U (where we use y to denote a generic point in $\mathbf{R}^3$). The main result of this paper is that the only solution of (1.3) belonging to $L^3(\mathbf{R}^3)$ is $U\equiv 0$.

We make a few remarks regarding the integrability condition $U\in L^3(\mathbf{R}^3)$. If one requires that u defined by (1.2) has finite kinetic energy and satisfies the natural energy equality

$$\int_{\mathbf{R}^3}\tfrac12|u(x,t_1)|^2\,dx=\int_{\mathbf{R}^3}\tfrac12|u(x,t_2)|^2\,dx+\int_{t_1}^{t_2}\int_{\mathbf{R}^3}\nu|\nabla u(x,t)|^2\,dx\,dt \tag{1.4}$$

for each $t_1<t_2<T$, one obtains $\int_{\mathbf{R}^3}|U|^2\,dy<+\infty$ and $\int_{\mathbf{R}^3}|\nabla U|^2\,dy<+\infty$, which implies $U\in L^3(\mathbf{R}^3)$ by standard imbedding theorems. (We remark that Leray suggests to seek for bounded solutions of (1.3) satisfying $\int_{\mathbf{R}^3}|U|^2\,dy<+\infty$, see [Le, p. 225]. Such solutions obviously belong to $L^3(\mathbf{R}^3)$.)

On the other hand, if one only requires that for some ball $B=B_R(0)$ and some $t_0<T$ we have (for u given by (1.2))

$$\operatorname*{ess\,sup}_{t_0<t<T}\int_B |u(x,t)|^2\,dx<+\infty \quad\text{and}\quad \int_{t_0}^T\int_B |\nabla u(x,t)|^2\,dx\,dt<+\infty, \tag{1.5}$$

one gets conditions which do not imply $U\in L^3(\mathbf{R}^3)$. Therefore our result does not exclude the possibility of self-similar singularities which satisfy the natural energy estimates only locally. Finally, we remark that our method is also applicable in higher dimensions (specifically $3\leqslant n\leqslant 7$), but we will not pursue the case $n\geqslant 4$ in this paper.

2. Preliminaries

We shall use the following notation for balls. For $x\in\mathbf{R}^n$ we denote $B_R(x)=\{y\in\mathbf{R}^n: |y-x|<R\}$, and for $(x,t)\in\mathbf{R}^n\times\mathbf{R}$ we denote $Q_R(x,t)=B_R(x)\times(t-R^2,t)$. If there is no danger of confusion, we will skip the center of the ball from the notation and write simply B_R or Q_R. We will use the following result.

PROPOSITION 2.1. *For each $\nu>0$ there exists $\varepsilon_0>0$ and $C_0,C_1,...>0$ such that the following statement is true: let $u=(u_1,u_2,u_3)$ and p be smooth functions satisfying the Navier–Stokes equations* (1.1) *in $Q_R=B_R\times(-R^2,0)$, and assume that*

$$R^{-2}\int_{Q_R}(|u(x,t)|^3+|p(x,t)|^{3/2})\,dx\,dt<\varepsilon_0.$$

Then, for each $k=0,1,...$,

$$\sup_{Q_{R/2}}|\nabla^k u|\leqslant C_k R^{-1-k},$$

where ∇ means, as usual, the gradient with respect to the space variables x_1,x_2,x_3.

Proof. The statement can be deduced from [CKN, Proposition 1, p. 775] (see also [Sch]), and from [Se1] (see also [Oh]) in the following way. (We remark that one could also use other results on the regularity of the Navier–Stokes equations, such as [Gi], [So], [Str1], [Ta] and [vW].) We see from the scaling properties of the Navier–Stokes equations (see, for example, [CKN, p. 774]) that it is enough to consider the case $R=1$. Using the local energy equality

$$\int_{B_R} |u(x,t)|^2\varphi(x,t)\,dx+2\nu\int_{-R^2}^t\int_{B_R}|\nabla u|^2\varphi\,dx\,ds$$
$$=\int_{-R^2}^t\int_{B_R}|u|^2(\varphi_t+\nu\Delta\varphi)\,dx\,ds+\int_{-R^2}^t\int_{B_R}(|u|^2+2p)u\cdot\nabla\varphi\,dx\,dt,$$

