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Worst-Case Design in the Time Domain: The Maximum Principle and the Standard H_∞ Problem*

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Abstract. This paper presents a time-domain, optimal-control approach to worst-case design, an alternative to frequency-domain H_∞ techniques. The generic linear-quadratic set-up of the "standard H_∞ problem" is discussed. The results include a characterization of suboptimal values, as well as a parametrization of all suboptimal compensators, in terms of two coupled indefinite Riccati equations. Both the usual infinite-horizon, time-invariant case and the finite-horizon, time-varying case, are treated. The latter is beyond the scope of frequency-domain analysis.

Key words. H_∞ control, Min-max optimization, Maximum principle analysis.

1. Introduction

One significant research thrust in systems and control during the past decade has been the study of worst-case design problems within "H_∞ control theory," a frequency-domain methodology which has allowed application of deep complex-function and operator-theoretical results. (See the surveys [FD] and [H] and the textbook [F].) Our goal is to present some results that suggest an alternative time-domain approach to worst-case design, following the lines of classical optimal-control theory. Advantages of this approach seem to include mathematical simplicity, as well as an expansion of the scope of worst-case design.

We focus on the "standard H_∞ problem" which sets a generic framework for linear-quadratic (LQ) worst-case design. Central issues, such as robust stabilization, model matching, and tracking, can easily be transformed into a "standard problem" form [F]; it is also a natural worst-case counterpart of the stochastic LQG optimization problem. Our results include a characterization of suboptimal values and a parametrization of suboptimal compensators in terms of two indefinite Riccati equations. In particular, low-order compensators are identified.

Detailed description of the "standard problem," the results, and references to related works are deferred to the following section. We use the remainder of this introduction for a brief discussion of some underlying ideas in our approach.

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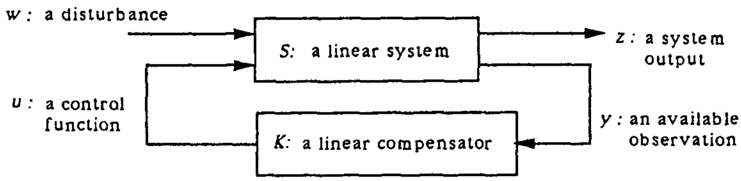

Fig. 1

Our notations are fairly standard. The space of square integrable functions $f: X \to Y$ is denoted by $L_2(X, Y)$. Notations of X and/or Y are suppressed when obvious (which will mostly be the case). L_2 norms and induced operator norms are denoted by $\|\cdot\|$. When $X = [t_0, t_1]$ then $\|\cdot\|_t$, $t_0 \le t < t_1$, are the norms in $L_2[t, t_1]$. The Euclidean norm in $\mathbb{R}^\alpha$, for an integer α, is denoted by $|\cdot|$, and $\langle\cdot, \cdot\rangle$ stands for both the inner product in $\mathbb{R}^\alpha$ and in L_2. The transpose of a matrix, as well as the adjoint of an operator, are denoted by a prime, e.g., A'. Given a positive-definite matrix M, we denote $|\cdot|_M$, $\|\cdot\|_M = \langle\cdot, M\cdot\rangle^{1/2}$.

Figure 1 provides a pictorial description of the "standard problem" set-up. Given S, the designer's goal is to minimize the closed-loop impact of disturbances (w) on the output (z) by an appropriate choice of the compensator, K. That impact is measured by the induced operator norm relative to L_2 signal norms. The design constraint is closed-loop internal stability. This can be summarized as an effort to attain the *optimal value* γ_0 in

$$\gamma_0 = \min_{\text{stabilizing } K} \max_{w \in L_2} \frac{\|z\|}{\|w\|}. \tag{1.1}$$

(In order to make the mapping $w \to z$ linear, and not merely affine, definition (1.1) is made with the technical assumption that the initial internal state in S is zero.)

H_∞ theory uses a frequency domain statement of the problem: stability means analyticity of associated transfer functions over the right half of the complex plane, and the induced operator norm of the mapping $w \to z$ is equal to the corresponding transfer function's H_∞ norm. The problem is thus analyzed by use of function-theoretic, and related algebraic and operator-theoretic, tools.

There is another view which leads to a different technical approach. The "standard problem" features a competition between disturbances and controls on the output norm. The following definitions and observation help in casting that competition in rigorous terms. Given $\gamma \in \mathbb{R}$ denote

$$J_\gamma(x_0, w, u) = \gamma^2\|w\|^2 - \|z\|^2,$$

where x_0 is the initial internal state in S and z is the output trajectory that corresponds to x_0, the disturbance w, and the control u. When defined, J_γ is an indefinite quadratic form in its three variables. If $x_0 = 0$ and $u = Ky$ is a closed-loop control, J_γ becomes a quadratic form in w alone. This next observation simply rewrites the definition in (1.1).

Observation 0. *$\gamma > \gamma_0$ if and only if there exists some internally stabilizing compensator K such that $J_\gamma(0, w, u = Ky)$ becomes a uniformly positive definite form in w, that* *is, such that*

$$J_\gamma(0, w, u = Ky) \geq \delta^2 \|w\|^2 \tag{1.2}$$

for some fixed $\delta \neq 0$ *and all* $w \in L_2$.

"Good controls" are therefore those that increase J_γ, whereas "bad disturbances" decrease its value. We derive most of our results as consequences of a detailed study of the MinMax problem

$$\underset{w \in L_2}{\text{Min}} \; \underset{u \in L_2}{\text{Max}} \; J_\gamma(x_0, w, u). \tag{1.3}$$

A natural framework for investigating (1.3) is that of time-domain, optimal-control theory. Indeed, the present discussion follows, step-by-step, classical LQ optimal-control analysis. It thus relies on inherently "dynamic" (rather than "algebraic") tools such as Pontryagin's maximum principle, the principle of dynamic programming, and some straightforward Lyapunov stability analysis.

Given the wide scope of optimal control theory, we believe that the approach suggested here may be instrumental in considerably expanding worst-case design research, well beyond the LQ, time-invariant, infinite-horizon confines of current H_∞ theory. As a first step, we show here how the LQ, finite-horizon, time-varying counterpart of the "standard problem" can easily be handled. This natural problem has been studied extensively in classical optimal control literature (see [KS], [LM], and [KFA]); it is just as relevant in worst-case design (see, e.g., [NJM]). Being outside the scope of frequency-domain analysis, this case has by and large been ignored in H_∞ research.

The paper is organized as follows. Following the description of the problem and main results for the time-invariant, infinite-horizon case, in Section 2, the complete proof is given in Section 3, which is the main part of the presentation. The extension of results and proofs for the time-varying, finite-horizon case are discussed in Section 4. Our developments are carried under certain simplifying assumptions. In Section 5 we briefly remark on the general case, where these assumptions may fail. To conclude, in Section 6, we recap some of the main ideas and techniques.

2. The Infinite-Horizon, Time-Invariant Standard Problem and Main Results

The system S in Fig. 1 is assumed to be governed by the following equations:

$$\begin{aligned} \dot{x} &= Ax + B_1 u + B_2 w, \\ y &= C_1 x + C_2 w, \\ z &= D_1 x + D_2 u. \end{aligned} \tag{2.1}$$

Here $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^l$, $y \in \mathbb{R}^k$, $z \in \mathbb{R}^j$ are the system's state, control, disturbance, observation, and output signals, respectively. The coefficients A, B_i, C_i, D_i, are matrices of appropriate dimensions. For the moment we assume that the system is defined over the positive time ray $[0, \infty)$, and that it is time-invariant.

Admissible feedback operators are those which can be realized as input–output mappings of linear systems:

$$\dot{p} = Mp + Ng, \qquad p(0) = 0, \qquad\qquad (2.2)$$
$$u = Qp + Ry.$$

An admissible feedback operator is *internally stabilizing* if it can be realized by a system of the form (2.2), so that the matrix

$$\mathscr{A} = \begin{bmatrix} A + B_1RC_1 & B_1Q \\ NC_1 & M \end{bmatrix}$$

is stable. Such a realization provides an *internally stabilizing feedback compensator* for (2.1). (Stability of $\mathscr{A}$ means that both the internal states x and p in S and K, and the output functions z, y, and u, react in an L_2-bounded fashion to disturbances introduced at any point in Fig. 1, as wall as to state perturbations.)

Given an internally stabilizing feedback compensator K we denote by T_K the closed-loop mapping $T_K: L_2 \to L_2$, $w \mapsto z$. Following from (1.1), a value γ is *strictly suboptimal* ($\gamma > \gamma_0$) if there exists such K that assures $\|T_K\| < \gamma$. (Here $\|\cdot\|$ is the induced operator norm. The operator T_K becomes linear when $x(0) = 0$.

The *standard problem* is this: find the optimal value γ_0 and internally stabilizing compensators K, such that $\|T_K\| = \gamma_0$. It is usually treated in a relaxed version: characterize all suboptimal values γ; given such γ, describe all internally stabilizing compensators K for which $\|T_K\| < \gamma$. This sets the framework to the present discussion.

The reader should notice the resemblance of our problem to the stochastic LQG problem: $\|z\|^2$ is a quadratic cost on the trajectory and the control. If the statistical distribution of distrubances is assumed known, the LQG framework suggests minimizing the average of $\|z\|^2$. In the present framework, however, this knowledge is not assumed. Instead, our goal is to reduce the worst value of $\|z\|^2$, say, for all unit energy disturbances. (Due to linearity, the Max in (1.1) can be taken over $\{w \in L_2: \|w\| = 1\}$.) The standard problem is therefore a natural worst-case counterpart of the LQG problem, as mentioned in the introduction.

