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Elimination in Control Theory*

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Abstract. For nonlinear systems described by algebraic differential equations (in terms of "state" or "latent" variables) we examine the converse to realization, *elimination*, which consists of deriving an externally equivalent representation not containing the state variables. The elimination in general yields not only differential equations but also differential *inequations*. We show that the application of differential algebraic elimination theory (which goes back to J. F. Ritt and A. Seidenberg) leads to an *effective* method for deriving the equivalent representation. Examples calculated by a computer algebra program are shown.

Key words. Equivalent system representations, Latent variable elimination, State elimination, Elimination theory, Differential polynomial algebras.

1. Introduction

Continuous-time nonlinear systems can be represented by several different types of differential equations. In particular, we often consider systems of equations of the type

$$P_i(w, \dot{w}, \ldots, w^{(\alpha)}, \zeta, \dot{\zeta}, \ldots, \zeta^{(\beta)}) = 0, \qquad i = 1, 2, \ldots, \tag{E}$$

where, as in Willems' work [W1], w stands for the external variables of the system and ζ denotes a set of *latent* variables whose introduction comes from modeling techniques. Alternatively, we also encounter external representations

$$p_i(w, \dot{w}, \ldots, w^{(\alpha)}) = 0, \qquad i = 1, 2, \ldots, \tag{E'}$$

where w still stands for the external variables which need not be, *a priori*, partitioned into input and output variables (see [W1]). Finally, we consider the standard *state* representations

$$\begin{cases} \dot{x}_i = F_i(u, x), & i = 1, 2, \ldots, n, \\ y_j = H_j(u, x), & j = 1, 2, \ldots, p, \end{cases} \tag{S}$$

where u and y are, respectively, inputs and outputs of the system and the x_i's are the state variables. A natural and fundamental question that arises is the one of

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equivalence of representations: in what precise sense do two systems of equations represent one and the same *system*? This is not an easy question, and a full answer to it seems to be lacking. Many papers in the literature treat the related problem of transforming one of the above representations to another one. Realization theory, which consists of deriving a state representation of type (S) from an external representation of type (E′), is one of these transformation processes. Because of the great popularity of state representations, realization theory is abundantly studied in control theory literature (for the nonlinear context, see, for example, [S9], [FW], [S6]–[S8], [J2], [F1], [V1]–[V3], [CL], [G], and [WS]; see also the criticisms in Fliess' work [F2], [FH]). The converse question to realization is precisely what we call *elimination*. More generally, elimination consists of deriving equivalent representations of type (E′) from those of type (E). The main point of our paper is to argue that it is natural to supplement (E′) by *inequations*. We explain this intuitively next.

In the context of constant coefficient linear systems, Proposition 3.3 of [W2] and p. 107 of [BY] show through matrix computations that an arbitrary constant linear system represented by (E) can be described by equations of type (E′). In the more general context of bilinear systems, [FR] and Theorem 1 of [S8] show that the previous conclusion holds. What about general nonlinear systems of type (E)? This question is investigated in [CMP] where the restricted form (S) of (E) is assumed, and in [V2]–[V3] where some constant rank conditions are assumed; the previously mentioned works rest on the implicit function theorem. The latter question is also addressed by Glad [G]. Glad and the author had recourse to differential algebra; the main difference between their respective works is that in [G] the underlying question of equivalence is not considered. Analogous questions for difference equations are treated in [S6] and [S7], using techniques from algebraic geometry.

From a differential algebraic geometry point of view, passing from (E) to (E′) is a *projection* operation. Let us associate to (E) its *differential algebraic set*, **V**(E), defined to be the set of *zeros* of (E), and similarly for (E′); then **V**(E′) is the *projection* of **V**(E) *onto* the w coordinates (more precise definitions are given below). A suitable definition of the *external behavior* of the system of equations (E) is the latter projection **V**(E′) of **V**(E); furthermore, two systems of equations of type (E) are said to be *externally equivalent* if and only if they have the same external behavior. What is clear from this point of veiw is that we cannot expect to get merely a set of equations as an equivalent representation of (E) which no longer invokes the latent variable ζ, since it is well known that the projection of an algebraic set is in general not an algebraic set (that is, **V**(E′) may not be defined by equations only, which is in contrast to the previous linear and bilinear cases). To obtain such an equivalent representation of (E), (E′) should include *inequations*. A simple example showing this is the hyperbola defined by the equation $xy = 1$ for which the elimination of y (that is, the projection of the hyperbola onto the abscissa axis) leads to the inequation $x \neq 0$. (See basic books on algebraic geometry, such as [M, Section 2.C.] or [S5, Section I.5.2.], for these algebraic questions.) Our main result then reads as follows: when eliminating the ζ from equations (E) we are led to a finite family of systems of equations and inequations of type

$$p_1(w, \dot{w}, \ldots, w^{(\alpha)}) = 0, \quad p_2(w, \dot{w}, \ldots, w^{(\beta)}) = 0, \quad \ldots, \quad \text{and} \quad q(w, \dot{w}, \ldots, w^{(\gamma)}) \neq 0.$$

This appears to be a new result in control theory. Every inequation defines a region, in terms of the external variables, in which equations (E) are externally equivalent to the set of equations associated to the given inequation. In other words, from a representation of type (E) we are led to a representation of type (E′) but valid only *locally*, since the external equivalence is valid only where the inequations hold. This locality aspect, often referred to in the literature, has here received a definite meaning.

