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TWO-BODY PROBLEM ON SPACES OF CONSTANT CURVATURE: I. DEPENDENCE OF THE HAMILTONIAN ON THE SYMMETRY GROUP AND THE REDUCTION OF THE CLASSICAL SYSTEM

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We consider the problem of two bodies with central interaction that propagate in a simply connected space with a constant curvature and an arbitrary dimension. We obtain the explicit expression for the quantum Hamiltonian via the radial differential operator and generators of the isometry group of a configuration space. We describe the reduced classical mechanical system determined on the homogeneous space of a Lie group in terms of orbits of the coadjoint representation of this group. We describe the reduced classical two-body problem.

1. Introduction

The simply connected constant-curvature spaces $\mathbb{S}^n$ and $\mathbb{H}^n$ possess isometry groups as wide as the isometry group of the space $\mathbb{E}^n$ and have no selected points or directions [1]. A geodesic flow on these spaces is equivalent to the energy-preserving motion of a classical particle in a Coulomb field in a Euclidean space [2–5]. The classical and quantum problems of a single particle propagating in the central potential field in such spaces were reviewed in [6]. (We also mention [7-10], which were not mentioned in [6].)

In contrast to the Euclidean case, the phase spaces $\mathbb{S}^n \times \mathbb{S}^n$ and $\mathbb{H}^n \times \mathbb{H}^n$ of two-body problems are not spaces of constant curvature. Only space isometries that preserve the interaction potential enter the symmetry group of such a problem a priori. However, this group does not suffice to ensure the integrability of a two-particle problem. At the same time, no "hidden" symmetries or other integrability tools are known for nontrivial potentials. Moreover, numerical experiments in [11, 12] supported the nonintegrability of the classical restricted two-body problem with natural potentials on the two-dimensional sphere.

The classical mechanical two-body problem was first considered in [6], where the method of the Hamiltonian reduction of systems with symmetries [13] was used to exclude the motion of a system as a whole. The description of reduced mechanical systems, their classification, and the existence conditions for a global dynamics were obtained using explicit analytic coordinate calculations on a computer. In [14], an analogous quantum mechanical system was considered in the two-dimensional case, i.e., on the spaces $\mathbb{S}^2$ and $\mathbb{H}^2$. There, the quantum mechanical Hamiltonian was expressed through the isometry group generators and the radial differential operator. The expression obtained is similar to the structure of the reduced Hamilton function. The idea arises to seek a general procedure for using the symmetry group to simultaneously simplify both the classical and quantum problems without performing cumbersome calculations. We present such a procedure in this paper. The obtained quantum mechanical Hamiltonian is useful for solving at least three problems.

First, we can derive the Hamilton function of a reduced classical mechanical system starting from the obtained quantum mechanical Hamiltonian describing the reduced classical mechanical system on the homogeneous space of a Lie group in terms of orbits of the coadjoint representation of this group (see Sec. 4). Second, using this expression, we can prove that Hamiltonians of a two-particle system with

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a singular interaction are self-adjoint. Third, using the group representation theory, we can reduce the problem of finding the energy levels of the Hamiltonian to a sequence of systems of ordinary differential equations enumerated by the irreducible representations of the isometry group. The two latter problems will be considered in a forthcoming paper.

2. Notation

The sphere $\mathbb{S}^n$ is described as the space $\mathbb{R}^n \cup \{\infty\}$ with the metric

$$g_s = \left(4R^2 \sum_{i=1}^{n} dx_i^2\right) \Big/ \left(1 + \sum_{i=1}^{n} x_i^2\right)^2, \tag{1}$$

where x_i, $i = 1, \dots, n$, are the Cartesian coordinates in $\mathbb{R}^n$ and R is the curvature radius. Let $\rho^s(\cdot, \cdot)$ denote the distance between two points in $\mathbb{S}^n$. The connected component of the isometry group of the space $\mathbb{S}^n$ with the left action is $\mathrm{SO}(n+1)$, while the Killing vector fields on $\mathbb{S}^n$,

$$\begin{aligned} X_{ij}^s &= x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad 1 \le i < j \le n, \\ Y_i^s &= \frac{1}{2}\left(1 + x_i^2 - \sum_{\substack{j=1 \\ j \ne i}}^{n} x_j^2\right) \frac{\partial}{\partial x_i} + x_i \sum_{\substack{j=1 \\ j \ne i}}^{n} x_j \frac{\partial}{\partial x_j}, \quad i = 1, \dots, n, \end{aligned} \tag{2}$$

correspond to a basis in the algebra $\mathrm{so}(n+1)$.