We work under some technical assumptions that simplify the discussion considerably. These assumptions are counterparts of standard separation and nonsingularity hypotheses in the LQG framework. They can be made without any loss of generality, and in Section 5 we (briefly) explain how to treat the general case.

Assumption A

(i) $D_2'[D_1, D_2] = [0, I]$,
(ii) $C_2[B_2', C_2'] = [0, I]$,
(iii) $D_1'D_1 \geq \varepsilon_1 I$ *for some* $\varepsilon_1 > 0$,
(iv) $B_2B_2' \geq \varepsilon_2 I$ *for some* $\varepsilon_2 > 0$.

This next theorem summarizes our results, pertinent to the time-invariant, infinite-horizon problem, as stated above.

Theorem I

(a) *The value $\gamma > 0$ is strictly suboptimal in (2.1) if and only if there exist negative definite matrices P_2 and P_2 which satisfy*

$$P_1 A + A' P_1 + P_1\left(B_1 B_1' - \frac{1}{\gamma^2} B_2 B_2'\right) P_1 = D_1' D_1 \tag{2.3}$$

and

$$P_2\left(A - \frac{1}{\gamma^2} B_2 B_2' P_1\right)' + \left(A - \frac{1}{\gamma^2} B_2 B_2' P_1\right) P_2 + P_2\left(C_1' C^1 - \frac{1}{\gamma^2} P_1 B_1 B_1' P_1\right) P_2 = B_2 B_2' \tag{2.4}$$

and such that the matrices

$$A_1 = A + \left(B_1 B_1' - \frac{1}{\gamma^2} B_2 B_2'\right) P_1$$

and

$$A_2 = A - \frac{1}{\gamma^2} B_2 B_2' P_1 + P_2\left(C_1' C_1 - \frac{1}{\gamma^2} P_1 B_1 B_1' P_1\right)$$

are stable.

(b) *K is an internally stabilizing compensator which assures the norm bound $\|T_K\| < \gamma$ if and only if it can be realized as*

$$\begin{aligned} \dot{p} &= (A_1 + P_2 C_1' C_1) p - \left(I + \frac{1}{\gamma^2} P_2 P_1\right) B_1 v + P_2 C_1' y, \\ q &= C_1 p + y, \qquad v = K_0 q, \\ u &= -B_1' P_1 p + v, \end{aligned} \tag{2.5}$$

where K_0 is an admissible operator realized by a stable system, and such that $\|K_0\| < \gamma$.

(c) *Assume that direct state observation is available. That is, substitute Assumptions* A(i) *and* A(iv) *by "$C_1 = I$ and $C_2 = 0$." Then:* (i) *γ is strictly suboptimal if and only if there exists a negative-definite solution to* (2.3) *and A_1 is stable, and* (ii) *If γ is strictly suboptimal, then the state-feedback $u = B_1' P_1 x$ is internally stabilizing and assures $\|T_K\| < \gamma$.*

The proof is given in Section 3.

The reader may note that when the choice $K_0 = 0$ is made in the parametrization (2.5), the resulting compensator is in Luenberger form. That is, the compensator is an "optimal state-feedback" applied to an "optimal state-estimator." Indeed, following from the proof it will be very easy to establish that $u = B_1' P_1 x$ is, in a sense, an optimal state-feedback control, and that $-p$ is an optimal estimate of x. This substantiates the (intuitive) relation of the "standard problem" to LQG optimization.

With that choice ($K_0 = 0$) the degree of K reaches a generic minimum, that is $\deg(K) = n$. It has recently been of great interest [LH], [VR] to find nth-order suboptimal compensators, and our parametrization appears to do the job. Moreover, an exceptionally good zeroth-order compensator is suggested in part (c): when the state x is available, the estimator part of K becomes redundant. Hence we are left with the "optimal state-feedback" $u = B_1'P_1x$.

A few words about related work are in order. The relation of H_∞ optimization to algebraic Riccati equations is studied in a series of papers by Khargonekar, Petersen, Rotea, and Zhou [P1], [KPR], [KPZ], [ZK]. I was particularly motivated by enlightening comments in [KPZ] that suggest a game-theoretic interpretation of H_∞ optimization. Part (c) of Theorem I originally appears in [KPZ]. Ball and Cohen characterized suboptimal values, and parametrized all suboptimal compensators in terms of the two algebraic Riccati equations, in [BC]. This line of research was later pursued by Verma and Romig [VR].

During the preparation of this paper, a revised version of [T], the work of Glover and Doyle [GD], and its extension, jointly with Khargonekar and Francis, became available in part, in the form of the conference note [DGKF]. I was encouraged and inspired by that report. Indeed, the present results are closely related to those of [DGKF]. (The reader may note that the second Riccati equation in [DGKF] is different from ours: in our notation it is

$$\tilde{P}_2A' + A\tilde{P}_2 + \tilde{P}_2\left(C_1'C_1 - \frac{1}{\gamma^2}D_1'D_1\right)\tilde{P}_2 = B_1B_2'. \tag{2.4^0}$$

Unlike our (2.4), equation (2.4)^0 is independent of (2.3). Yet the relation between the two sets is simple, namely

$$\tilde{P}_2 = \left(\frac{1}{\gamma^2}P_1 + P_2^{-1}\right)^{-1}$$

and the obvious coupling between (2.3) and (2.4) is substituted in [DGKF] by the condition $\tilde{P}_2^{-1} > (1/\gamma^2)\tilde{P}_1$.)

A very different, Hankel-extension approach has been taken by Limebeer *et al.* [LKS].

Finally, let us mention that our results are reminiscent of long-established game-theoretic observations [M1], [M2], [RL]. This fact stands in support of the game-theoretic, optimal-control approach taken in the present discussion.

3. Proof of Theorem I

3.1. *First Half of Necessity in Part* (*a*)

We start with the implication "if $\gamma > \gamma_0$, then there exists $P_1 > 0$, a solution to (2.3), such that A_1 is stable." Our starting point is Observation 0, above, and the MinMax problem (1.3).

Assume $\gamma > \gamma_0$. Fix $x(0) = x_0 \in \mathbb{R}^n$ and $w \in L_2$, and consider the following LQ optimal-control problem:

$$\operatorname*{Max}_{u \in L_2} J_\gamma(x_0, w, u) = \gamma^2\|w\|^2 - \operatorname*{Min}_{u \in L_2} \|z\|^2. \tag{3.1}$$

By Assumptions A(i) and A(iii), $\|z\|^2 = \|u\|^2 + \|D_1x\|^2$ is a nonsingular quadratic cost on the control and the state of S. Since $\gamma > \gamma_0$, there exists some closed-loop stabilizing control. In particular, our system is stabilizable.

Classical LQ optimal-control theory therefore tells us the following (see, e.g., Chapter 3 of [LM]): there exists a negative-definite matrix L, that satisfies the following algebraic Riccati equation:

$$LA + A'L + LB_1B_1'L = D_1'D_1, \tag{3.2}$$

such that $A_L = A + B_1B_1'L$ is a stable matrix. Let

$$r(t) = \int_t^{\infty} e^{A_L'(\tau-t)}LB_2w(\tau)\,d\tau, \tag{3.3}$$

let x be the solution of

$$\dot{x} = A_Lx + B_1B_1'r + B_2w, \qquad x(0) = x_0, \tag{3.4}$$

and set

$$\eta = Lx + r. \tag{3.5}$$

Then η satisfies the adjoint Hamilton–Jacobi equation

$$\dot{\eta} = D_1'D_1x - A'\eta \tag{3.6}$$

and $x(t)$, $r(t)$, and $\eta(t) \to 0$ as $t \to \infty$. It thus follows from Pontryagin's maximum principle that the unique optimal control in (3.1) is $u = B_1'\eta$, and that x is the associated optimal trajectory.

Throughout this part of the proof it is assumed that u, x, and z are the optimal control, state, and output trajectories associated with (x_0, w): We denote $(x, u, \eta) = \mathscr{F}(x_0, w)$ and $z = \mathscr{G}(x_0, w)$, and make note of the following direct consequence of the classical results.