where φ is a suitable "cut-off" function, we can estimate $\sup_{t\in(-R_1^2,0)}\int_{B_{R_1}}|u(x,t)|^2\,dx$ and $\int_{Q_{R_1}}|\nabla u|^2\,dx\,dt$ in terms if $\int_{Q_R}(|u|^3+|p|^{3/2})\,dx\,dt$ for a suitable $R_1<R$, for example $R_1=\frac{7}{8}R$. Now we can apply [CKN, Proposition 1, p. 775], to obtain the required estimate for $k=0$ in $Q_{3R/4}$. (Strictly speaking, this proposition is proved only for $\nu=1$, but it is easy to see that it holds for any $\nu>0$. The constants in the estimates depend on ν, of course. Also, the proposition as stated in [CKN] gives the estimates on the ball whose radius is the half of the original radius, but it is easily seen that the factor $\frac{1}{2}$ can be replaced by any factor strictly less than 1. Of course, the constants in the estimates depend on the factor we choose.) To prove the estimates for $k=1,2,\ldots$, we use the estimates we have obtained for $k=0$ together with the estimate for $\int_{Q_{R_1}}|\nabla u|^2\,dx\,dt$, and follow the proof of higher regularity in Serrin's paper [Se1]. Serrin does not write explicit estimates, but we can follow his proof line-by-line and replace the statements about smoothness properties of the functions which appear in the proof by the corresponding estimates. The elliptic and parabolic estimates needed to execute this procedure are listed in the appendix. Since this works without any additional tricks, we feel that it is not necessary to reproduce this purely mechanical procedure here in detail. $\square$

We recall some well-known facts about the equation for the pressure

$$-\Delta P=\partial_j\partial_k U_jU_k \quad \text{in } \mathbf{R}^n. \tag{2.2}$$

(Here and in what follows we use the usual convention and sum over the repeated indices.) If $U\in L^{2q}(\mathbf{R}^n)$ for some $q>1$, we can use the classical Riesz transformation to solve (2.2). Let us recall, that for $j=1,\ldots,n$ the Riesz transformation R_j is the singular integral operator given by the Fourier multiplier $-i\xi_j/|\xi|$, see [St] for details.

From the classical results regarding the operators R_j (see e.g. [CZ] or [St]) we infer that P defined by

$$P=R_jR_k(U_jU_k)$$

satisfies $\|P\|_{L^q(\mathbf{R}^n)}\leqslant c_q\|U\|^2_{L^{2q}(\mathbf{R}^n)}$ and solves (2.2) in the sense of distributions. If moreover U is smooth, then P is smooth by the classical regularity theory for the Laplace operator.

Of course, for a given U, the equation (2.2) determines P only up to a harmonic function, and we have to specify the behavior of P at infinity to get uniqueness. For example the requirement $P \in L^q(\mathbf{R}^n)$ obviously guarantees uniqueness since the only harmonic function belonging to $L^q(\mathbf{R}^n)$ is the identical zero.

LEMMA 2.2. *For $j=1,\dots,n$ let R_j denote the Riesz transform given by the Fourier multiplyer $-i\xi_j/|\xi|$. For $j,k=1,\dots,n$ let $F_{jk} \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$. Finally, let ϕ be a smooth, compactly supported function on $\mathbf{R}^n$ depending only on $|x|$ and satisfying $\phi(0)=1$. Then*

$$\lim_{\varepsilon \to 0_+} \int_{\mathbf{R}^n} R_j R_k (F_{jk})(x) \phi(\varepsilon x)\, dx = -\frac{1}{n} \int_{\mathbf{R}^n} F_{jj}(x)\, dx.$$