After the latter fact is clarified, it remains to find an *effective* method to determine the external behavior of a system defined by equations (E) (or, in what amounts to the same thing, we need an effective method for deriving from (E) an externally equivalent representation which no longer contains a given latent variable). Fortunately, such a method is readily available to us via *differential algebraic elimination theory*, which is a theory developed at the beginning of differential algebra studies in the 1930s, with Ritt and, later, Seidenberg (and others). So we just have to apply one of the known algorithms to obtain a mechanical method for deriving external equivalent representations of a given system of equations (E). This applies immediately to the elimination question in control theory. (This is true at least for systems which can be described by algebraic differential equations with coefficients in a differential field of characteristic zero; for some systems described by differential equations where expressions with transcendental elementary functions occur, it seems possible, according to works by Wu *et al.*, to reduce the problem of elimination to the previous algebraic case by some technique of variable transformations. For more details see [WW].) This is what we do in this paper. Instead of the powerful algorithm developed by Ritt [R1] [R2] (and revisited by Seidenberg [S4]), we adapt an algorithm due to the latter author (who published it in 1956) for the following reason: it is simple and it rests on elementary notions of differential algebra, especially differential polynomial algebras; we do not claim, however, that this algorithm is the best one in any efficiency sense.

Related results for discrete-time nonlinear systems, obtained by Sontag [S6], [S7], have already been applied to identification and other areas by many authors (see [DD] and [LB]). Thus present work should also be applicable to practical problems. Note however that the work [S6] and [S7] did *not* notice the importance of adding inequations, which is the main point of the present work.

We recall that differential algebra was introduced into systems theory in 1985 by Fliess [F2]. For the reader who is not familiar with this part of mathematics, we recall the few notions necessary for what follows (for more details see [R1], [R2], or [K]). We then give details on the algorithm and some illustrative examples.

The material of this paper, which was announced in [D1], is part of a doctoral dissertation [D2].

2. Basic Differential Algebra

Differential algebra was initiated by Ritt [R1], [R2] during the 1930s. Algebraic language was then known as being particularly suitable for describing properties of algebraic equations. The purpose was therefore to extend these basic methods in order to apply them to differential equations. The easiest differential equations we can deal with are certainly algebraic ones, i.e., those defined by *differential polynomials*. Well-known algebraic concepts are extended to differential ones essentially by adjoining differential operators to algebraic laws operations such as addition and multiplication. Basic differential algebra then consists of the study of the resulting structures. Differential polynomials turn out to be the suitable differential algebraic objects generalizing polynomials. We first briefly recall the notions of differential ring, field, etc.

Let $\mathbf{R}$ be a ring (that we call the *underlying* ring). A map ∂ of $\mathbf{R}$ into itself is called a *derivation* on $\mathbf{R}$ if and only if the following two axioms are satisfied:

1. For all $a, b \in \mathbf{R}$, $\partial(a + b) = \partial a + \partial b$.
2. For all $a, b \in \mathbf{R}$, $\partial(ab) = \partial(a)b + a\partial b$.

A ring $\mathbf{R}$ equipped with a derivation is called an (*ordinary*) *differential ring* and is still denoted by the same symbol. A ring is always a differential one, for the zero map (which sends to zero any element of $\mathbf{R}$) is clearly a derivation; the resulting differential ring structure is in no way different from the underlying ring structure. This indicates the generalization of algebra provided by differential algebra. If we replace the word *ring* by *field* in the above definitions we obtain the notion of *differential field*. In an analogous way as for rings, we define *differential modules, differential linear spaces, differential algebras*, and so on.

In what follows, rings and algebras are assumed to be *commutative and with unit element*, algebras are *associative*, and fields are *commutative*. For brevity, "*d*-..." denotes "*differential* ..." and the derivation is indicated by a dot.

3. Differential Polynomial Algebras

We next indicate the standard notations that we adopt here. $\mathbf{R}[(Z_i)_{i\in I}]$ is the polynomial algebra in the family of indeterminates $(Z_i)_{i\in I}$ indexed by the arbitrary set I, with coefficients in the ring $\mathbf{R}$. The elements of $\mathbf{R}[(Z_i)_{i\in I}]$ are the *finite* sums of *finite* products of the Z_i with coefficients in $\mathbf{R}$. Formally, let $E^{(J)}$ be the set of families of elements of E, all zero (assume that E possesses an element called zero) but a finite number of them, indexed by the arbitrary set J. The monomials of $\mathbf{R}[(Z_i)_{i\in I}]$ are

$$Z^\mu = \prod_{i\in I} Z_i^{\mu_i},$$

where $\mu = (\mu_i) \in \mathbb{N}^{(I)}$; and the elements of $\mathbf{R}[(Z_i)_{i\in I}]$ are

$$P = \sum_{\mu\in\mathbb{N}^{(I)}} a_\mu Z^\mu,$$

where $a = (a_\mu) \in \mathbf{R}^{(\mathbb{N}^{(I)})}$ is the family of coefficients of P; these are all zero but for a finite number.