Proposition 1. *$\mathscr{F}: \mathbb{R}^n \times L_2 \to L_2 \times L_2 \times L_2$ and $\mathscr{G}: \mathbb{R}^n \times L_2 \to L_2$ are continuous linear operators.*

We now try to solve the "Min" part in (1.3). Denote

$$J_\gamma^0(x_0, w) = \gamma^2\|w\|^2 - \|\mathscr{G}(x_0, w)\|^2,$$
$$J_\gamma^*(x_0) = \operatorname*{Inf}_{w \in L_2} J_\gamma^0(x_0, w) = \operatorname*{Inf}_{w \in L_2}\operatorname*{Max}_{u \in L_2} J_\gamma(x_0\ w, u),$$

and

$$|||w||| = J_\gamma^0(0, w)^{1/2}.$$

The following are key observations

Proposition 2. *$|||\cdot|||$ defines a norm on L_2, that is equivalent to the usual L_2 norm.*

Proof. By Proposition 1, $|||w|||^2$ is a quadratic form is w. By Observation 0, there exist an internally stabilizing K and $\delta \neq 0$, such that

$$J_\gamma(0, w, u = Ky) \geq \delta^2\|w\|^2. \tag{3.7}$$

Following from the definition of $\mathscr{G}$ in (3.1), therefore

$$\gamma^2 \|w\|^2 \geq |||w|||^2 \geq J_\gamma(0, w, u = Ky) \geq \delta^2 \|w\|^2. \qquad \blacksquare$$

Proposition 3. *For each $x_0 \in \mathbb{R}^n$ there corresponds a unique disturbance, $w^* \in L_2$, such that $J_\gamma^0(x_0, w^*) = J_\gamma^*(x_0)$. Moreover, $J_\gamma^0(x_0) \in (-\infty, 0]$ and $J_\gamma^*(x_0) = 0$ if and only if $x_0 = 0$.*

Proof.

$$\begin{aligned} J_\gamma^0(x_0, w) &= \gamma^2 \|w\|^2 - \|\mathscr{G}(x_0, w)\|^2 \\ &\geq \gamma^2 \|w\|^2 - (\|\mathscr{G}(x_0, 0)\| + \|\mathscr{G}(0, w)\|)^2 \\ &\geq \gamma^2 \|w\|^2 - (\|\mathscr{G}(x_0, 0)\| + (\gamma^2 - \delta^2)^{1/2} \|w\|)^2 \\ &= \gamma^2 \|w\|^2 (1 - (\|\mathscr{G}(x_0, 0)\|/\gamma\|w\| + (1 - \delta^2/\gamma^2)^{1/2})^2). \end{aligned} \tag{3.8}$$

The rightmost term in (3.8) becomes positive for large $\|w\|$, hence $J_\gamma^*(x_0) > -\infty$. Take $w = 0$, then $J_\gamma^0(x_0, 0) = -\|\mathscr{G}(x_0, 0)\|^2 \leq 0$, and equality holds only if $x_0 = 0$. So $J_\gamma^*(x_0) \leq 0$, and <0 if $x_0 \neq 0$. If $x_0 = 0$, then $J_\gamma^*(x_0) = \inf_w |||w|||^2 = 0$.

Now, assume $\{w_\alpha\} \subset L_2$ is a minimizing sequence. Then

$$\begin{aligned} |||w_\alpha - w_\beta|||^2 &= \gamma^2 \|w_\alpha - w_\beta\|^2 - \|\mathscr{G}(0, w_\alpha - w_\beta)\|^2 \\ &= \gamma^2 \|w_\alpha - w_\beta\|^2 - \|\mathscr{G}(x_0, w_\alpha) - \mathscr{G}(x_0, w_\beta)\|^2 \\ &= \gamma^2 \|w_\alpha\|^2 - \|\mathscr{G}(x_0, w_\alpha)\|^2 + \gamma^2 \|w_\beta\|^2 - \|\mathscr{G}(x_0, w_\beta)\|^2 \\ &\quad - 2(\gamma^2 \langle w_\alpha, w_\beta \rangle - \langle \mathscr{G}(x_0, w_\alpha), \mathscr{G}(x_0, w_\beta) \rangle) \\ &= J_\alpha^0(x_0, w_\alpha) + J_\gamma^0(x_0, w_\beta) \\ &\quad - 2(\gamma^2 \langle w_\alpha, w_\beta \rangle - \langle \mathscr{G}(x_0, w_\alpha), \mathscr{G}(x_0, w_\beta) \rangle). \end{aligned} \tag{3.9}$$

Also

$$\begin{aligned} J_\gamma^0(x_0, \tfrac{1}{2}(w_\alpha + w_\beta)) &= \tfrac{1}{4}\gamma^2 \|w_\alpha + w_\beta\|^2 - \|\mathscr{G}(x_0, \tfrac{1}{2}(w_\alpha + w_\beta))\|^2 \\ &= \tfrac{1}{4}(\gamma^2 \|w_\alpha + w_\beta\|^2 - \|\mathscr{G}(x_0, w_\alpha) + \mathscr{G}(x_0, w_\beta)\|^2) \\ &= \tfrac{1}{4}(\gamma^2 \|w_\alpha\|^2 - \|\mathscr{G}(x_0, w_\alpha)\|^2 + \gamma^2 \|w_\beta\|^2 - \|\mathscr{G}(x_0, w_\beta)\|^2 \\ &\qquad + 2(\gamma^2 \langle w_\alpha, w_\beta \rangle - \langle \mathscr{G}(x_0, w_\alpha), \mathscr{G}(x_0, w_\beta) \rangle)) \\ &= \tfrac{1}{4}(J_\gamma^0(x_0, w_\alpha) + J_\gamma^0(x_0, w_\beta) \\ &\qquad + 2(\gamma^2 \langle w_\alpha, w_\beta \rangle - \langle \mathscr{G}(w_0, w_\alpha), \mathscr{G}(x_0, w_\beta) \rangle)). \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we get

$$\begin{aligned} |||w_\alpha - w_\beta|||^2 &= 2(J_\gamma^0(x_0, w_\alpha) + J_\gamma^0(x_0, w_\beta) - 2J_\gamma^0(x_0, \tfrac{1}{2}(w_\alpha + w_\beta))) \\ &\leq 2(J_\gamma^0(x_0, w_\alpha) + J_\gamma^0(x_0, w_\beta) - 2J_\gamma^*(x_0)) \to 0. \end{aligned}$$

Therefore $\{w_\alpha\}$ is a Cauchy sequence in the $(L_2, |||\cdot|||)$ topology. By Proposition 2, it is a Cauchy sequence in the usual L_2 sense, and $w_\alpha \to w^*$ for some $w^* \in L_2$. Moreover, since any shuffling of minimizing sequences creates yet another minimizing sequence, our reasoning implies that w^* is unique.

Finally, the continuity of $\mathscr{G}$ implies

$$\begin{aligned} J_\gamma^0(x_0, w^*) &= \gamma^2 \|w^*\|^2 - \|\mathscr{G}(x_0, w^*)\|^2 \\ &= \lim_\alpha (\gamma^2 \|w_\alpha\|^2 - \|\mathscr{G}(x_0, w_\alpha)\|^2) \\ &= \lim_\alpha J_\gamma^0(x_0, w_\alpha) = J^*(x_0). \end{aligned}$$ ∎

The reader may note that our reasoning here mimics very similar arguments that have been used in establishing existence and uniqueness of LQ optimal controls (see, e.g., Chapter 3 of [LM]).

We define a mapping $\mathscr{H}$ by $w^* = \mathscr{H}x_0$. Then, given x_0 we denote $(x^*, u^*, \eta^*) = \mathscr{F}(x_0, w^*)$ and $z^* = \mathscr{G}(x_0, w^*)$. The pair (w^*, u^*) is the unique solution to the MinMax problem (1.3). We now prove a "maximum principle" characterization of w^*.

Proposition 4. *$w = w^*$ if and only if $w = -(1/\gamma^2)B_2'\eta$ where $(x, u, \eta) = \mathscr{F}(x_0, w)$.*

Proof. (*Only if.*) Denote $w^0 = -(1/\gamma^2)B_2'\eta^*$, $(x^0, u^0, \eta^0) = \mathscr{F}(x_0, w^0)$, and $z^0 = \mathscr{G}(x_0, w^0)$. Using the fact that $u^* = B_1'\eta^*$, we get

$$\gamma^2 |w^0(t)|^2 - |z^0(t)|^2 + 2\frac{d}{dt}\langle \eta^*(t), x^0(t)\rangle = -|(z^0 - z^*)(t)|^2 + |z^*(t)|^2 - \gamma^2 |w^0(t)|^2 \tag{3.11}$$

and

$$\gamma^2 |w^*(t)|^2 - |z^*(t)|^2 + 2\frac{d}{dt}\langle \eta^*(t), x^*(t)\rangle = \gamma^2 |(w^0 - w^*)(t)|^2 + |z^*(t)|^2 - \gamma^2 |w^0(t)|^2. \tag{3.12}$$

Subtract (3.11) from (3.12) and integrate over $[0, \infty)$; since $(x^* - x^0)(0) = 0$ and $\eta^*(t)$, $x^*(t)$ and $x^0(t) \to 0$ as $t \to \infty$, there holds

$$J_\gamma^*(x_0) = J_\gamma^0(x_0, w^*) = J_\gamma^0(x_0, w^0) + \gamma^0 \|w^0 - w^*\|^2 + \|z^0 - z^*\|^2.$$

The minimality of $J_\gamma^*(x_0)$ implies $w^* = w^0$.

The proof of the *if* part is straightforward. ∎

Corollary 5. *$\mathscr{H}\colon \mathbb{R}^n \to L_2$ is a linear (hence continuous) operator.*

Denote

$$P_1 x_0 = L x_0 + \int_0^\infty e^{A_L' t} L B_2 (\mathscr{H} x_0)(t)\, dt. \tag{3.13}$$

By the corollary, $P_1\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping and it admits a matrix representation.

Proposition 6. *Given* x_0, $w^* = \mathscr{H}x_0$ *and* $(x^*, u^*, \eta^*) = \mathscr{F}(x_0, \mathscr{H}x_0)$; *then* $\eta^* = P_1x^*$, $u^* = B_1'P_1x^*$ *and* $w^* = -(1/\gamma^2)B_2'P_1x^*$.

Proof. It follows from (3.3) and (3.5) that $\eta^*(0) = P_1x^*(0)$. As it was established that $u^* = B_1'\eta^*$ and $w^* = -(1/\gamma^2)B_2'\eta^*$, the proposition holds at $t = 0$.