Proof. Let $\phi_\varepsilon(x) = \phi(\varepsilon x)$. Denoting by $\widehat{\ }$ the Fourier transformation (with the kernel given by $e^{2\pi i x\cdot\xi}$) we have from the Parseval formula

$$\int_{\mathbf{R}^n} R_j R_k (F_{jk}) \phi_\varepsilon \, dx = \int_{\mathbf{R}^n} -\frac{\xi_j \xi_k}{|\xi|^2} \widehat{F}_{jk}(\xi) \bar{\hat{\phi}}_\varepsilon(\xi)\, d\xi = \int_{\mathbf{R}^n} -\frac{\xi_j \xi_k}{|\xi|^2} \widehat{F}_{jk}(\xi) \bar{\hat{\phi}}\left(\frac{\xi}{\varepsilon}\right) \varepsilon^{-n}\, d\xi.$$

By using the substitution $\xi/\varepsilon \mapsto \xi$, the last integral can be rewritten as

$$\int_{\mathbf{R}^n} -\frac{\xi_j \xi_k}{|\xi|^2} \widehat{F}_{jk}(\varepsilon\xi) \bar{\hat{\phi}}(\xi)\, d\xi.$$

Since $\widehat{F}_{jk}$ is bounded and continuous, and $\hat{\phi}$ is rapidly decreasing, this integral converges to

$$\widehat{F}_{jk}(0) \int_{\mathbf{R}^n} -\frac{\xi_j \xi_k}{|\xi|^2} \bar{\hat{\phi}}(\xi)\, d\xi$$

as $\varepsilon \to 0_+$. Using the fact that $\hat{\phi}$ is radial and integrating first over the spheres and then over the radii, we see that

$$\int_{\mathbf{R}^n} -\frac{\xi_j \xi_k}{|\xi|^2} \bar{\hat{\phi}}(\xi)\, d\xi = -\frac{\delta_{jk}}{n} \int_{\mathbf{R}^n} \bar{\hat{\phi}}(\xi)\, d\xi = -\frac{\delta_{jk}}{n} \phi(0) = -\frac{\delta_{jk}}{n},$$

where δ_{jk} is the usual Kronecker symbol. Since $\delta_{jk}\widehat{F}_{jk} = \widehat{F}_{jj}$ and $\widehat{F}_{jk}(0) = \int_{\mathbf{R}^n} F_{jk}(x)\, dx$, the result follows. □

3. The main theorem

By a weak solution of (1.3) we mean a function $U=(U_1,U_2,U_3)$ on $\mathbf{R}^3$ which belongs *locally* to $W^{1,2}$, is divergence-free, and satisfies

$$\int_{\mathbf{R}^3} (\nu \nabla U \cdot \nabla \varphi + aU \cdot \varphi + a(y\cdot\nabla)U \cdot \varphi + (U\cdot\nabla)U \cdot \varphi)\, dy = 0$$

for each smooth, compactly supported, divergence-free vector field $\varphi=(\varphi_1,\varphi_2,\varphi_3)$ in $\mathbf{R}^3$. Using the regularity theory for the linear Stokes operator (see e.g. [Ca]), standard imbeddings and the bootstrapping argument, we see that every weak solution U of (1.3) is smooth. (The proof is more or less the same as the proof of regularity for steady solutions of the Navier–Stokes equations in three dimensions, cf. [La] or [Ga].)

We note that the pressure does not explicitly appear in the definition of the weak solution. Therefore the following lemma will be useful.

LEMMA 3.1. *Let $U=(U_1,U_2,U_3)\in L^3(\mathbf{R}^3)$ be a weak solution of* (1.3). *Assume that P is defined by $P=R_jR_k(U_jU_k)$, where we use the notation introduced in* §2. *Then both U and P are smooth, P belongs to $L^{3/2}(\mathbf{R}^3)$, and moreover*

$$-\nu\Delta U+aU+a(y\cdot\nabla)U+(U\cdot\nabla)U+\nabla P=0 \quad \textit{in } \mathbf{R}^3. \tag{3.1}$$