To define the differential polynomial algebra denoted by $\mathbf{R}\{(Z_i)_{i\in I}\}$ with $\mathbf{R}$ a differential ring, we let it set-theoretically be the $\mathbf{R}$-algebra $\mathbf{R}[(Z_i^{(\nu)})_{(i,\nu)\in I\times\mathbb{N}}]$. The differential monomials are then the objects

$$Z^{\mu} = \prod_{(i,v) \in I \times \mathbb{N}} Z_i^{(v)\mu_{i,v}},$$

where $\mu = (\mu_{i,v}) \in \mathbb{N}^{(I \times \mathbb{N})}$. We can now easily define a derivation on $\mathbf{R}\{(Z_i)_{i \in I}\}$ which extends the one on $\mathbf{R}$. Let

$$(Z^{\mu})^{\cdot} = \sum_{(i,v) \in I \times \mathbb{N}} \left(\mu_{i,v} Z_i^{(v)\mu_{i,v}-1} \cdot \prod_{(j,\lambda) \in I \times \mathbb{N}, (j,\lambda) \neq (i,v)} Z_j^{(\lambda)\mu_{j,\lambda}} \right) \cdot Z_i^{(v+1)}$$

for any $\mu \in \mathbb{N}^{(I \times \mathbb{N})}$; and

$$\dot{P} = \sum_{\mu \in \mathbb{N}^{(I)}} (\dot{a}_{\mu} Z^{\mu} + a_{\mu} (Z^{\mu})^{\cdot})$$

for any $P = \sum a_{\mu} Z^{\mu} \in \mathbf{R}\{(Z_i)_{i \in I}\}$. It is straightforward to check that this is a derivation on $\mathbf{R}\{(Z_i)_{i \in I}\}$ which extends the one on $\mathbf{R}$. Then $\mathbf{R}\{(Z_i)_{i \in I}\}$ together with this derivation is called the *differential polynomial algebra* in the family of indeterminates $(Z_i)_{i \in I}$ with coefficients in $\mathbf{R}$. Its elements are the *d-polynomials*, and the Z_i's are more precisely the *d-indeterminates*. We denotes $Z^0 = 1$, $Z_i^{(0)} = Z_i$, $\dot{Z}_i = Z_i^{(1)}$, and so forth. As usual, we identify $\mathbf{R}\{(Z_i)_{i \in I}\}$ with $\mathbf{R}$ if I is the empty set, with $\mathbf{R}\{Z\}$ if I consists of a single element, and with $\mathbf{R}\{Z_1, Z_2, \ldots, Z_n\}$ if I consists of n elements.

We now show a few results which generalize known algebraic situations and which are used later. First, some definitions on $\mathbf{R}\{Z\}$. For any $P \in \mathbf{R}\{Z\} \backslash \mathbf{0}$ (where $\mathbf{0}$ is the zero subalgebra of $\mathbf{R}\{Z\}$), we define

$$\omega(P) = \begin{cases} (0, d) & \text{if } P \in \mathbf{R}[Z] \backslash \mathbf{0} \text{ and } d = d_Z^{\circ} P \text{ (degree of } P \text{ in } Z), \\ (r, d) & \text{if } P \in \mathbf{R}[Z, Z^{(1)}, Z^{(2)}, \ldots, Z^{(r)}] \backslash \mathbf{R}[Z, Z^{(1)}, Z^{(2)}, \ldots, Z^{(r-1)}], \\ & r \geq 1, \text{and } d = d_{Z^{(r)}}^{\circ} P. \end{cases}$$

Therefore we have defined a map

$$\omega \colon \mathbf{R}\{Z\} \backslash \mathbf{0} \to \mathbb{N}^2$$

which will play a central role in what follows (as will be seen in the Proposition below, ω curiously behaves like the *degree function* in usual polynomial procedures). For any $P \in \mathbf{R}\{Z\} \backslash \mathbf{0}$, the integer r in $\omega(P) = (r, d)$ is called the *order* of P; and the integer d the *degree* of P. The order of P is merely the highest derivative of Z that appears in P; and the degree of P is the degree in $Z^{(r)}$ where r is the order of P. For any $P \in \mathbf{R}\{Z\} \backslash \mathbf{0}$, the *initial* of P is the coefficient (element of $\mathbf{R}\{Z\}$) in P of $Z^{(r)d}$ where $(r, d) = \omega(P)$, and the *separant* of P is the d-polynomial $\partial P / \partial Z^{(r)}$ (it can easily be seen that if $\mathbf{R}$ is a d-ring having the ring of integers $\mathbb{Z}$ as a subring (such d-rings are said to be of *characteristic zero* or to be *Ritt algebras*), then whenever P is in $\mathbf{R}\{Z\} \backslash \mathbf{R}$ the separant of P is merely the *common* initials of the derivatives $P^{(i)}$ $(i \geq 1)$ of P). In the following, $\mathbb{N}^2$ is assumed to be *lexicographically* ordered. That is, $(n, m) < (n', m')$ if and only if either $n < n'$ or $n = n'$ and $m < m'$. According to this ordering of $\mathbb{N}^2$, it is clear that, for any P in $\mathbf{R}\{Z\} \backslash \mathbf{R}$, the initial I and the separant S of P respectively satisfy:

either $I = 0$ or $\omega(I) < \omega(P)$,
either $S = 0$ or $\omega(S) < \omega(P)$.

We now quote the following important result, similar to the division algorithm in usual one-indeterminate polynomial algebras (see Proposition 10, Section IV.1.6, of [B]) for a nondifferential version, see also [R1], [R2], or Section I.9 of [K] for another division algorithm valid for partial differential polynomials).

Proposition. *Let* $\mathbf{R}$ *be a d-ring,* $P, Q \in \mathbf{R}\{Z\}$. *If the separant* S *of* Q *is not zero and if* I *denotes the initial of* Q, *then there exist* $P^* \in \mathbf{R}\{Z\}$, $\iota, \sigma \in \mathbb{N}$ *such that*

$$I^\iota S^\sigma P \equiv P^* \bmod [Q] \qquad \text{with either} \quad P^* = 0 \text{ or } P^* \neq 0 \quad \text{and} \quad \omega(P^*) < \omega(Q).$$

That is, assuming P *and* $Q \neq 0$ *and* $\omega(Q) = (r, d)$ *and* $\omega(P) = (r', d')$, *there exist d-polynomials* $U_0, \ldots, U_{r'-r}$, P^*, *and natural integers* ι *and* σ *such that*

$$I^\iota S^\sigma P = U_0 Q^{(r'-r)} + U_1 Q^{(r'-r-1)} + \cdots + U_{r'-r} Q + P^*$$

$$\text{with either} \quad P^* = 0 \text{ or } P^* \neq 0 \quad \text{and} \quad \omega(P^*) < \omega(Q).$$

Proof. To prove this we proceed by steps.