Having arrived at $x^*(t)$, at some later time $t > 0$, the considerations may start afresh: define the problem (1.3) over (t, ∞) with the initial state $x^*(t)$, etc. Then the conclusion follows that the unique MinMax control and disturbance satisfy $u^*(t) = B_1'P_1x^*(t)$ and $w^*(t) = (1/\gamma^2)B_1'P_1x^*(t)$, for all $t \geq 0$.

It remains to show that $\eta^* = P_1x^*$ throughout. In the previous paragraph we have established

$$(\mathscr{H}x^*(t))(\tau) = -\frac{1}{\gamma^2}B_2'P_1x^*(t + \tau). \tag{3.14}$$

Substitute $x^*(t)$ for x_0 in (3.13); then (3.14) implies

$$\begin{aligned} P_1x^*(t) &= Lx^*(t) - \frac{1}{\gamma^2}\int_0^\infty e^{A_L'\tau}LB_2B_2'P_1x^*(t + \tau)\,d\tau \\ &= Lx^*(t) - \frac{1}{\gamma^2}\int_t^\infty e^{A_L'(\tau - t)}LB_2B_2'P_1x^*(\tau)\,d\tau. \end{aligned} \tag{3.15}$$

Denote $\eta = P_1x^*$. Then it follows from (3.15) that η is a solution to the adjoint Hamilton-Jacobi equation (3.6), with $\eta(0) = P_1x^*(0)$. Thus $\eta = \eta^*$ throughout. ∎

Corollary 7. *The matrix* $A_1 = A + (B_1B_1' - (1/\gamma^2)(B_1B_1')P_1$ *is stable.*

Proof. Following from the previous observation, $x^*(t)$ satisfies

$$\dot{x}^*(t) = A_1x^*(t). \tag{3.16}$$

Given any initial state x_0 in (3.16), $z^* = \mathscr{G}(x_0, \mathscr{H}x_0)$ is an L_2 function. By Assumption A(iii), $\|z^*\|^2 \geq \varepsilon_1\|x^*\|^2$. So solutions of (3.16) all belong to L_2. That is, A_1 is stable. ∎

Proposition 8. P_1 *is a negative-definite solution of the first Riccati equation* (2.3).

Proof. It follows from (3.15) an (3.16) that

$$\frac{d}{dt}P_1e^{A_1t} = (D_1D_1' - A'P_1)e^{A_1t}. \tag{3.17}$$

Obviously, there also holds

$$\frac{d}{dt}P_1e^{A_1t} = P_1A_1e^{A_1t} = \left(P_1A + P_1\left(B_1B_1' - \frac{1}{\gamma^e}B_2B_2'\right)P_1\right)e^{A_1t}. \tag{3.18}$$

Combine (3.17) and (3.18) at $t = 0$ to establish that P_1 satisfies (2.3). ∎

It follows from results of Potter [P2, Theorems 1 and 2] that P_1 is symmetric if all the eigenvalues of A_1 belong to some half of the complex plane. (Potter's discussion is made under the assumption that the associated Hamiltonian matrix has only simple eigenvalues. This assumption can easly be relaxed.) By Corollary 7 that condition is met, so P_1 is symmetric.

Using the symmetry of P_1 we get

$$\frac{d}{dt}\langle x^*, P_1 x^*\rangle(t) = |z^*(t)|^2 - \gamma^2 |w^*(t)|.$$

The stability of (3.16) implies that $x^*(\infty) = 0$, whence

$$\langle x_0, P_1 x_0\rangle = \gamma^2 \|w^*\|^2 - \|z^*\|^2 = J_\gamma^*(x_0). \tag{3.19}$$

By Observation 3, the right-hand side of (3.19) is negative whenever $x_0 \neq 0$.

This concludes the first half of the proof of necessity in part (a).

3.2. *An Intermezzo: On the Maximum Principle*

Let $x_0 \in \mathbb{R}^n$ be an initial state in (2.1), and let u and w be L_2 control and disturbance such that $x \in L_2$ and $\lim_{t\to\infty} x(t) = 0$. We denote $u_0 = u - B_1' P_1 x$ and $w_0 = w + (1/\gamma^2) B_2' P_1 x$. These are the momentary deviations of u and w from their MinMax optimal values. Then integration of $(d/dt)\langle x, P_1 x\rangle$ along $[0, \infty)$ yields

$$J_\gamma(x_0, w, u) = \gamma^2 \|w\|^2 - \|z\|^2 = \langle x_0, P_1 x_0\rangle + \gamma^2 \|w_0\|^2 - \|u_0\|^2. \tag{3.20}$$

Equation (3.20) manifests the maximum principle in our setting, and plays a fundamental role in the analysis hereafter. (It seems to be just as important in the work of [DGKF].)

It is worthwhile to point out the geometric interpretation of the maximum principle, so we digress and elaborate on this. Let x^* be the MinMax optimal trajectory of (2.1), with $x^*(0) = x_0$, $u^* = B_1' P_1 x^*$, and $w^* = (1/\gamma^2) B_2' P_1 x^*$. Then set $\Delta x = x - x^*$, $\Delta u = u - u^*$, $\Delta w = w - w^*$, $\Delta z = z - z^*$. It is then easy to see that $(\Delta u)_0 = \Delta u - B_1' P_1 \Delta x = u - B_1' P_1 x = u_0$ and likewise, that $(\Delta w)_0 = w_0$. Apply (3.20) to the Δ-trajectory: since $\Delta x(0) = 0$, it implies

$$J_\gamma(0, \Delta w, \Delta u) = \gamma^2 \|w_0\|^2 - \|u_0\|^2.$$

We have also established (see (3.19) that

$$J_\gamma(x_0, w^*, u^*) = \langle x_0, P_1 x_0\rangle.$$

Thus (3.20) can be rewritten as

$$J_\gamma(x_0, w, u) = J_\gamma(x_0, w^*, u^*) + J_\gamma(0, \Delta w, \Delta u) \tag{3.21}$$

with the implication

$$\gamma^2\langle w^*, \Delta w\rangle - \langle z^*, \Delta z\rangle = 0. \tag{3.22}$$

Equation (3.22) tells us (as is well known in LQ optimal-control and game theory) that the quadratic form J_γ induces a geometry on the space of inputs, relative to which the optimal input pair is orthogonal to all other inputs. Equation (3.21) implies that, provided stability is maintained, the order of the "Min" and "Max" in the optimization problem (1.3) can be interchanged. Thus it proves the "saddle point" property of (w^*, u^*).

3.3. *Proof of Part* (*c*)

It remains to establish here:

Proposition 9. *Assume that P_1 is a negative definite solution of the first Riccati equation* (2.3), *such that A_1 is a stable matrix. Then $u = B_1'P_1x$ is an internally stabilizing state-feedback that assures the norm bound $\|T_K\| < \gamma$.*

Proof. If $u = B_1'P_1x$, then there holds

$$\dot{x} = A_3x + B_2w, \tag{3.23}$$

where $A_3 = A + B_1B_1'P_1$ satisfies the Lyapunov equation

$$P_1A_3 + A_3'P_1 = D_1'D_1 + P_1\left(B_1B_1' + \frac{1}{\gamma^2}B_2B_2'\right)P_1. \tag{3.24}$$

By Assumption A(iii), the right-hand side of (3.24) is $\geq \varepsilon_1$I. Thus, it is a standard observation [LM, p. 198] that (3.24) implies stability of A_3, hence internal stability with the suggested state-feedback.

Since A_3 is stable, the conditions under which (3.20) is valid are satisfied with any L_2 disturbance and the control feedback $u = B_1'P_1x$. In particular, when $x(0) = 0$,

$$J_\gamma(0, w, u = B_1'P_1x) = \gamma^2\|w_0\|^2. \tag{3.25}$$

Substituting $w_0 - (1/\gamma^2)B_2B_2'P_1x$ for w, (3.23) can be rewritten as

$$\dot{x} = A_1x + B_2w_0. \tag{3.26}$$

The stability of A_1 implies continuity of the mapping $w_0 \to w$, whereby there exists $\delta \neq 0$ such that $\gamma^2\|w_0\|^2 \geq \delta^2\|w\|^2$ for all $w_0 \in L_2$. Thus $J_\gamma(0, w, u = B_1'P_1x) \geq \delta^2\|w\|^2$, and the proposition follows from Observation 0; in fact $\|T_K\|^2 \leq \gamma^2 - \delta^2$. ■

3.4. *Completion of the Proof of Part* (*a*): *The Second Riccati Equation*

Our goal here is to identify a modified system where γ is strictly suboptimal, if it is strictly suboptimal in (2.1), and where the role of (2.4) is analogous to that of (2.3) in (2.1).

Indeed, assume that γ is strictly suboptimal in (2.1), and let K be an internally stabilizing compensator in that system, with $\|T_K\|^2 \leq \gamma^2 - \delta^2$ (for some $\delta \neq 0$). Given $u = Ky$ and $w \in L_2$, we maintain the notation $u_0 = u - B_2'P_1x$ and $w_0 = w + (1/\gamma^2)B_2'P_1x$, as in Section 3.2. We shall eventually show that the desired modified system is a transposed form of a realizaton of the closed-loop mapping T_K^0: $w_0 \to u_0$.

Here is an intuitive motivation: as follows from Observation 0, we are interested in $\mathrm{Min}_{w \in L_2} J_\gamma(x_0, w, u = Ky)$. Having (3.20) at hand, we can alternatively study $\mathrm{Min}_{w_0 \in L_2, u = Ky}(\gamma^2\|w_0\|^2 - \|u_0\|^2)$, which is an estimation problem. As is well known, estimation problems are solved via the treatement of associated LQ optimal-control problems in the transposed system.