Proof. Above we have proved all the statements of the lemma except for the equation (3.1). To prove (3.1), we let

$$F=-\nu\Delta U+aU+a(y\cdot\nabla)U+(U\cdot\nabla)U+\nabla P.$$

We must prove that $F\equiv 0$. Our assumptions imply that $\operatorname{curl} F=0$ and $\operatorname{div} F=0$ in $\mathbf{R}^3$, hence $\Delta F=0$ in $\mathbf{R}^3$. Let ϕ be a radial (i.e. depending only on $|x|$), smooth, compactly supported function in $\mathbf{R}^3$ with $\int_{\mathbf{R}^3}\phi\,dx=1$. Since $\Delta F=0$, we have for each $\varepsilon>0$ (cf. [St, p. 275])

$$F(0)=\int_{\mathbf{R}^3}F(x)\varepsilon^3\phi(\varepsilon x)\,dx.$$

For each multiindex $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ we have $\Delta(D_\alpha F)=0$, and hence, for each $\varepsilon>0$,

$$D_\alpha F(0)=\int_{\mathbf{R}^3}D_\alpha F(x)\varepsilon^3\phi(\varepsilon x)\,dx=(-1)^{|\alpha|}\int_{\mathbf{R}^3}F(x)\varepsilon^{3+|\alpha|}(D_\alpha\phi)(\varepsilon x)\,dx. \tag{3.2}$$

Since F is analytic (see e.g. [Mo, p. 166]), it is enough to prove that $D_\alpha F(0)=0$ for each α with $|\alpha|\geqslant 0$. From formula (3.2) we see that this will follow if we prove that for each smooth, compactly supported function ψ in $\mathbf{R}^3$ we have

$$\lim_{\varepsilon\to 0_+}\varepsilon^3\int_{\mathbf{R}^3}F(x)\psi(\varepsilon x)\,dx=0.$$

Using the definition of F, we see that we must prove

$$\lim_{\varepsilon\to 0_+}\varepsilon^3\int_{\mathbf{R}^3}(-\nu\Delta U+aU+a(y\cdot\nabla)U+(U\cdot\nabla)U+\nabla P)\psi(\varepsilon y)\,dy=0.$$

This can be done term-by-term, by using integration by parts and the assumption $U\in L^3(\mathbf{R}^3)$, together with the fact $P\in L^{3/2}(\mathbf{R}^3)$. For example, we have

$$\begin{aligned}\int_{\mathbf{R}^3}(y\cdot\nabla)U(y)\psi(\varepsilon y)\,dy &= -\int_{\mathbf{R}^3}3U(y)\psi(\varepsilon y)\,dy-\int_{\mathbf{R}^3}U(y)(\varepsilon y_k)(\partial_k\psi)(\varepsilon y)\,dy\\ &= -\int_{\mathbf{R}^3}3U(y)\psi(\varepsilon y)\,dy-\int_{\mathbf{R}^3}U(y)\tilde{\psi}(\varepsilon y)\,dy,\end{aligned}$$

where $\tilde{\psi}(y)=y_k\partial_k\psi(y)$. Using the Hölder inequality, we can write

$$\left|\int_{\mathbf{R}^3}U(y)\psi(\varepsilon y)\,dy\right|\leqslant\|U\|_{L^3(\mathbf{R}^3)}\left\{\int_{\mathbf{R}^3}|\psi(\varepsilon y)|^{3/2}\,dy\right\}^{2/3}=\varepsilon^{-2}\|U\|_{L^3(\mathbf{R}^3)}\|\psi\|_{L^{3/2}(\mathbf{R}^3)}.$$

Hence

$$\lim_{\varepsilon\to 0_+}\varepsilon^3\int_{\mathbf{R}^3}U(y)\psi(\varepsilon y)\,dy=0.$$

The other terms can be dealt with in the same way, and the proof is easily finished. □