1. *Case* $P = 0$ *or* $P \neq 0$ *and* $\omega(P) < \omega(Q)$. The existence of the U, P^*, ι, and σ is obvious if $P = 0$ or $P \neq 0$ and $\omega(P) < \omega(Q)$ for it then suffices to take U to be zero, P^* to be P, and ι and σ also to be zero. Therefore we assume $P \neq 0$ and $\omega(P) \geq \omega(Q)$.

2. *Case* $P \neq 0$ *and* P *and* Q *have the same order.* We then show the following.

If A, B *are both in* $\mathbf{R}\{Z\}\backslash\mathbf{R}$ *with* $\omega(A) = (s, t') \geq (s, t) = \omega(B)$ *(*A *and* B *are of the same order,* s*) and* J *is the initial of* B, *then there exist* V, R, *and* j *such that*

$$J^j A = VB + R \qquad \text{with either} \quad R = 0 \text{ or } R \neq 0 \quad \text{and} \quad \omega(R) < \omega(B)$$

(such V *and* R *are unique whenever* $\mathbf{R}$ *is integral and we fix* j, *e.g.,* $j = \max(t' - t + 1, 0)$*).*

Let $A_0 = A$. Let $J_0 =$ initial of A_0 and t_0 be the difference between the respective degrees in $Z^{(s)}$ of A_0 and B, and

$$R_0 = JA_0 - J_0 Z^{(s)^{t_0}} B.$$

If either $R_0 = 0$ or $R_0 \neq 0$ and $\omega(R_0) < \omega(B)$, then the process ends. Otherwise, we nevertheless have

$$d^\circ_{Z^{(s)}} R_0 < d^\circ_{Z^{(s)}} A_0.$$

We now set $A_1 = R_0$. We then reiterate the above process.

We thus repeat the latter at most $(t' - t)$ times, until we have

$$R_i = 0 \text{ or } R_i \neq 0 \quad \text{and} \quad \omega(R_i) < \omega(B)$$

since at each step $d^\circ_{Z^{(s)}} R_i$ is decreased at least by one. The quantities j, V, and R mentioned above are then given by

$$j = i + 1, \qquad V = J^i J_0 Z^{(s)^{t_0}} + J^{i-1} J_1 Z^{(s)^{t_1}} + \cdots + J_i Z^{(s)^{t_i}}, \qquad R = R_i.$$

Notice that if the initial of A_i contains J as a factor, then the corresponding exponent j can be taken to be 0.

3. *General case.* Now let $\omega(P) = (r', d')$ and $\omega(Q) = \omega(r, d)$ with $r' > r$. $Q^{(r'-r)}$ is of order r' and of degree 1 (in $Z^{(r')}$). By case 2, for $\sigma_0 = d' - 1 + 1 = d'$, there exist Q_0, R_0 such that

$$S^{\sigma_0}P = Q_0 Q^{(r'-r)} + R_0 \qquad \text{with} \quad R_0 \in \mathbf{R}[Z, Z^{(1)}, \ldots, Z^{(r'-1)}]$$

(since $Q^{(r'-r)}$ is of degree 1 in $Z^{(r')}$). If R_0 is of order $\leq r$, the proof is complete. Otherwise, there exist Q_1, R_1, and σ_1 such that

$$S^{\sigma_1}R_0 = Q_1 Q^{(r'-r-1)} + R_1 \qquad \text{with} \quad R_1 \in \mathbf{R}[Z, Z^{(1)}, \ldots, Z^{(r'-2)}];$$

we then have

$$S^{\sigma_0+\sigma_1}P = S^{\sigma_1}Q_0 Q^{(r'-r)} + Q_1 Q^{(r'-r-1)} + R_1 \qquad \text{with} \quad R_1 \in \mathbf{R}[Z, Z^{(1)}, \ldots, Z^{(r'-2)}].$$

We repeat this process i times, $i \leq r' - r$, until

$$S^{\sigma_i}R_{i-1} = Q_i Q^{(i)} + R_i \qquad \text{with} \quad R_i \in \mathbf{R}[Z, Z^{(1)}, \ldots, Z^{(r)}].$$

Then, for $\sigma = \sum_{j=0}^{i} \sigma_j$, we have

$$S^{\sigma}P = U_0' Q^{(r'-r)} + U_1' Q^{(r'-r-1)} + \cdots + U_i' Q^{(r'-r-i)} + R_i \text{ with } R_i \in \mathbf{R}[Z, Z^{(1)}, \ldots, Z^{(r)}],$$

where the U' have obvious values. By case 2 again, we have

$$I^t S^{\sigma}P = U_0 Q^{(r'-r)} + \cdots + U_i Q^{(r'-r-i)} + P^* \text{ with either } P^* = 0 \text{ or } P^* \neq 0 \text{ and } \omega(P^*) < \omega(Q).$$