The hyperbolic space $\mathbb{H}^n$ is a unit ball $D^n \subset \mathbb{R}^n$ with the metric

$$g_h = \left(4R^2 \sum_{i=1}^{n} dx_i^2\right) \Big/ \left(1 - \sum_{i=1}^{n} x_i^2\right)^2, \qquad \sum_{i=1}^{n} x_i^2 < 1. \tag{3}$$

Let $\rho^h(\cdot, \cdot)$ denote the distance between two points in the space $\mathbb{H}^n$. The connected component of the isometry group with the left action is then the group $\mathrm{SO}(1, n)$ with the Lie algebra $\mathrm{so}(1, n)$, and the Killing vector fields are

$$\begin{aligned} X_{ij}^h &= x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad 1 \le i < j \le n, \\ Y_i^h &= \frac{1}{2}\left(1 - x_i^2 + \sum_{\substack{j=1 \\ j \ne i}}^{n} x_j^2\right) \frac{\partial}{\partial x_i} - x_i \sum_{\substack{j=1 \\ j \ne i}}^{n} x_j \frac{\partial}{\partial x_j}, \quad i = 1, \dots, n. \end{aligned} \tag{4}$$

3. Representing free Hamiltonians

We now consider the configuration spaces of the two-body problems $Q_s = \mathbb{S}^n \times \mathbb{S}^n$ and $Q_h = \mathbb{H}^n \times \mathbb{H}^n$; the respective Hamiltonians are

$$\widehat{H}_{s,h} = -\frac{1}{2m_1}\Delta_1 - \frac{1}{2m_2}\Delta_2 + U(\rho^{s,h}) \equiv \widehat{H}_0^{s,h} + U(\rho^{s,h}), \tag{5}$$

where Δ_1 and Δ_2 are the Beltrami–Laplace operators of the first and second particle in either the space $\mathbb{S}^n$ or $\mathbb{H}^n$ and U is a central potential.

The general principle of quantum mechanics states [15] that the operator $\widehat{H}_{s,h}$ must be determined on the proper everywhere dense subspace of the space $\mathcal{L}^2(Q_{s,h}, d\mu_{s,h})$ of functions integrated with the square on the space $Q_{s,h}$. This subspace must be such that the operator becomes self-adjoint; the corresponding measure $d\mu_s$ or $d\mu_h$ is the product of two invariants w.r.t. the action of the respective group $\mathrm{SO}(n+1)$ or $\mathrm{SO}(1,n)$ measure on the space $\mathbb{S}^n$ or $\mathbb{H}^n$.

To express the total Hamiltonian $\widehat{H}_{s,h}$ through the radial differential operator and generators of the isometry group, it suffices to find such an expression for the free Hamiltonian. We recall [16, 17] that the Beltrami–Laplace operator Δ acting on the space $\mathbb{S}^n$ or $\mathbb{H}^n$ is a self-adjoint operator with the domains of definition

$$W_s^{2,2} := \{\phi \in \mathcal{L}^2(\mathbb{S}^n, d\mu_s) \mid \Delta\phi \in \mathcal{L}^2(\mathbb{S}^n, d\mu_s)\},$$

$$W_h^{2,2} := \{\phi \in \mathcal{L}^2(\mathbb{H}^n, d\mu_h) \mid \Delta\phi \in \mathcal{L}^2(\mathbb{H}^n, d\mu_h)\}.$$

The action of an operator Δ must be considered in the sense of distributions. The operator Δ on $\mathbb{S}^n$ is essentially self-adjoint on the space $C^\infty(\mathbb{S}^n)$ of smooth functions, and the operator Δ on $\mathbb{H}^n$ is essentially self-adjoint on the space $C_0^\infty(\mathbb{H}^n)$ of finite smooth functions. Hence, the free Hamiltonian $\widehat{H}_0^{s,h}$ is self-adjoint on the product $W_{s,h} := W_{s,h}^{2,2} \otimes W_{s,h}^{2,2}$ of two copies of spaces $W_{s,h}^{2,2}$ respectively corresponding to the first and second particles.