LEMMA 3.2. *Let $U=(U_1,U_2,U_3)$ and P be as in Lemma 3.1. Then, for each $k=0,1,2,\ldots$, we have*

$$|\nabla^k U(y)|=O(|y|^{-3-k}),\quad |y|\to\infty$$

and

$$|\nabla^k P(y)|=O(|y|^{-2-k}),\quad |y|\to\infty.$$

Proof. Let $T\in\mathbf{R}$. For $x\in\mathbf{R}^3$ and $t<T$ we let

$$u(x,t)=\frac{1}{\sqrt{2a(T-t)}}\,U\left(\frac{x}{\sqrt{2a(T-t)}}\right) \tag{3.3}$$

and

$$p(x,t)=\frac{1}{2a(T-t)}\,P\left(\frac{x}{\sqrt{2a(T-t)}}\right). \tag{3.4}$$

The equation (3.1) implies that u and p satisfy the Navier–Stokes equations (1.1) in $\mathbf{R}^3\times(-\infty,T)$. We have $U\in L^3(\mathbf{R}^3)$ and $P\in L^{3/2}(\mathbf{R}^3)$, and hence

$$\lim_{\varrho\to\infty}\int_{|y|\geqslant\varrho}(|U|^3+|P|^{3/2})\,dy=0.$$

Using this, we check easily by an elementary change of variables that for each $x_0\in\mathbf{R}^3\setminus\{0\}$ there exists $R>0$ such that

$$\sup_{T-R^2<t<T}\int_{B_R(x_0)}(|u(x,t)|^3+|p(x,t)|^{3/2})\,dx<\varepsilon_0, \tag{3.5}$$

where ε_0 has the same meaning as in Proposition 2.1. Since (3.5) obviously implies

$$R^{-2}\int_{Q_R(x_0,T)}(|u(x,t)|^3+|p(x,t)|^{3/2})\,dx\,dt<\varepsilon_0,$$

we see from Proposition 2.1 that

$$\sup_{Q_{R/2}(x_0,T)}|\nabla^k u(x,t)|\leqslant C_k R^{-1-k} \tag{3.6}$$

for each $k=0,1,2,\dots$. Let $0<\varrho_1'<\varrho_1<\varrho_2<\varrho_2'$ and let $t_1<T$. Since u is smooth in $\mathbf{R}^3\times(-\infty,T)$ and x_0 was an arbitrary point of $\mathbf{R}^3\setminus\{0\}$, we see from (3.6) and the obvious compactness argument that the functions $\nabla^k u(x,t)$ are bounded in $\Omega_{\varrho_1',\varrho_2'}\times(t_1,T)$, where $\Omega_{\varrho_1',\varrho_2'}=\{x\in\mathbf{R}^3:\varrho_1'<|x|<\varrho_2'\}$. Let also $\Omega_{\varrho_1,\varrho_2}=\{x\in\mathbf{R}^3:\varrho_1<|x|<\varrho_2\}$. We claim that, for $k=0,1,\dots$, the space derivatives $\nabla^k p$ are bounded in $\Omega_{\varrho_1,\varrho_2}\times(t_1,T)$. To see this, we note that for each $t<T$ we have

$$\int_{\mathbf{R}^3}|p(x,t)|^{3/2}\,dx=\int_{\mathbf{R}^3}|P(y)|^{3/2}\,dy. \tag{3.7}$$

Since for each $t<T$ we have

$$-\Delta p(x,t)=\partial_i\partial_j(u_i(x,t)u_j(x,t))\quad\text{in }\mathbf{R}^3 \tag{3.8}$$

we see from the classical elliptic estimates (cf. [Br]) that

$$\|p(\cdot,t)\|_{C^{k,\alpha}(\Omega_{\varrho_1,\varrho_2})}\leqslant c_k\sum_{i,j}\|u_i(\cdot,t)u_j(\cdot,t)\|_{C^{k,\alpha}(\Omega_{\varrho_1',\varrho_2'})}+\tilde c_k\|p(\cdot,t)\|_{L^{3/2}(\mathbf{R}^3)},$$

where we use the usual notation for the Hölder norms. Since, as we have seen above, we control the terms on the right-hand side, we see that $\nabla^k p$ is bounded in $\Omega_{\varrho_1,\varrho_2}\times(t_1,T)$ for each k, as claimed.