The proposition is thus proved. ■

We come back to the general polynomial algebra and specify what we call a zero of a system of d-polynomials. Let (Ξ) be the following system of d-polynomials of $\mathbf{R}\{(Z_i)_{i\in I}\}$:

$$P_1, P_2, \ldots, P_t, \tag{Ξ}$$

let $\mathbf{A}$ be a differential $\mathbf{R}$-algebra. A *zero* (or *solution*) of (Ξ) over $\mathbf{A}$ is a tuple $\zeta = (\zeta_i)_{i\in I}$ of elements ζ_i of $\mathbf{A}$ such that

$$P_1(\zeta) = 0, \quad P_2(\zeta) = 0, \quad \ldots, \quad P_t(\zeta) = 0.$$

We also say that (Ξ) *is satified by* ζ, and (Ξ) has a zero if there is a differential $\mathbf{R}$-algebra over which we can find some ζ which is a zero of (Ξ). We need to invoke systems of d-polynomials of type

$$P_1, P_2, \ldots, P_t; Q; \tag{Σ}$$

by definition, a zero of such a system is an element $\zeta = (\zeta_i)_{i\in I}$ over a differential $\mathbf{R}$-algebra such that

$$P_1(\zeta) = 0, \quad P_2(\zeta) = 0, \quad \ldots, \quad P_t(\zeta) = 0 \quad \text{and} \quad Q(\zeta) \neq 0.$$

If I is finite ($\mathbf{R}\{(Z_i)_{i\in I}\} = \mathbf{R}\{Z_1, Z_2, \ldots, Z_n\}$), $\mathbf{R}$ is a field $\mathbf{k}$, and if we consider only solutions over differential $\mathbf{k}$-algebras which are d-fields then, after Rabinowitch, we can use the famous trick that by adjoining the indeterminate Z_{n+1}, and setting

$$P_{t+1} = Z_{n+1}Q(Z_1, Z_2, \ldots, Z_n) - 1,$$

if $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ is a zero of (Σ'), then, for $\zeta_{n+1} = 1/Q(\zeta_1, \zeta_2, \ldots, \zeta_n)$, $(\zeta_1, \zeta_2, \ldots, \zeta_{n+1})$ is a zero of

$$P_1, P_2, \ldots, P_t, P_{t+1}. \tag{$\Xi\Sigma$}$$

Conversely, if $(\zeta_1, \zeta_2, \ldots, \zeta_{n+1})$ is a zero of $(\Xi\Sigma)$, of course $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ is a zero of (Σ); that is, we can *replace* (Σ) by a system of polynomials of type (Ξ). The point is that, to make the induction work in the elimination procedure below, we need systems of polynomials such as (Σ) instead of (Ξ). This is made clear in the next two sections.

4. On Elimination Theory

Let

$$P_i(w, \zeta) = 0 \tag{$\mathscr{E}$}$$

be a system of algebraic equations (without derivations and where $w = (w_1, w_2, \ldots, w_s)$ and $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)$) with coefficients in an algebraically closed field $\mathbf{k}$. To eliminate the ζ from the equations $(\mathscr{E})$ amounts to finding the equations of the projection $((w, \zeta) \to w)$ of the algebraic set $\mathbf{V}(\mathscr{E})$ (defined to be the subset of $\mathbf{k}^{s+n}$ of common zeros (w, ζ) of the P in $(\mathscr{E})$) onto $\mathbf{k}^n$. It is a basic well-known fact in algebraic geometry that the projection of an algebraic set is not algebraic in general (recall the hyperbola example in the introduction, and [M, Section 2.C], [S5, Section I.5.2], or any other book on that field). Elimination of the variable ζ results in a finite family of systems of algebraic equations and inequations in the w with coefficients in $\mathbf{k}$. Constructive algebraic methods of elimination to derive the equations of the projection of $\mathbf{V}(\mathscr{E})$ from those of $(\mathscr{E})$ (see the theory of resultant polynomials in [V4], [L], and [J1], and [T] and [S2] for algebraic elimination theory over real closed fields) are also well known (see [S3] for example). Since algebraic equations are particular cases of differential algebraic equations,

$$P_i(w, \zeta) = 0, \tag{E}$$

the latter phenomenon still takes place in the differential context. Moreover, the above theoretical justification can be extended to differential algebraic equations. For this the ground differential field $\mathbf{k}$ (which contains the coefficients of equations (E)) is replaced by another differential field $\mathscr{U}$ called its *universal differential field extension* (a complete definition of this object would be too long to give here; see [K] where existence of $\mathscr{U}$ is also proved) which plays a role analogous to the algebraically closed field $\mathbf{k}$ in the nondifferential case. The *differential algebraic set* in $\mathscr{U}^{s+n}$ associated to the differential algebraic equations (E) is analogous of the algebraic set in $\mathbf{k}^{s+n}$ associated to $(\mathscr{E})$. Examples of systems of equations of type (E) (even of type (S)), which are not nondifferential ones can easily be found; one is given later in the text. Algorithms extending to the differential context the known effective methods of elimination in the nondifferential case also exist. One is found in the pioneering work by Ritt. Seidenberg has revisited this problem of constructivity of elimination theory in his notable paper [S4] where he gives (among other things) a simple method valid for ordinary differential equations that we shall adapt to the field of control theory. It is clear from what precedes that an elimination algorithm treats systems of equations *and* inequations

$$\begin{cases} P_1(w,\zeta) = 0, \\ P_2(w,\zeta) = 0, \\ \quad\vdots \\ P_t(w,\zeta) = 0, \\ Q(w,\zeta) \neq 0 \end{cases} \tag{Σ}$$

instead of systems of merely equations such as (E). Departing from systems of equations of type (E), we therefore choose to add the trivial inequation $1 \neq 0$ to (E).