Let submanifolds F_r^s and F_r^h of the respective spaces Q_s and Q_h correspond to a constant value r of the respective functions $\tan(\rho^s/2R)$ and $\tanh(\rho^h/2R)$. The submanifolds F_0^s and F_∞^s are diffeomorphic to $\mathbb{S}^n$ (the value $r = \infty$ corresponds to two diametrically opposite points on the sphere $\mathbb{S}^n$), and F_0^h is diffeomorphic to $\mathbb{H}^n$. For $0 < r < \infty$, the submanifold F_r^s is a homogeneous Riemannian space of the group $\mathrm{SO}(n+1)$ with the stationary subgroup $K = \mathrm{SO}(n-1)$. For $0 < r < 1$, the submanifold F_r^h is a homogeneous Riemannian space of the group $\mathrm{SO}(1,n)$ with the stationary subgroup K.

Up to a zero measure set, $Q_s = \mathbb{R}_+ \times (\mathrm{SO}(n+1)/K)$, where $\mathbb{R}_+ = (0,\infty)$, and $Q_h = I \times (\mathrm{SO}(1,n)/K)$, where $I = (0,1)$. The operators $\widehat{H}_0^{s,h}$ are the Beltrami–Laplace operators for the metric $\tilde{g}_{s,h} = 2m_1 g_{s,h}^{(1)} + 2m_2 g_{s,h}^{(2)}$ on $Q_{s,h}$, where the metrics $g_{s,h}^{(1)}$ and $g_{s,h}^{(2)}$ have either form (1) or (3) and are determined on different copies of the spaces $\mathbb{S}^n$ or $\mathbb{H}^n$ corresponding to the first and second particles.

3.1. The Hamiltonian on the sphere $\mathbb{S}^n$. Given the point $\mathbf{x}_0 \in F_r$, we can identify the layer F_r with the factor space $\mathrm{SO}(n+1)/\mathrm{SO}(n-1)$ using the formula $\mathbf{x} = gK\mathbf{x}_0$, where gK is the left coset of the element g in the group $\mathrm{SO}(n+1)$. Let $(r, y_1, \ldots, y_{2n-1})$ be local coordinates in the neighborhood W of the point $\mathbf{x}_0 \in Q_s$ such that $(y_1, \ldots, y_{2n-1})$ are the coordinates in any nonempty open subset $W \cap F_r$ of the space Q_s. The metric $\tilde{g}_s$ in W then becomes

$$\tilde{g}_s = g_{rr}(r)\, dr^2 + \sum_{i,j=1}^{2n-1} g_{ij}(r, y_1, \ldots, y_{2n-1})\, dy_i\, dy_j.$$

The second term in this formula is the restriction of a metric g_f from the layer F_r to the set $U \cap F_r$. Using the standard expression for the Beltrami–Laplace operator in the local coordinates, we obtain

$$\Delta_{\tilde{g}_s} = (g_{rr} \det g_{ij})^{-1/2} \frac{\partial}{\partial r}\left(\sqrt{g^{rr} \det g_{ij}}\, \frac{\partial}{\partial r}\right) + \Delta_{g_f}. \tag{6}$$

To express the operator Δ_{g_f} on F_r through the generators of the Lie group $\mathrm{SO}(n+1)$, we expand this operator to the group $\mathrm{SO}(n+1)$ using the construction in [18]. Let Γ be a Lie group and Γ_0 be its compact subgroup. The group Γ acts from the left on the homogeneous space Γ/Γ_0. Left-invariant differential operators on the space Γ/Γ_0 can be represented by left-invariant operators on the group Γ that are simultaneously invariant w.r.t. the right action of the group Γ_0. This representation is determined unambiguously up to operator terms vanishing when acting on functions that are right invariant w.r.t. the action of Γ_0.