We can now take the space derivatives of the Navier–Stokes equations to see that for each $k=0,1,2\dots$ the derivatives $\nabla^k\partial u/\partial t=\partial(\nabla^k u)/\partial t$ are bounded in $\Omega_{\varrho_1,\varrho_2}\times(t_1,T)$. Since

$$\int_{\Omega_{\varrho_1,\varrho_2}}|u(x,t)|^3\,dx\to 0\quad\text{as } t\to T_-$$

(which can be seen for (3.3) and the assumption $U\in L^3(\mathbf{R}^3)$), we see from the above estimates and the Arzela–Ascoli theorem that $\nabla^k u(\cdot,t)\to 0$ as $t\to T_-$ uniformly in $\Omega_{\varrho_1,\varrho_2}$ for each $k=0,1,\dots$. Therefore we can write, for $x\in\Omega_{\varrho_1,\varrho_2}$,

$$\nabla^k u(x,t)=-\int_t^T\nabla^k\frac{\partial}{\partial t}u(x,s)\,ds,$$

and by the estimates for $\nabla^k \partial u(x,t)/\partial t$ above we have, for $k=0,1,\dots$,

$$|\nabla^k u(x,t)| \leqslant M_k(T-t) \quad \text{in } \Omega_{\varrho_1,\varrho_2}\times(t_1,T), \tag{3.9}$$

where M_k are suitable constants. We have seen above that we also have, for $k=0,1,\dots$,

$$|\nabla^k p(x,t)| \leqslant \widetilde{M}_k \quad \text{in } \Omega_{\varrho_1,\varrho_2}\times(t_1,T). \tag{3.10}$$

It is now a matter of an elementary calculation to verify that (3.9) together with (3.3) implies $|\nabla^k U(y)|=O(|y|^{-3-k})$ as $|y|\to\infty$, and that (3.10) together with (3.4) implies that $|\nabla^k P(y)|=O(|y|^{-2-k})$ as $|y|\to\infty$. The proof is finished. □

LEMMA 3.3. *Assume that $U=(U_1,U_2,U_3)$ and P are smooth functions satisfying* (1.3) *in $\mathbf{R}^3$. Then the quantity*

$$\Pi(y)=\tfrac{1}{2}|U(y)|^2+P(y)+ay\cdot U(y)$$

satisfies the maximum principle, i.e. for each bounded domain $\Omega\subset\mathbf{R}^3$ we have

$$\sup_{y\in\Omega}\Pi(y)\leqslant \sup_{y\in\partial\Omega}\Pi(y).$$

Proof. We let $\widetilde{U}(y)=U(y)+ay$ and $\widetilde{P}(y)=P(y)-\frac{1}{2}a^2|y|^2$. An easy calculation shows that (1.3) is equivalent to

$$\left.\begin{aligned} -\nu\Delta\widetilde{U}+(\widetilde{U}\cdot\nabla)\widetilde{U}+\nabla\widetilde{P}&=0\\ \operatorname{div}\widetilde{U}&=3a\end{aligned}\right\}\quad \text{in } \mathbf{R}^3.$$

Multiplying the first equation by $\widetilde{U}$ we get

$$-\nu\Delta\widetilde{U}\cdot\widetilde{U}+(\widetilde{U}\cdot\nabla)\big(\tfrac{1}{2}|\widetilde{U}|^2+\widetilde{P}\big)=0.$$

Hence

$$\begin{aligned}-\nu\Delta\big(\tfrac{1}{2}|\widetilde{U}|^2+\widetilde{P}\big)+(\widetilde{U}\cdot\nabla)\big(\tfrac{1}{2}|\widetilde{U}|^2+\widetilde{P}\big)&=-\nu|\nabla\widetilde{U}|^2-\nu\Delta\widetilde{P}\\ &=-\nu|\nabla U+aI|^2-\nu\Delta P+3\nu a^2\\ &=-\nu|\nabla U|^2+\nu(\partial_i U_j)(\partial_j U_i)\leqslant 0,\end{aligned}$$

where we denote by I the identity matrix. Since $\frac{1}{2}|\widetilde{U}|^2+\widetilde{P}=\Pi$, we see that

$$-\nu\Delta\Pi+(\widetilde{U}\cdot\nabla)\Pi\leqslant 0 \quad \text{in } \mathbf{R}^3,$$

and the result follows from the standard maximum principle for elliptic equations (see e.g. [GT]). □

Remark. The fact that the quantity $\frac{1}{2}|u|^2+p$ satisfies a maximum principle for the steady Navier–Stokes equations is well-known (see e.g. [Se2, p. 261], or [GW]), and has played an important role in recent results ([FR1], [FR2] and [Str2]) regarding the regularity of solutions of the steady Navier–Stokes equations in higher dimensions.