Let us denote by $W_0 \subset L_2$ the image of the closed-loop mapping from L_2 to L_2 taking $w \to w_0$.

Observation 10. *The mapping T_K^0: $L_2 \to L_2$, $w_0 \to u$ is linear and closed. Moreover, T_K^0 is bounded if and only if $W_0 = L_2$, in which case $\|T_K^0\| < \gamma$.*

Proof. The operator T_K^0 is linear and closed since it is realized by the following linear system (denoted S_0):

$$\begin{aligned} \dot{x} &= \left(A - \frac{1}{\gamma^2} B_2 B_2' P_1\right) x + B_1 u + B_2 w_0, \\ y &= C_1 x + C_2 w_0, \qquad u = Ky, \\ u_0 &= B_1' P_1 x + u. \end{aligned} \tag{3.27}$$

(Notice that $C_2 w_0 = C_2 w$, by Assumption A(ii). So x, y, and u are exactly as in (2.1), given $w = w_0 - (1/\gamma^2) B_2' P_1 x$.)

Assume $x(0) = 0$. By Observation 0, the left-hand side of (3.20) is then not smaller then $\delta^2 \|w\|^2$. Since K is internally stabilizing in (2.1), the mapping $w \to w_0$ from L_2 to L_2 is continuous. So there exists some $\eta \neq 0$ such that $\delta^2 \|w\|^2 \geq \eta^2 \|w_0\|^2$. Therefore, for all $w_0 \in W_0$, the right-hand side of (3.20) is not smaller than $\eta^2 \|w_0\|^2$. Assume $W_0 = L_2$. Then it is Observation 0, again, that tells us that $\|T_K^0\| < \gamma$.

Conversely, assume that T_K^0 is bounded. Then trajectories of (3.27) are governed by the equation

$$\dot{x} = A_1 x + (B_1 T_K^0 + B_2) w_0. \tag{3.28}$$

Since T_K^0 is bounded and A_1 is stable, the mapping $w_0 \to x$ is continuous. Consequently, $w_0 \to w = w_0 - (1/\gamma^2) B_2 B_2' P_1 x$ is a continuous inverse of $w \to w_0$. Hence $W_0 = L_2$. ■

Proposition 11. *The feedback K is internally stabilizing in* (3.27). (*In particular, T_K^0 is bounded, $W_0 = L_2$, and $\|T_K^0\| < \gamma$.*)

Proof. By (3.20), if $w_0 = 0$, then J_γ is nonpositive. Under the assumption that the proposition is false and K is not internally stabilizing in (3.27), we construct an L_2 trajectory where $w_0 = 0$ and yet J_γ is positive. A contradiction then proves that the proposition is true. This will take some work.

Let the internal structure of K be as in Section 2:

$$\begin{aligned} \dot{p} &= Mp + Ny, \\ u &= Qp + Ry. \end{aligned}$$

We recall that since K is internally stabilizing in (2.1), the following matrix is stable:

$$\mathscr{A} = \begin{bmatrix} A + B_1 R C_1 & B_1 Q \\ N C_1 & M \end{bmatrix}.$$

Denote

$$\mathscr{B} = \begin{bmatrix} B_2 + B_1 R C_2 \\ N C_2 \end{bmatrix}.$$

In order for K to be internally stabilizing in (3.27), we need that $\mathscr{A}_0 = \mathscr{A} + \mathscr{B}[-(1/\gamma^2)B_2'P_1, 0]$ be stable as well.

Assume now, in contradiction to the proposition, that $\mathscr{A}_0$ is unstable and let $d_0 = \begin{pmatrix} x_0 \\ p_0 \end{pmatrix}$ be a nonzero unstable vector for $\mathscr{A}_0$. For $T > 0$, let the dynamics of

$$d_T(t) = \begin{pmatrix} x_T(t) \\ p_T(t) \end{pmatrix}$$

be governed by the differential equation

$$\dot{d} = \mathscr{A}_0 d \tag{3.29}$$

along the interval $[0, T]$, with $d_T(0) = d_0$.

Since d_0 is an unstable vector for $\mathscr{A}_0$, the $L_2[0, T]$ norm $\|d_T\|$ grows to infinity with the increase of T. We claim that the same holds for $\|x_T\|$ and $\|w_T = -(1/\gamma^2)B_2'P_1 x_T\|$. Indeed, (3.29) can be rewritten as

$$\dot{d} = \mathscr{A}d + \mathscr{B}w, \tag{3.30}$$

with the feedback law $w = -(1/\gamma^2)B_2B_2'P_1 x$. Were $\|x_T\|$, and thereby $\|w_T\|$, uniformly bounded for all T, the stability of $\mathscr{A}$ would imply uniform boundedness of $\|d_T\|$, in contradiction to our assumption.

With d_T and w_T we associate, in the natural way, a control function $u_T = [RC_1, Q]\,d_T + RC_2 w_T\,(= Ry_T + Qp_T)$ and an output $z_T = D_1 x_T + D_2 u_T$.

Let us extend the definitions of all these trajectories through (T, ∞), as follows. In a very similar way to what we did in Section 3.1, we establish existence and uniqueness of a solution $\zeta^* \in L_2$ to the optimization problem

$$\min_{\zeta \in L_2} J_\gamma(x_T(T), \zeta, u = [RC_1, Q]d + RC_2\zeta) \tag{3.31}$$

subject to the differential equation (3.30) (with ζ substituting for w). Let $w_T(t) - \zeta^*(t - T)$ for $t > T$, let d_T be the continued solution of (3.30), and let u_T and z_T be extended accordingly.

We now denote $d_T = d_{T,0} + d_{T,1}$, so that $d_{T,0}$ is the trajectory of the homogeneous part of (3.30) (i.e., with $w = 0$), initiated at $d_{T,0}(0) = d_0$, and so that $d_{T,1}$ is the solution of (3.30) with $w = w_T$ and $d_{T,1}(0) = 0$. With these trajectories we associate, as above, $u_{T,0}$, $z_{T,0}$, $u_{T,1}$, and $z_{T,1}$. We immediately notice that $x_{T,1}$, w_T, and $u_{T,1}$ satisfy (2.1) with the feedback control $u_{T,1} = Ky_{T,1}$. Thus

$$J_\gamma(0, w_T, u_{T,1}) \geq \delta^2 \|w_T\|^2, \tag{3.32}$$

and the right-hand side of (3.32) becomes arbitrarily large with the growth of T.

The trajectories $d_{T,0}$ and $z_{T,0}$ are actually independent of T. And since $\mathscr{A}$ is a stable matrix, these are L_2 functions. Thus $\|z_{T,1}\|$ grows to infinity as T grows, and it becomes the dominant part in $\|z_T\|$. Consequently,

$$\lim_{T \to \infty} \left(a(T) := \frac{\gamma^2 \|w_T\|^2 - \|z_T\|^2}{\gamma^2 \|w_T\|^2 - \|z_{T,1}\|^2} \right) = 1. \tag{3.33}$$

Combining (3.32) and (3.33), for large T we get

$$J_\gamma(x_0, w_T, u_u) \geq a(T)\delta^2 \|w_T\|^2 > 0. \tag{3.34}$$

Let us now define trajectories x, u, w, and z in L_2 as follows: along $[0, T]$, $x = x_T$, $u = u_T$, $w = w_T$, and $z = z_T$. For $r > T$ we let x continue as a solution of

$$\dot{x} = A_1 x \tag{3.35}$$

with $u = B_1' P_1 x$, $w = -(1/\gamma^2) B_2' P_1 x$, and $z = D_1 x + D_2 u$.

The matrix A_1 is stable, so our trajectory satisfies the assumptions for (3.20). Moreover, $w_0 = 0$ throughout, as desired. (Recall the explanation at the beginning of this proposition's proof.) Now, it was established in Section 3.1 that

$$\begin{aligned}
\gamma^2 \|w\|^2_{[T,\infty)} - \|z\|_{[T,\infty)} &= J_\gamma^*(x(T)) \\
&= \operatorname*{Min}_{\zeta \in L_2} \operatorname*{Max}_{\xi \in L_2} J(x(T), \zeta, \xi) \\
&\geq \operatorname*{Min}_{\zeta \in L_2} J_\gamma(x_T(T), \zeta, u = [RC_1 Q]d + RC_2\zeta) \\
&= \gamma^2 \|w_T\|^2_{[T,\infty)} - \|z_T\|^2_{[T,\infty)}.
\end{aligned}$$

Invoking both (3.20) and (3.34) we thus get

$$\begin{aligned}
0 &\geq \langle x_0, P_1 x_0 \rangle - \|u_0\|^2 = J_\gamma(x_0, w, u) \\
&\geq J_\gamma(x_0, w_T, u_T) \geq a(T)\delta^2 \|w_T\|^2 > 0;
\end{aligned}$$

which is the contradiction we sought. ∎

Through the remainder of this section, we use $\#$ to denote system transposition. Thus $T_K^{0\#}$ is realized by

$$\begin{aligned}
\dot{\tilde{x}} &= \left(A - \frac{1}{\gamma^2} B_2 B_2' P_1\right)' \tilde{x} + C_1' \tilde{u} - P_1 B_1 \tilde{w}, \\
\tilde{y} &= B_1' \tilde{x} + \tilde{w}, \qquad \tilde{u} = K^{\#} \tilde{y}, \\
\tilde{z} &= B_2' \tilde{x} + C_1' \tilde{u},
\end{aligned} \tag{3.36}$$

where $K^{\#}$ is the transposed compensator

$$\begin{aligned}
\dot{\tilde{p}} &= M' \tilde{p} + Q' \tilde{y}, \\
\tilde{u} &= N' \tilde{p} + R' \tilde{y}.
\end{aligned}$$

Observation 12. *If γ is strictly suboptimal in* (3.36), *then there exists a negative definite solution P_2 for* (2.4), *such that A_2 is stable.*

Proof. Just check that (2.4), given (3.36), is the counterpart of (2.3), given (2.1). ∎

The proof of part (a) is therefore completed with

Proposition 13. *$K^{\#}$ is internally stabilizing in* (3.36) *and* $\|T_K^{0\#}\| < \gamma$.