Before doing this we should notice the following. The notion of a differential algebraic set allows us to define, in an easy way, the control theory concept of *external behavior* of a system and, furthermore, the idea of *external equivalence* of representations of systems. Let a system be given by its equations (E); its *external behavior* is nothing but the image of $\mathbf{V}(\mathscr{E})$ through the projection mentioned above. A rather simple definition of external equivalence is the following: two representations of a system are said to be externally equivalent if their external behaviors are equal on any universal differential field extension $\mathscr{U}$ of the ground field $\mathbf{k}$. This definition can be made more subtle, but would need more sophisticated algebra (see [D2] for references).

From the classical result recalled above, we borrow the terminology (first suggested in [R1]) of a family of resultant systems for a system of d-polynomials with respect to some indeterminates.

Definition. Let

$$P_1, P_2, \ldots, P_t; Q \tag{Σ}$$

be a system of d-polynomials of $\mathbb{Z}\{W_1, W_2, \ldots, Z_1, Z_2, \ldots\}$ in *finitely* many indeterminates. A *finite* family of systems of d-polynomials

$$p_{j_1}, \ldots, p_{j_{k_j}}; q_j, \tag{R_j}$$

where the p are in $\mathbb{Z}\{W_1, W_2, \ldots\}$, is called a *family of resultant systems* for (Σ) in the Z if and only if, for any d-field $\mathbf{k}$ of characteristic zero, any *d*-extension field $\mathbf{K}$ of $\mathbf{k}$, and any values w in $\mathbf{k}$ of the W, a necessary and sufficient condition for the existence of ζ in a d-extension field of $\mathbf{K}$, such that (w, ζ) is a zero of (Σ), is that the w is a zero of (R_j) at least for one j.

There are some immediate properties related to this notion:

Remark 1. It is clear that for any system of d-polynomials (Σ), any d-polynomial $P \in \mathbb{Z}\{W_1, W_2, \ldots, Z_1, Z_2, \ldots\}$, a family of resultant systems for (Σ) is obtained by *joining* the respective families of resultant systems for

$$P_1, P_2, \ldots, P_t, P; Q, \tag{Σ'}$$

$$P_1, P_2, \ldots, P_t; P \cdot Q \tag{Σ''}$$

together; that is, if (R'_j) and (R''_j) are the respective families of resultant systems for (Σ') and (Σ''), then (R_j) is a family of resultant systems for (Σ), where (R_j) consists of all the systems of d-polynomials that occur in (R'_j) or in (R''_j). To describe the use of this property, we say (Σ) *splits into* (Σ') *and* (Σ'').

Remark 2. If the P_i's do not involve the Z, a family of resultant systems for (Σ) is obtained as follows: we regard Q as a d-polynomial of $\mathbf{A}\{Z_1, Z_2, \ldots\}$ where $\mathbf{A} = \mathbb{Z}\{W_1, W_2, \ldots\}$ and denote its coefficients by $\tilde{Q}_1, \tilde{Q}_2, \ldots$ (they are in $\mathbf{A}$ and finite in number); then the

$$P_1, P_2, \ldots, P_t; \tilde{Q}_j \tag{R_j}$$

form a family of resultant systems for (Σ) since we are in a field of characteristic zero.

5. The Elimination Procedure

We can now state and prove the following fundamental theorem of elimination theory.

Theorem. *Any system of differential polynomials*

$$P_1, P_2, \ldots, P_t; Q, \tag{Σ}$$

with $P_1, P_2, \ldots, P_t, Q \in \mathbb{Z}\{W_1, W_2, \ldots, Z_1, Z_2, \ldots, Z_n\}$ *(the* W *are finite in number) possesses a family of resultant systems in the* Z:

$$p_{j_1}, \ldots, p_{j_{t_j}}; q_j \tag{R_j}$$

which we can compute within a finite number of steps, using only the coefficients of the P *and* Q *and d-field operations* $(+, \cdot,$ *and derivation).*

We have followed [S4], making slight changes, in the proof below.

Proof. We first show by induction on n that it suffices to have a procedure for eliminating one variable. Let us assume such a procedure to be available. Regarding $Z_1, Z_2, \ldots, Z_{n-1}$ as indeterminate coefficients, i.e., as additional W, we then have a family of resultant systems in Z_n

$$p_{j_1}, \ldots, p_{j_{t_j}}; q_j, \tag{R_j}$$

where $p_{j_1}, \ldots, p_{j_{t_j}}, q_j \in \mathbb{Z}\{W_1, W_2, \ldots, Z_1, Z_2, \ldots, Z_{n-1}\}$. By the induction hypothesis, there is, for each (R_j), a family of resultant systems of $(\mathrm{R}_{j,1})$ in $Z_1, Z_2, \ldots, Z_{n-1}$; it is then clear that the $(\mathrm{R}_{j,1})$ are a family of resultant systems of (Σ) in $Z_1, Z_2, \ldots, Z_n$. It now remains to prove the theorem for $n = 1$. We denote Z_1 by Z and $\mathbb{Z}\{W_1, W_2, \ldots, Z\}$ by $\mathbb{Z}\{W, Z\}$ where W stands for all the indeterminate coefficients.

Procedure 0: *Notation.* Renaming the polynomials, we assume $\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_\tau$ and $P_1, P_2, \ldots, P_t$ to be, respectively, the previous P_i's which do not involve Z and those which involve Z, with possibly $t = 0$, i.e., the case where no P_i involves Z. The system of d-polynomials (Σ) in the theorem then becomes

$$\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_\tau, P_1, P_2, \ldots, P_t; Q. \tag{Σ}$$

We also assume P_1 to be of the least ω, that is, $\omega(P_1) \leq \omega(P_i)$ for all i. We set

$$\omega(\Sigma) = \omega(P_1) = (r, d) \quad \text{and} \quad \pi(\Sigma) = (r, d, t).$$

I and S also stand for the initial and the separant of P_1, respectively. We make an induction on the triplet $\pi(\Sigma)$ (which is permitted since $\mathbb{N}^3$ equipped with the lexicographical order is a well-ordered set). We continue through procedures 1, 2, or 3 according to the respective cases $t = 0$, $t \geq 2$, or $t = 1$.