THEOREM 1. *Let U be a weak solution of* (1.3) *belonging to* $L^3(\mathbf{R}^3)$. *Then* $U\equiv 0$ *in* $\mathbf{R}^3$.

Proof. Let $P=R_jR_k(U_jU_k)$, where we use the notation introduced in §2. By Lemma 3.1 the functions U and P are smooth and satisfy (1.3). Hence $\Pi(y)=\frac{1}{2}|U(y)|^2+P(y)+ay\cdot U(y)$ satisfies the maximum principle by Lemma 3.3. Since $\Pi(y)=O(|y|^{-2})$ as $|y|\to\infty$ by Lemma 3.2, the maximum principle (applied to large balls) implies that $\Pi\leqslant 0$ in $\mathbf{R}^3$. Let ϕ be as in Lemma 2.2 and assume moreover that $\phi\geqslant 0$ in $\mathbf{R}^3$. For $\varepsilon>0$ let $\phi_\varepsilon(y)=\phi(\varepsilon y)$. Since $\Pi\leqslant 0$ in $\mathbf{R}^3$, we have $\int_{\mathbf{R}^3}\Pi\phi_\varepsilon\,dy\leqslant 0$ for each $\varepsilon>0$. Since ϕ_ε is radial and $\operatorname{div} U=0$, we have $\int_{\mathbf{R}^3}a(y\cdot U)\phi_\varepsilon\,dy=0$ for each $\varepsilon>0$. Since $U_jU_k\in L^1(\mathbf{R}^3)\cap L^2(\mathbf{R}^3)$ by Lemma 3.2, we see from Lemma 2.2 that $\lim_{\varepsilon\to 0+}\int_{\mathbf{R}^3}P\phi_\varepsilon\,dy=-\frac{1}{3}\int_{\mathbf{R}^3}|U|^2\,dy$. We infer that

$$\int_{\mathbf{R}^3}\tfrac{1}{2}|U|^2\,dy-\int_{\mathbf{R}^3}\tfrac{1}{3}|U|^2\,dy\leqslant 0,$$

and hence $U\equiv 0$. The proof is finished. □

An alternative proof of Theorem 1. We feel that the following alternative way of proving Theorem 1 is of interest. We let $Q(y)=P(y)+ay\cdot U(y)$. The system (1.3) can be rewritten as

$$\begin{gathered}-\nu\Delta U_j+U_k\frac{\partial U_j}{\partial y_k}+\frac{\partial Q}{\partial y_j}+ay_k\left(\frac{\partial U_j}{\partial y_k}-\frac{\partial U_k}{\partial y_j}\right)=0,\\ \operatorname{div} U=0.\end{gathered}\tag{3.11}$$

After multiplying the first equation by $y_j/|y|$, we obtain

$$\frac{y_j}{|y|}\cdot\frac{\partial Q}{\partial y_j}=\nu\Delta U_j\frac{y_j}{|y|}-U_k\frac{\partial U_j}{\partial y_k}\cdot\frac{y_j}{|y|}.\tag{3.12}$$

Denoting $\varrho=|y|$ and using Lemma 3.2, we see that $\partial Q/\partial\varrho=O(\varrho^{-5})$ as $\varrho\to\infty$. Since also $Q(y)=O(|y|^{-2})$ as $|y|\to\infty$ by Lemma 3.2, we see (by integrating along the rays through the origin) that

$$Q(y)=O(|y|^{-4}),\quad |y|\to\infty.$$

Multiplying (3.11) by y_j and integrating over $\mathbf{R}^3$ (which we can because all the terms are integrable by the decay estimates for Q above and the decay estimates for U and P established in Lemma 3.2), we obtain

$$\int_{\mathbf{R}^3}\left(-\nu y_j\Delta U_j+y_jU_k\frac{\partial U_j}{\partial y_k}+y_j\frac{\partial Q}{\partial y_j}\right)dy=0.$$