Indeed, functions on the factor space Γ/Γ_0 are in one-to-one correspondence with functions on the group Γ that are invariant w.r.t. the right action of the subgroup Γ_0. This correspondence is described by the formula $\lambda\colon f \to \tilde{f} := f \circ \pi$, where π is the canonical projection $\Gamma \to \Gamma/\Gamma_0$ and f is a function on the factor space Γ/Γ_0. Let D be a differential operator on Γ that is left-invariant w.r.t. the group Γ and simultaneously right invariant w.r.t. Γ_0, and let f be a smooth function on the factor space Γ/Γ_0. If D_u is a differential operator that acts on the factor space Γ/Γ_0 and is invariant w.r.t. the left action of Γ, then the formula $\widetilde{D_u f} = D\tilde{f}$ yields the correspondence $D \to D_u$.

Let $e_1, \ldots, e_N$ be a basis of the Lie algebra of the group Γ, $N := \dim \Gamma$, and let L_γ and R_γ denote the respective left and right shifts by the element γ. The algebra of left-invariant differential operators on the group Γ over the field $\mathbb{R}$ is generated by left-invariant vector fields $e_1^l, \ldots, e_N^l$, where $e_i^l(\gamma) = dL_\gamma(e_i)$, $\gamma \in \Gamma$, $i = 1, \ldots, N$ [18].

Now let $\Gamma = \mathrm{SO}(n+1)$, $\Gamma_0 = K$, $e_i^r(\gamma) = dR_\gamma(e_i)$, $i = 1, \ldots, N$, $N = (n+1)(n+2)/2$, and $\mathbf{x}_0 = (r_1, \underbrace{0, \ldots, 0}_{n-1}, r_2, \underbrace{0, \ldots, 0}_{n-1}) \in \mathbb{S}^n \times \mathbb{S}^n$, where

$$r_1 = \tan\left(\frac{m_2}{m_1 + m_2} \arctan r\right), \qquad r_2 = -\tan\left(\frac{m_1}{m_1 + m_2} \arctan r\right).$$

The set of Killing vectors X_{ij}^s, Y_i^s, $i, j = 1, \ldots, n$, on the space $\mathbb{S}^n \times \mathbb{S}^n$, which correspond to (2), coincides (up to permutations) with the set

$$\left\{\tilde{e}_i^r(\gamma x_0) = \frac{d}{d\tau}\bigg|_{\tau=0} \exp(\tau e_i)\gamma \mathbf{x}_0\right\}_{i=1}^N, \qquad \mathbf{x}_0 = \mathbf{x}_0(r), \quad 0 < r < \infty, \tag{7}$$

under a proper choice of the basis $e_1^l, \ldots, e_N^l$. Let Δ_f be a second-order differential operator on the group Γ such that $(\Delta_f)_u = \Delta_{g_f}$. This operator is then left invariant and can be expressed in the form[2]

$$\Delta_f\big|_\gamma = \sum_{i,j=1}^N c^{ij} e_i^l(\gamma) e_j^l(\gamma) + \sum_{i=1}^N c^i e_i^l(\gamma),$$

where c^{ij}, c^i are constant on the layer F_r. Let e be the unit element of the group Γ. Obviously, $e_i^r(\mathrm{e}) = e_i^l(\mathrm{e})$, $i = 1, \ldots, N$, and

$$\Delta_f\big|_{\mathrm{e}} = \sum_{i,j=1}^N c^{ij} e_i^r(\mathrm{e}) e_j^r(\mathrm{e}) + \sum_{i=1}^N c^i e_i^r(\mathrm{e}). \tag{8}$$

Therefore,

$$\Delta_{g_f}\big|_{\mathbf{x}_0} = \sum_{i,j=1}^N c^{ij} \tilde{e}_i^r(x_0) \tilde{e}_j^r(x_0) + \sum_{i=1}^N c^i \tilde{e}_i^r(x_0) =: \Delta_{g_f}^{(2)}\big|_{\mathbf{x}_0} + \Delta_{g_f}^{(1)}\big|_{\mathbf{x}_0}.$$