Proof. The closed-loop transfer functions in (3.27) and (3.36) are the transpose of each other. As is well known [F], the induced operator norm of an input–output operator is equal to the H_∞ norm of the associated transfer function. Hence $\|T_K^{0\#}\| = \|T_K^0\|^2 < \gamma$. Internal stability in (3.36), given $K^{\#}$, is equivalent to stability of $\mathscr{A}'_0$; the latter was established in Proposition 11. ■

3.5. Necessity in Part (b): The Parametrization

Proposition 14. *Every internally stabilizing feedback operator K in* (2.1) *that assures $\|T_K\| < \gamma$ can be realized in the form* (2.5).

Proof. Let K be an internally stabilizing compensator with $\|T_K\| < \gamma$, let the constructions of Section 3.4 be in force, and set $\tilde{w}_0 = \tilde{w} - (1/\gamma^2)B'_1P_1P_2\tilde{x}$ and $\tilde{u}_0 = \tilde{u} - C_1P_2\tilde{x}$ in (3.36). Applying Observation 10 and Proposition 11 to (3.36) we see that the mapping $K_0^{\#}: \tilde{w}_0 \to \tilde{u}_0$ is realized by a stable system, and that $\|K_0^{\#}\| < \gamma$.

Consider now the following compensator $K_1^{\#}$, as an alternative to $K^{\#}$ in (3.36). Later we shall show that the input–output mappings in $K^{\#}$ and in $K_1^{\#}$ are identical.

$$
\begin{aligned}
\dot{\tilde{p}} &= (A'_1 + C'_1C_1P_2)\tilde{p} + C'_1\tilde{v} - P_1B_1\tilde{y}, \qquad \tilde{p}(0) = 0,\\
\tilde{q} &= -B'_1\left(I + \frac{1}{\gamma^2}P_1P_2\right)\tilde{p} + \tilde{y}, \qquad \tilde{v} = K_0^{\#}\tilde{q},\\
\tilde{u} &= C_1P_2\tilde{p} + \tilde{v}.
\end{aligned}
\tag{3.37}
$$

We can easily check that when $\tilde{x}(0) = 0$ and $\tilde{u} = K_1^{\#}\tilde{y}$ then $\tilde{p} = \tilde{x}$ in the closed-loop system. (Indeed, verify that $\tilde{e} = \tilde{x} - \tilde{p}$ satisfies a homogeneous differential equation with $\tilde{e}(0) = 0$.) Therefore,

$$
\begin{aligned}
\tilde{q} &= -B'_1\left(I + \frac{1}{\gamma^2}P_1P_2\right)\tilde{p} + \tilde{y}\\
&= -B'_1\left(I + \frac{1}{\gamma^2}P_1P_2\right)\tilde{x} + (B'_1\tilde{x} + \tilde{w})\\
&= -\frac{1}{\gamma^2}B'_1P_1P_2\tilde{x} + \tilde{w} = \tilde{w}_0,
\end{aligned}
$$

whence

$$
\tilde{u} = C'_1P_2\tilde{x} + K_0^{\#}\tilde{w}_0. \tag{3.38}
$$

By definition of $K_0^{\#}$, (3.38) means $u = K^{\#}\tilde{y}$. In short, when $\tilde{x}(0) = 0$ the input–output operators $K^{\#}$ and $K_1^{\#}$ coincide over the image of the closed-loop mapping $\tilde{w} \to \tilde{y} = \tilde{w} + B'_1\tilde{x}$ in (3.36).

But when $\tilde{x}(0) = 0$, that mapping is a Volterra operator of the second type, and is therefore L_2-invertible when restricted to any finite time interval [0, T]. Causality thus implies that $K^{\#}$ and $K_1^{\#}$ agree throughout. The proposition follows since $K_1^{\#}$ is the transpose of (2.5). ■

3.6. *The Proof of Sufficiency in Parts* (*a*) *and* (*b*)

We assume the existence of negative definite matrices, P_1 and P_2, which satisfy the Riccati equations (2.3) and (2.4), and such that the associated matrices A_1 and A_2 are stable.

Proposition 15. *Assume that K is a compensator of the form* (2.5). *Then K is internaly stabilizing in* (2.1) *and assures the closed-loop bound $\|T_K\| < \gamma$. In particular, $\gamma > \gamma_0$ under our assumptions.*

Proof. Recall the form (2.5) of K:

$$\dot{p} = (A_1 + P_2C_1'C_1)p - \left(I + \frac{1}{\gamma^2}P_2P_1\right)B_1v + P_2C_1'y,$$

$$q = C_1p + y, \qquad v = K_0q,$$

$$u = -B_1'P_1p + v,$$

with K_0 being the input–output mapping of a stable linear system such that $\|K_0\| < \gamma$. Assume K is implemented, let x and p be the respective states of (2.1) and of (2.5), and let r be the internal state in K_0. Internal stability means exponential decay of the triplet $\begin{pmatrix} x \\ p \\ r \end{pmatrix}(t)$, given any initial state $\begin{pmatrix} x \\ p \\ r \end{pmatrix}(0)$, and $w = 0$. Since K_0 is assumed stable, exponential decay of the pair $\begin{pmatrix} x(t) \\ p(t) \end{pmatrix}$, hence of $q(t) = C_1(p(t) + x(t))$, implies exponential decay of $r(t)$. We therefore focus only on x and p. In fact, it will be more convenient to work with $d = \begin{pmatrix} x \\ x + p \end{pmatrix}$.

When $w = 0$ there holds

$$\dot{d} = \mathscr{A}d + \mathscr{B}v, \qquad v = K_0\mathscr{C}d, \tag{3.39}$$

where the matrices $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are

$$\mathscr{A} = \begin{bmatrix} A + B_1B_1'P_1 & -B_1B_1'P_1 \\ \frac{1}{\gamma^2}B_2B_2'P_1 & A - \frac{1}{\gamma^2}B_2B_2'P_1 + P_2C_1'C_1 \end{bmatrix},$$

$$\mathscr{B} = \begin{bmatrix} I \\ -\frac{1}{\gamma^2}P_2P_1 \end{bmatrix}B_1 \quad \text{and} \quad \mathscr{C} = [0, C_1].$$

(These notations are different from those made in previous sections!) We denote two more matrices:

$$\mathscr{V} = -\begin{bmatrix} \frac{1}{\gamma_0^2}P_1 & 0 \\ 0 & P_2^{-1} \end{bmatrix}$$

and

$$\mathscr{W} = \begin{bmatrix} \frac{1}{\gamma^2} D_1' D_1 + \frac{1}{\gamma^2} P_1 B_2 B_2' P_1 & \frac{1}{\gamma^2} P_1 B_2 B_2' P_2^{-1} \\ \frac{1}{\gamma^2} P_2^{-1} B_2 B_2' P_1 & P_2^{-1} B_2 B_2' P_2^{-1} \end{bmatrix}$$

Obviously, $\mathscr{V}$ is positive definite. From Assumptions A(iii) and A(iv), so is $\mathscr{W}$.

There holds

$$\frac{d}{dt}\langle d, \mathscr{V} d\rangle(t) = -\langle d, \mathscr{W} d\rangle(t) - \frac{1}{\gamma^2}|(v - B_1' P_1 p)(t)|^2 - |C_1(x + p)(t)|^2 + \frac{1}{\gamma^2}|v(t)|^2. \tag{3.40}$$

In view of the casualty of K_0, the relation $v = K_0 C_1(x + p)$, and the norm bound $\|K_0\| < \gamma$, there also holds

$$\frac{1}{\gamma^2}\int_0^t |v(\tau)|^2 \, d\tau < \int_0^t |C_1(x + p)(\tau)|^2 \, d\tau \tag{3.41}$$

for all $t > 0$. Introducing (3.41) into an integrated form of (3.40), we have

$$\langle d, \mathscr{V} d\rangle|_0^t < -\int_0^t \langle d, \mathscr{W} d\rangle(\tau) \, d\tau.$$

Therefore $d(t)$ decays exponentially, as claimed.

It remains to establish that $\|T_K\| < \gamma$. For that purpose we reverse the constructions of earlier subsections.

The counterpart of (3.20) in the transposed system (3.36) is

$$\gamma^2 \|\tilde{w}\|^2 - \|\tilde{z}\|^2 = \gamma^2 \|\tilde{w}_0\|^2 - \|\tilde{u}_0\|^2 + \langle \tilde{x}(0), P_2 \tilde{x}(0)\rangle. \tag{3.42}$$

Assume that $\tilde{x}(0) = 0$. We established in Section 3.5 that then $\tilde{u}_0 = K_0^{\#} \tilde{w}_0$. Also, by our assumption $\|K_0^{\#}\| = \|K_0\| < \gamma$. Thus there exists $\delta \neq 0$ such that the right-hand side of (3.42) is not smaller than $\delta^2 \|\tilde{w}_0\|^2$.