Integrating by parts (which, again, we can because of the decay estimates we have for U and Q), we obtain

$$\int_{\mathbf{R}^3}(|U|^2+3Q)\,dy=0.$$

On the other hand, $|U|^2+2Q\leqslant 0$ in $\mathbf{R}^3$ by Lemma 3.2 and Lemma 3.3 (see also our first proof of the Theorem 1), and hence

$$\int_{\mathbf{R}^3}(|U|^2+2Q)\,dy\leqslant 0.$$

Subtracting $\int_{\mathbf{R}^3}\left(\frac{2}{3}|U|^2+2Q\right)dy=0$ from this inequality, we obtain $\int_{\mathbf{R}^3}\frac{1}{3}|U|^2\,dy\leqslant 0$ and the proof is finished. □

Appendix

Here we recall the estimates which can be used for the proof of Proposition 2.1.

PROPOSITION A.1. *Let $Q=B_1(0)\times(-1,0)\subset\mathbf{R}^n\times\mathbf{R}$ and $Q'=B_{1-\delta}\times(-(1-\delta)^2,0)$, where $0<\delta<1$ is a fixed number. (Typically we imagine that δ is small.) Let $1<p<+\infty$ and $f=(f_1,...,f_n)\in L^p(Q)$. Assume that $v\in L^2(Q)$ is a weak solution of the heat equation*

$$\frac{\partial v}{\partial t}-\Delta v=\frac{\partial f_j}{\partial x_j}\quad \text{in } Q.$$

Then v satisfies the following estimates (in which we use c to denote the "generic constant" which can depend on p and δ):

$$\|Dv\|_{L^p(Q')}\leqslant c\|f\|_{L^p(Q)}+c\|v\|_{L^2(Q)},$$
$$\|v\|_{L^q(Q')}\leqslant c\|f\|_{L^p(Q)}+c\|v\|_{L^2(Q)}\quad \text{if } \frac{1}{q}=\frac{1}{p}-\frac{1}{n+2}>0,$$
$$\|v\|_{C^\alpha_{\mathrm{par}}(Q')}\leqslant c\|f\|_{L^p(Q)}+c\|v\|_{L^2(Q)}\quad \text{if } \alpha=1-\frac{n+2}{p}>0,$$

where $C^\alpha_{\mathrm{par}}(Q')$ denotes the space of functions which are α-Hölder continuous with respect to the parabolic metric in which the distance between (x,t) and (x',t') is given by $|x-x'|+|t-t'|^{1/2}$.

Proof. These estimates are well-known and can be obtained for example by combining the L^p-estimates established in [LUS, Chapter IV, §9, Theorem 9.1] with the imbeddings established in [LUS, Chapter II, §3, Lemmas 3.2 and 3.3]. □

PROPOSITION A.2. *Let $B=B_1(0)\subset\mathbf{R}^n$ and let $B'=B_{1-\delta}(0)$, where $0<\delta<1$ is a fixed number. (Typically we imagine that δ is small.) Let $u=(u_1,...,u_n)\in L^2(B)$. Then, for $k=1,2...$ we have*

$$\|D^k u\|_{L^p(B')}\leqslant c\|D^{k-1}\operatorname{curl} u\|_{L^p(B)}+c\|D^{k-1}\operatorname{div} u\|_{L^p(B)}+c\|u\|_{L^2(B)}$$

and

$$\|D^k u\|_{C^\alpha(B')} \leqslant c\|D^{k-1}\operatorname{curl} u\|_{C^\alpha(B)} + c\|D^{k-1}\operatorname{div} u\|_{C^\alpha(B)} + c\|u\|_{L^2(B)},$$

where $1<p<\infty$, $0<\alpha<1$, *and* c *denotes the "generic constant" which can depend on* δ, p *and* α.

Proof. This is a well-known consequence of the classical L^p- and C^α-estimates for the Laplace equation. □

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