Procedure 1: $t = 0$. The procedure for eliminating Z ends here. For it then suffices to take the (R_j) to be

$$\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_\tau; \tilde{Q}_j, \tag{R_j}$$

where the $\tilde{Q}$ are the elements of $\mathbb{Z}\{W\}$ which occur as coefficients of Q when we write Q as an element of $\mathbb{Z}\{W\}\ \{Z\}$ (see Remark 2).

Procedure 2: $t \geq 2$. According to Remark 1, (Σ) splits into the following four systems of d-polynomials:

$$\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_\tau, I, S, P_1, P_2, \ldots, P_t; Q, \tag{Σ'}$$

$$\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_\tau, I, P_1, P_2, \ldots, P_t; S \cdot Q, \tag{Σ''}$$

$$\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_\tau, S, P_1, P_2, \ldots, P_t; I \cdot Q, \tag{Σ'''}$$

$$\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_\tau, P_1, P_2, \ldots, P_t; I \cdot S \cdot Q. \tag{Σ''''}$$

We recall that the polynomial S cannot vanish (identically) since we have assumed characteristic zero. For (Σ'), (Σ''), and (Σ''') the index function π has been decreased (lexicographically) since $\omega(I)$ and $\omega(S)$ are both less than (r, d); therefore we go through procedure 0. At last, let (Σ_a'''') be

$$\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_\tau, P_1, P_2^*, \ldots, P_t^*; I \cdot S \cdot Q, \tag{Σ_a''''}$$

where the P^* are the remainders of the divison of each P_i, $i \geq 2$, respectively by P_1 according to the polynomial division algorithm previously presented. Since (Σ'''') and (Σ_a'''') have the same families of resultant systems, we may replace (Σ'''') by (Σ_a''''). We have for all i, $i \geq 2$, either $P_i^* = 0$ or $\omega(P_i^*) < \omega(P_1)$; therefore we again return to procedure 0.

Procedure 3: $t = 1$. In this case, (Σ) splits into

$$\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_\tau, I, P_1\text{-}I \cdot Z^{(r)^d}; Q \tag{Σ'}$$

and

$$\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_\tau, P_1; I \cdot Q. \tag{Σ''}$$

We dispose of (Σ') by induction since, anyway, $\omega(\Sigma')$ has been decreased. For (Σ''), we distinguish three cases:

(a) If Q is of order strictly less than P_1, then we can replace (Σ'') by

$$\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_\tau; I \cdot Q. \qquad (\Sigma''_a)$$

For, if (w, ζ) is a zero of (Σ'') over some d-field $\mathbf{L}$ (of characteristic zero), then it is clear that (w, ζ) is a zero of (Σ''_a). Conversely, let (w, ζ) be a zero of (Σ''_a) over $\mathbf{L}$. As an element of $\mathbf{L}(Z, Z^{(1)}, \ldots, Z^{(r-1)})\,[Z^{(r)}]$ (with coefficients in $\mathbf{L}$ depending on w; and where $Z, Z^{(1)}, \ldots, Z^{(r-1)}$ are (nondifferential) indeterminates), $P_1(Z^{(r)})$ has an irreducible factor which we still denote by the same symbol $P_1(Z^{(r)})$ and which has a zero $\zeta_{1,r}$ in some field extension of the field $\mathbf{L}(Z, Z^{(1)}, \ldots, Z^{(r-1)})$. There is one and only one derivation in $\mathbf{L}' = \mathbf{L}(Z, Z^{(1)}, \ldots, Z^{(r-1)}, \zeta_{1,r})$ which extends the one in $\mathbf{L}$ and such that $Z^{(1)}$ is the derivation of Z, for each i, $2 \le i \le r-1$, $Z^{(i)}$ is the derivation of $Z^{(i-1)}$, and $\zeta_{1,r}$ is the derivation of $Z^{(r-1)}$ (for more details see [S1]). We then can set

$$\zeta_1 = Z,$$

so that the iterated derivatives of ζ_1 in the differential field $\mathbf{L}'$ thus obtained are

$$\dot{\zeta}_1 = Z^{(1)}, \quad \ldots, \quad \zeta_1^{(r-1)} = Z^{(r-1)}, \qquad \zeta_1^{(r)} = \zeta_{1,r}, \quad \zeta_1^{(r+1)} = \dot{\zeta}_{1,r}, \quad \zeta_1^{(r+2)} = \ddot{\zeta}_{1,r}, \quad \ldots.$$

Therefore

$$P_1(w, \zeta_1) = 0,$$

that is, (w, ζ_1) is a zero of (Σ'') since

$$I(Z, Z^{(1)}, \ldots, Z^{(r-1)})Q(Z, Z^{(1)}, \ldots, Z^{(r-1)}) \neq 0.$$

(b) Except for the trivial case in which the d-polynomials $\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_\tau$ cannot have a common zero, (Σ'') has a zero if and only if P_1 contains a factor not occurring in Q(P_1 and Q being considered as elements of the (nondifferential) polynomial algebra $\mathbb{Z}\{W_1, W_2, \ldots\}\,[Z, Z^{(1)}, \ldots, Z^{(r-1)}]$); in other words, (Σ'') has a zero if and only if P_1 does not divide (again in the nondifferential polynomial algebra $\mathbb{Z}\{W_1, W_2, \ldots\}\,[Z, Z^{(1)}, \ldots, Z^{(r-1)}]$) Q^d. Denoting by Q^* the remainder of the division of Q^d by P_1, (Σ'') can then be replaced by

$$\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_\tau; I \cdot Q^*. \qquad (\Sigma''_b)$$