Now we reuse an argument from the proof of Observation 10, converted to the present context: there holds (when $\tilde{x}(0) = 0$)

$$\dot{\tilde{x}} = A_2' \tilde{x} + (C_1' K_0^{\#} - P_1 B_1)\tilde{w}_0.$$

Since A_2 is stable and the mapping $K_0^{\#}: L_2 \to L_2, \tilde{w}_0 \to \tilde{w}$ is bounded, it follows that $\tilde{w}_0 \to \tilde{w} = \tilde{w}_0 + (1/\gamma^2) B_1' P_1 P_2 \tilde{x}$ from $L_2 \to L_2$ is a bounded operator. So $\delta^2 \|\tilde{w}_0\|^2 \geq \eta^2 \|\tilde{w}\|^2$ for some fixed $\eta \neq 0$, and (by Observation 0) $\|T_K^0\|^2 = \|T_K^{0\#}\|^2 \leq \gamma^2 - \eta^2$.

Now we are in a similar situation in the original setting of (2.1): T_K replaces $T_K^{0\#}$, T_K^0 is the counterpart of $K_0^{\#}$, and A_1 replaces A_2. Using (3.20) and (3.28) we conclude that $\|T_K^0\| < \gamma$ implies $\|T_K\| < \gamma$.

This completes the proof of Theorem I. ∎

4. The Finite-Horizon, Time-Varying Case

Consider the system S when restricted to some finite time interval $[t_0, t_1]$, and allow time-varying L_∞-coefficients (e.g., $A = A(t) \in L_\infty[t_0, t_1]$) both in (2.1) and in admissible compensators. The statement of the standard problem in this context remains essentially the same as before, with the understanding that now $L_2 = L_2[t_0, t_1]$. Stability, however, ceases to be a factor in our considerations: linear bounded-coefficient systems are stable over finite time spans, for any reasonable notion of stability.

Before giving our results (which are natural counterparts of those in Theorem I) we need the following definitions. Let $\Omega: L_2[t_0, t_1] \to L_2[t_0, t_1]$ be the time-reversing operator, $\Omega f(t) = f(t_0 + t_1 - t)$. It is a unitary isometry. Let T be a bounded operator on L_2. We denote $T^{\#} = \Omega T' \Omega$. Obviously, $\#$ is a norm-preserving conjugation: $T^{\#\#} = T$ and $\|T^{\#}\| = T\|$. When $M(t)$ is a time-varying L_∞-matrix, the consistent definition is $M^{\#}(t) = \Omega M'(t) = M'(t_0 + t_1 - t)$. These definitions are needed since the previous notion of $\#$, via transposition of transfer functions, is not available here.

We easly note that if an operator T has a linear-varying system realization over $[t_0, t_1]$, say

$$\dot{p} = Mp + Nf,$$
$$g = Qp + Rf,$$

where M, N, Q, and R are L_∞ matrices, then $T^{\#}$ is realized by the $\#$-transpose system

$$\dot{\tilde{p}} = M^{\#}\tilde{P} + Q^{\#}\tilde{f},$$
$$\tilde{g} = N^{\#}\tilde{p} + R^{\#}\tilde{f}.$$

Theorem II

(a) *The value $\gamma > 0$ is a strictly suboptimal in (2.1) over $[t_0, t_1]$ if and only if there exist uniformly bounded, negative definite solutions, P_1 and P_2, to the following two dynamic matrix Riccati equations:*

$$\dot{P}_1 = D_1'D_1 - P_1A - A'P_1 - P_1\left(B_1B_1' - \frac{1}{\gamma^2}B_2B_2'\right)P_1, \qquad P_1(t_1) = 0, \quad (4.1)$$

$$\dot{P}_1 = -B_2B_2' + P_2\left(-\frac{1}{\gamma^2}B_2B_2'P_1\right)' + \left(A - \frac{1}{\gamma^2}B_2B_2'P_1\right)P_2$$
$$+ P_2\left(C_1'C_1 - \frac{1}{\gamma^2}P_1B_1B_1'P_1\right)P_2, \qquad P_2(t_0) = 0 \quad (4.2)$$

(b) *An admissible compensator assures the closed-loop norm bound $\|T_K\| < \gamma$ if and only if it can be realized in the form*

$$\dot{p} = (A_1 + P_2C_1'C_1)p - \left(I + \frac{1}{\gamma^2}P_2P_1\right)B_1v + P_2C_1'y, \qquad p(t_0) = 0,$$
$$q = C_1p + y, \qquad v = K_0q, \quad (4.3)$$
$$u = -B_1'P_1p + v,$$

where K_0 is an admissible feedback operator with $\|K_0\| < \gamma$.

(c) *If the system's state is available; i.e., if Assumptions* A(ii) *and* A(iv) *are replaced by "$C_1 = I$ and $C_2 = 0$," then γ is strictly suboptimal if and only if P_1 exists, as claimed, in which case the state feedback $u = B_1'P_1x$ assures $\|T_K\| < \gamma$ in a closed loop.*

Proof. The main lines of the proof are those of the proof of Theorem I. We are therefore content to highlight parts where changes are needed.

1. Instead of a single MinMax problem (1.3), we consider a parametrized set of problems

$$\min_{w \in L_2[t,t_1]} \max_{u \in L_2[t,t_1]} J_{\gamma,t}(x_t, w, u), \tag{4.4}$$

where $x(t) = x_t \in \mathbb{R}^n$ is a midinterval constraint in (2.1) and where

$$J_{\gamma,t}(x_t, w, u) = \gamma^2 \|w\|_t^2 - \|z\|_t^2.$$

(Recall that $\|\cdot\|_t$ is the norm in $L_2[t, t_1]$.) In this context, future intervals become shorter when time advances.)

2. The Riccati equation (3.2) becomes dynamics

$$\dot{L} = D_1'D_1 - LA - A'L - LB_1B_1'L,$$

with the terminal-time conditon $L(t_1) = 0$. The fundamental matrix $\Phi_L(\tau, t)$, generated by $A_L = A + B_1B_1L$, will substitute for $e^{A_L(\tau-t)}$, t_1 for ∞, and t_0 for 0 in the constructions (3.3) and (3.4).

3. Instead of two fixed operators $\mathscr{F}$ and $\mathscr{G}$, we need two families of operators $\{\mathscr{F}(t)\}$ and $\{\mathscr{G}(t)\}$: given $(t, x_t, w) \in (t_0, t_1) \times \mathbb{R}^n \times L_2[t_0, t_1]$, let r, x, and η be defined by (3.3), (3.4), and (3.5), modified as explaned in part 2, and with the initial condition in (3.4) substituted by the midinterval constraint $x(t) = x_t$. Let $u = B_1'\eta$ and let $z = D_1x + D_2u$. Then for any $\tau \in [t_0, t_1)$ the triplet $(x, u, z)|_{[\tau,t_1]}$ is optimal in $\min_u \|z\|_\tau^2$ given the data $(\tau, x(\tau), w|_{[\tau,t_1]})$. In particular, $(x, u, z)|_{[t,t_1]}$ is optimal given $(t, x_t, w|_{[t,t_1]})$.

We thus define $(x, u, \eta) = \mathscr{F}(t)(x_t, w)$ and $z = \mathscr{G}(t)(x_t, w)$. These operators satisfy

Proposition 1°. *$\{\mathscr{F}(t)\}$ and $\{\mathscr{G}(t)\}$ are uniformly compact families of linear operators, and the functions $t \to \mathscr{F}(t)$ and $t \to \mathscr{G}(t)$ are norm continuous.*

(The easy proof is left to the reader.)

4. In accordance with (4.4), the definitions of J_γ^0, J_γ^*, and $|||\cdot|||$ become time parametrized:

$$J_{\gamma,t}^0(x_0, w) = \gamma^2 \|w\|_t^2 - \|\mathscr{G}(t)(x_0, w)\|_t^2,$$

$$J_{\gamma,t}^*(x_0) = \inf_w J_{\gamma,t}^0(x_0\ w),$$

and

$$|||w|||_t = J_{\gamma,t}^0(0, w)^{1/2}.$$

Propositions 2, 3, and 4 remain valid with essentially unchanged proofs. But now, the disturbance w_t^* that minimizes $J_{\gamma,t}^0$ is determined only over $[t, t_1]$. We extend its definition to the entire interval as $w_t^*(\tau) = 0$ for $\tau < t$ (other consistent extensions are also possible) and denote $w_t^* = \mathscr{H}(t)x_t$.