We can find the coefficients c^{ij} as follows. We can treat an ordered set of vectors

$$\{Y_1^s(\mathbf{x}_0), \ldots, Y_n^s(\mathbf{x}_0), X_{12}^s(\mathbf{x}_0), \ldots, X_{1n}^s(\mathbf{x}_0)\}$$

[2]Here, we identify left-invariant vector fields on Γ and the elements of $T_e\Gamma$.

as a basis in the space $T_{\mathbf{x}_0}F_r$. If $\{Y^1,\dots,Y^n,X^2,\dots,X^n\}$ is the dual basis, then

$$g_f\big|_{\mathbf{x}_0} = aY^1\otimes Y^1 + \sum_{i=2}^{n}\Big[Y^1\otimes(\alpha_i Y^i+\beta_i X^i) + \sum_{j=2}^{n}(\alpha_{ij}Y^i\otimes Y^j + \beta_{ij}X^i\otimes X^j + \gamma_{ij}Y^i\otimes X^j)\Big],$$

where

$$a = \tilde g\big|_{\mathbf{x}_0}\big(Y_1^s(\mathbf{x}_0), Y_1^s(\mathbf{x}_0)\big) = 2R^2(m_1+m_2),$$

$$\alpha_i = \tilde g\big|_{\mathbf{x}_0}\big(Y_1^s(\mathbf{x}_0), Y_i^s(\mathbf{x}_0)\big) = 0,$$

$$\beta_i = \tilde g\big|_{\mathbf{x}_0}\big(Y_1^s(\mathbf{x}_0), X_{1i}^s(\mathbf{x}_0)\big) = 0,$$

$$\alpha_{ij} = \tilde g\big|_{\mathbf{x}_0}\big(Y_i^s(\mathbf{x}_0), Y_j^s(\mathbf{x}_0)\big) = 2R^2\sum_{k=1}^{2}\frac{m_k(1-r_k^2)^2}{(1+r_k^2)^2}\delta_{ij}, \tag{9}$$

$$\beta_{ij} = \tilde g\big|_{\mathbf{x}_0}\big(X_{1i}^s(\mathbf{x}_0), X_{1j}^s(\mathbf{x}_0)\big) = 8R^2\sum_{k=1}^{2}\frac{m_k r_k^2}{(1+r_k^2)^2}\delta_{ij},$$

$$\gamma_{ij} = \tilde g\big|_{\mathbf{x}_0}\big(Y_i^s(\mathbf{x}_0), X_{1j}^s(\mathbf{x}_0)\big) = 4R^2\sum_{k=1}^{2}\frac{m_k r_k(1-r_k^2)}{(1+r_k^2)^2}\delta_{ij}, \quad i,j=2,\dots,n.$$

We therefore obtain

$$\Delta_{g_f}^{(2)}\big|_{\mathbf{x}_0} = \frac{1}{a}\big(Y_1^s(\mathbf{x}_0)\big)^2 + \sum_{i=2}^{n}\Big[A_s\big(X_{1i}^s(\mathbf{x}_0)\big)^2 + C_s\big(Y_i^s(\mathbf{x}_0)\big)^2 + B_s\big\{X_{1i}^s(\mathbf{x}_0), Y_i^s(\mathbf{x}_0)\big\}\Big], \tag{10}$$

where $\{\cdot,\cdot\}$ denotes the anticommutator and

$$A_s = \frac{m_1(1-r_1^2)^2(1+r_2^2)^2 + m_2(1+r_1^2)^2(1-r_2^2)^2}{8R^2 m_1 m_2 (r_1-r_2)^2(1+r_1r_2)^2},$$

$$B_s = -\frac{m_1 r_1(1-r_1^2)(1+r_2^2)^2 + m_2 r_2(1-r_2^2)(1+r_1^2)^2}{4R^2 m_1 m_2 (r_1-r_2)^2(1+r_1r_2)^2},$$

$$C_s = \frac{m_1 r_1^2(1+r_2^2)^2 + m_2 r_2^2(1+r_1^2)^2}{2R^2 m_1 m_2 (r_1-r_2)^2(1+r_1r_2)^2}.$$