(c) If Q is of order strictly greater than P_1, then again by the polynomial division algorithm above we can replace Q by Q' in (Σ'') and go back to one of the cases (a) or (b) above. Q' is the remainder of the division of Q by P_1; it satisfies $\omega(Q') < \omega(P)$. Thus the theorem is completely proved. ■

6. Examples

Here, we give some examples as illustrations of the procedure. The calculations have been made by a computer algebra system program; details can be found in [D2].

Example 1. The following linear example is borrowed from Example 4.2, Chapter

2, of [R3]; ζ_1, ζ_2 are the variables to be eliminated:

$$\begin{cases} \dot{\zeta}_1 + \zeta_1 + \dddot{\zeta}_2 + 2\ddot{\zeta}_2 = \ddot{u} + u, \\ \ddot{\zeta}_1 + 3\dot{\zeta}_1 + 2\zeta_1 + \zeta_2^{(4)} + 4\dddot{\zeta}_2 + 4\ddot{\zeta}_2 + \dot{\zeta}_2 + 2\zeta_2 = \dddot{u} + 2\ddot{u} + \dot{u} + 3u, \\ y = -\ddot{\zeta}_1 - 3\dot{\zeta}_1 - \zeta_1 - \zeta_2^{(4)} - 4\dddot{\zeta}_2 - 4\ddot{\zeta}_2 + \zeta_2 + \dddot{u} + 2\ddot{u} + \dot{u} + 2u. \end{cases}$$

Here is the program output, which is an externally equivalent representation of the previous system:

$$\ddot{y} + 3\dot{y} + 2y = 2\dot{u} + 3u.$$

Example 2. We show in this example the existence of state representation from which, if the state variable is eliminated, the result is a system of equations *and* inequations. We consider the following system of equations:

$$\begin{cases} \dot{\zeta}_1 = \zeta_1\zeta_2, \\ \dot{\zeta}_2 = \zeta_2, \\ y = \zeta_2^2 + \zeta_2. \end{cases}$$

An externally equivalent representation of the previous system which no longer contains either ζ is the following:

$$\begin{cases} (\dot{y} - 2y)(\dot{y} - 2y - 1) - y = 0, \\ \qquad\qquad 2\dot{y} - 4y - 1 \neq 0. \end{cases}$$

Example 3. The following example illustrates a *cancellation* problem which remains to be solved (if possible):

$$\begin{cases} \dot{\zeta} = u\zeta^2 + u^2\zeta, \\ y = \zeta^2. \end{cases}$$

Elimination of ζ leads to a family of three resultant systems whose disjunction describes the external behavior of the initial system:

Resultant system 1:

$$y = 0.$$

Resultant system 2:

$$\begin{cases} \dot{y} = -2u^2y, \\ -2uy = 0, \\ 4y \neq 0. \end{cases}$$

Resultant system 3:

$$\begin{cases} 4u^4y^2 - 4u^2y\dot{y} - 4u^2y^3 + \dot{y}^2 = 0, \\ \qquad\qquad 8u^2y^2(2u^2y - \dot{y}) \neq 0. \end{cases}$$

Notice that there is some redundancy in the result delivered by the procedure: the disjunction of the three resultant systems is equivalent to the unique following equation:

$$(\dot{y} - 2u^2y)^2 - 4u^2y^3 = 0.$$

To solve this problem we have to find a method of cancellation for the redundant systems that appear in the calculations. This is a very difficult problem not examined here.

7. Conclusion

The long-standing control theory problem of elimination has been examined using techniques from differential algebraic elimination theory. Seidenberg's algorithm is suggested as a mechanical means for deriving externally equivalent representations of a system defined by differential algebraic equations with coefficients in a differential field of characteristic zero (recall that this conclusion may be extended to certain equations involving transcendental elementary functions, following work by Wu *et al.*). We could also have used Ritt's algorithm [S4].

Three questions are brought to mind: one concerns the uniqueness of the family of resultant systems, the second concerns the number of systems of equations and inequations which form the family, and the third relates to the necessary and sufficient conditions for the equations (E) to possess a family of resultant systems consisting of one and only one system of equations (without an inequation). To the first question, the answer is clear: neither equations of an arbitrary system nor the family of resultant systems are in any way unique. Given equations (E), by the elimination procedure we get a family of resultant systems (R_j). It is clear that, for any j, if we add to (or remove from) the corresponding (R_j) d-polynomials which are a linear combination of the P_i (in (R_j)) in the ambient differential polynomial algebra, we then obtain a new family of resultant systems for the *same* system. The answer to the second question is not provided by the procedure (see the last example). Nevertheless, it seems that we can without difficulty rewrite the procedure in order to get rid of many (but perhaps not all) possible redundant equations; this would simply be too long. What is clear is that we obtain enough systems of equations and inequations to describe exactly the external behavior of (E); this is by definition of the family of resultant systems. The third question is a difficult one. In other words, the question is to describe the class of d-algebraic sets (those defined by equations of type (E)) of affine spaces $\mathbf{k}^{s+n}$ whose projections onto $\mathbf{k}^s$ are d-algebraic sets, $\mathbf{k}$ being an arbitrary d-field.

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