5. We need a stronger statement than Corollary 5.

Corollary 5⁰. *$\{\mathscr{H}(t)\}$ is a uniformly compact family of linear operators and the function $t \to \mathscr{H}(t)$ is norm continuous.*

Proof. Linearity of each $\mathscr{H}(t)$ follows, as before, from the uniqueness of w_t^* and from Proposition 4 ($w^* = (1/\gamma^2)B_2'\eta^*$). Uniform boundedness follows from the uniform boundedness of the family $\{\mathscr{G}(y)\}$, and from the inequality (3.8). Uniform compactness then follows from the uniform compactness of $\{\mathscr{F}(t)\}$ and the uniform boundedness of $\{\mathscr{H}(t)\}$, via

$$\mathscr{H}(t)x_t = -\frac{1}{\gamma^2}B'\eta|_{[t,t_1]}, \qquad (x, u, \eta) = \mathscr{F}(t)(x_t, \mathscr{H}(t)x_t).$$

Assume $t_\alpha \to t$ and $\{\xi_\alpha\} \subset \{\xi \in \mathbb{R}^n: |\xi| = 1\}$. Without loss of generality, $\xi_\alpha \to \xi$. Let $w_\alpha = \mathscr{H}(t_\alpha)\xi_\alpha$. By Proposition 5^0, $\{w_\alpha\}$ is a compact sequence in L_2, and has an accumulaton point, say w. Following from Proposition 4 and the uniform continuity of $\{\mathscr{F}(t)\}$, it follows that $w|_{[t_0,t)} = 0$ and $w|_{[t,t_1]} = -(1/\gamma^2)B_2'\eta|_{[t,t_1]}$, where $(x, u\ \eta) = \mathscr{F}(t)(\xi, w)$. That is, $w = \mathscr{H}(t)\xi$. This assures the norm convergences $\mathscr{H}(t_\alpha) \to \mathscr{H}(t)$. ■

6. The constant matrix P_1 is now replaced by a matrix-valued function, defined by

$$P_1(t)\xi = L(t)\xi + \int_t^{t_1} \Phi_L'(\tau, t)L(\tau)B_2(\tau)(\mathscr{H}(t)\xi)(\tau)\, d\tau.$$

Following from Corollary 5^0, the function $t \to P_1(t)$ is continuous over the closed interval $[t_0, t_1]$. In particular, it is uniformly bounded. Proposition 6 ($\eta^*(t) = P_1(t)x^*(t)$) remains valid with essentially an unchanged proof. Corollary 7 (A_1 is stable) is redundant.

7. In the proof of Proposition 8 (P_1 is a negative-definite solution of the first Riccati equation) we have to establish differentiability of $P_1(t)$. Indeed, let $\Phi_1(t, \tau)$ be the fundamental matrix generated by A_1. Then it follows from the facts that $\dot{x}^* = A_1x^*$ and that $\eta^* = P_1x^*$, that $t \to P_1(t)\Phi_1(t, \tau)$ is differentiable. Hence so is

$$P_1(t) = P_1(t)\Phi_1(t, \tau)\Phi_1(\tau, t).$$

8. The reader should bear in mind that in the remaining parts of the proof of Theorem II, #-conjugation is as defined at the beginning of this section. That is $T^\# = \Omega T'\Omega$ for an operator $T: L_2 \to L_2$, and $M^\#(t) = M'(t_0 + t_1 - t)$ for a matrix $M(t)$. With that modification the rest of the proof remains practically unchanged (except of course, those parts pertinent to stability are omitted). ■

5. The Case where Assumption A Fails

The straightforward ideas in this section are taken from [KPZ], where more detail is available.

Assumptions A(i)–(iv) can be divided into two pairs: A(i) and A(iii), which are pertinent to the first Riccati equation and the associated optimization problem (1.3), and A(ii) and A(iv), which play an analogous role in the transposed system, and are pertinent to the second Riccati equation.

Let us focus, then, on A(i) and A(iii). The assumption $D_2'D_1 = 0$ is merely a matter of convenience: this way $|z|^2 = |D_1x|^2 + |D_2u|^2$. If it fails, choose L such that $D_2'(D_1 + D_2L) = 0$. Then introduce an artificial input, v, so that $u = Lx + v$. The formalism of the problem and its basic properties remain essentially unchanged when $A^0 = A + B_1L$ replaces A, and $D_1^0 = D_1 + D_2L$ replaces D_1. In this setting $|z|^2 = |D_1^0x|^2 + |D_2v|^2$.

Assume that $D_2'D_1 = 0$. Then full column rank of D_2 is necessary for nonsingularity of the optimal-control problem $\min_{u \in L_2} \|z\|^2$, while full column rank of D_1 assures that input–output boundedness implies internal stability (for then $\|z\|^2 \geq \varepsilon_1 \|x\|^2$). Both are essential properties. Suppose they fail. Then we use the following.

Observation. *Let Q and R be positive-definite matrices. Then γ is strictly suboptimal in (2.1) if and only if there exist $\varepsilon, \delta \neq 0$ and an internally stabilizing feedback $u = Ky$, so that*

$$\gamma^2 \|w\|^2 - \|z\|^2 - \varepsilon^2(\|x\|_Q^2 + \|u\|_R^2) \geq \delta^2 \|w\|^2. \tag{5.1}$$

Proof. Assume γ were strictly suboptimal. Then there exists an internally stabilizing K such that $\|T_K\|^2 < \gamma^2 - 2\delta^2$ for some $\delta \neq 0$. That is

$$\gamma^2 \|w\|^2 - \|z\|^2 \geq 2\delta^2 \|w\|^2. \tag{5.2}$$

By internal stability, u: $L_2 \to L_2$, $w \to x$, are bounded mappings. Thus

$$\varepsilon^2(\|x\|_Q^2 + \|u\|_R^2) \leq \delta^2 \|w\|^2 \tag{5.3}$$

for some $\varepsilon \neq 0$. Combine (5.2) and (5.3) to get (5.1). Conversely, assume that (5.1) holds. Then it is obvious that $\|T_K\| < \gamma$, and γ is strictly suboptimal. ■

Given D_1 and D_2 we can construct (see [KPZ] for detail) matrices Q and R, such that given $\varepsilon \neq 0$ there exist D_1^ε and D_2^ε satisfying $D_2^{\varepsilon\prime} D_1^\varepsilon = 0$ and

$$|D_1^\varepsilon x|^2 + |D_2^\varepsilon u|^2 = |z|^2 + \varepsilon(|x|_Q^2 + |u|_R^2).$$

Thus, both D_1^ε and D_2^ε have full column rank. Moreover, by the observation, γ is strictly suboptimal in the original setting of (2.1) if and only if there exists some $\varepsilon \neq 0$ small enough, so that γ is suboptimal when D_1^ε and D_2^ε substitute for D_1 and D_2.

Finally, when D_2 has full column rank, we can always substitute $v = (D_2'D_2)^{1/2}u$ for u, and I for D_2, to get $\|v\|^2 = \|D_2u\|^2$.

As mentioned above, the main role of Assumptions A(ii) and A(iv) comes when treating the transposed system (3.27). There they are the exact counterparts of A(i) and A(iii). The preceding short discussion explains how to handle a situation where either of them fails. There is one case where A(ii) (explicitly, $B_2C_2' = 0$) was used before, which is in deriving the exact form of (3.24). The modificatons required are insignificant: simply write $y = (C_1 + (1/\gamma^2)C_2B_2'P_1)x + C_1w_0$, and modify the definitions of $\mathscr{A}_0$ and $\mathscr{B}$ accordingly, in the statement and the proof of Proposition 11.

6. Conclusions

The proofs presented above, although very basic mathematically, and relying on well-known ideas, are rather lengthy. This stems from the intricate nature of the problem (see [BC], [GD], and [VR]) and from the fact that we could not rely on previous results on worst-case design using optimal control theory. In these concluding notes we wish briefly to recap some of the main ideas and techniques employed.

1. Translating the problem into an LQ game-theoretic formalism, the analysis is based on a search for a compensator which provides the "best control" in response to the "worst disturbance." It is later established that this compensator and a family of its perturbations, also guarantee acceptable behavior in response to other disturbances.

2. Following the usual pattern in optimal-control theory, the output-feedback problem is divided into a state-feedback problem and a state-estimation problem. Again, as usual, the latter is converted into a dual optimal-control problem.

3. The first part of the proof aims at establishing existence, uniqueness, and a "maximum principle" characterization of the "worst (MinMax) disturbance." Our analysis mimics that commonly used in classical LQ optimization. In particular, the idea of using a quadratic-cost index as an alternative norm ($|||\cdot|||$) dates back to the early 1960s (see, e.g., the proof of Theorem 1 of [LM], pp. 174–177).

4. Integration of $(d/dt)\langle x, Px\rangle$ is also an old and common technique (see, e.g., p. 200 of [LM]). It relies on the underlying geometry of the maximum principle. In classical optimal-control theory it is used in establishing $\langle x_0, Px_0\rangle = J^*(x_0)$. This is the use made in our proof of Proposition 8 ($P_1 < 0$). We also invoke it in obtaining the two fundamental equalities, (3.20) and (3.42), that enable relating $\|T_K\|$ to $\|T_K^0\|$ and $\|T_K^{0\#}\|$ to $\|K_0^\#\|$.

5. In the interplay between the systems (2.1) and (3.27) (or (3.23)) and between (3.36) and (3.37), continuity of the mapping $w \leftrightarrow w_0$ and $\tilde{w} \leftrightarrow \tilde{w}_0$ is crucial. Here we use either the (given) internal stability of the system (in the directions $w \rightarrow w_0$ and $\tilde{w} \rightarrow \tilde{w}_0$) or the stability of A_1 and of A_2 (for the converse). Essentially the same argument is repeated in the proofs of Observation 10 and Propositions 9, 15, and implicitly, in 14.

6. The solutions of suitable Riccati equations are natural candidates for Lyapunov kernels. In the proof of Proposition 15 (internal stability with the parametrized compensators) we just fill in the necessary detail.

7. Rectangular (nonsquare) systems and nonminimal realizations are troublesome when frequency-domain analysis is employed. Notice that the dimensions of a system play no role in our developments in the time domain. The price that we pay is in our stability analysis: when nonminimal realizations are allowed, input–output stability is weaker than internal stability. We had to ensure that the latter, stronger, property prevails.

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