The functions A_s, B_s, and C_s can be expressed through the coordinate r,

$$A_s(r) = \frac{1}{2R^2}\left(\frac{(1+r^2)^2}{8mr^2} + \frac{1-r^4}{8mr^2}\cos\zeta + \frac{1+r^2}{4m_1m_2r}(m_1-m_2)\sin\zeta\right),$$

$$B_s(r) = \frac{1}{2R^2}\left(\frac{m_2-m_1}{m_1m_2r}(1+r^2)\cos\zeta + \frac{1-r^4}{2mr^2}\sin\zeta\right),$$

$$C_s(r) = \frac{1}{2R^2}\left(\frac{(1+r^2)^2}{8mr^2} - \frac{1-r^4}{8mr^2}\cos\zeta - \frac{1+r^2}{4m_1m_2r}(m_1-m_2)\sin\zeta\right),$$

$$\zeta = 2\frac{m_1-m_2}{m_1+m_2}\arctan r, \qquad m = \frac{m_1m_2}{m_1+m_2}.$$

The operators $\Delta_{g_f}\big|_{\mathbf{x}_0}$ and $\Delta^{(2)}_{g_f}\big|_{\mathbf{x}_0}$ (10) are invariant w.r.t. reflections of the sphere $\mathbb{S}^n$, $T_k\colon x_k \to -x_k$, $x_j \to x_j$, $j \neq k$; the operator $\Delta^{(1)}_{g_f}\big|_{\mathbf{x}_0}$ is then also invariant w.r.t. these transformations. However, this is possible only for vanishing first-order operators with constant coefficients, and we have $c^i = 0$, $i = 1, \dots, N$.

Letting $Y_1^{s,l}$, $X_i^{s,l}$, and $Y_i^{s,l}$ denote the left-invariant vector fields on the group $\mathrm{SO}(n+1)$ that correspond to the respective vectors $Y_1^s(\mathbf{x}_0)$, $X_{1i}^s(\mathbf{x}_0)$, and $Y_i^s(\mathbf{x}_0)$, $i = 2, \dots, n$, we obtain

$$\Delta_f = \frac{1}{a}(Y_1^{s,l})^2 + \sum_{i=2}^{n}\left[A_s(X_i^{s,l})^2 + C_s(Y_i^{s,l})^2 + \frac{1}{4}B_s\{X_i^{s,l}, Y_i^{s,l}\}\right]. \tag{11}$$

We thus find the operator Δ_f up to terms annihilated by the functions that are right invariant w.r.t. the subgroup Γ_0. Direct calculations show that this operator is right invariant w.r.t. the subgroup K.

We now find the first term in expression (6) for the operator $\Delta_{\tilde g}$. At the point $\mathbf{x}_0$, we have

$$\frac{\partial}{\partial r} = \frac{m_2}{m_1+m_2}\frac{1+r_1^2}{1+r^2}\frac{\partial}{\partial r_1} - \frac{m_1}{m_1+m_2}\frac{1+r_2^2}{1+r^2}\frac{\partial}{\partial r_2}$$

and therefore

$$g_{rr} = \tilde g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = \frac{8R^2 m_1 m_2}{(m_1+m_2)(1+r^2)^2}. \tag{12}$$

By virtue of formulas (9), we obtain

$$\Delta_{\tilde g} = \frac{(1+r^2)^n}{8mR^2 r^{n-1}}\frac{\partial}{\partial r}\left(\frac{r^{n-1}}{(1+r^2)^{n-2}}\frac{\partial}{\partial r}\right) + \Delta_{g_f},$$

where the first term is the radial part of the Hamiltonian of a single particle with the mass m.

The explicit expression for the measure $d\mu_s$, which corresponds to the metric $\tilde g$ on the space Q_s at the point $\mathbf{x}_0$, is (up to a constant multiplier)

$$d\mu_s\big|_{\mathbf{x}_0} = \frac{r^{n-1}}{(1+r^2)^n}dr \wedge Y^1 \wedge \dots \wedge Y^n \wedge X^2 \wedge \dots \wedge X^n.$$

The measure $d\mu_s$ is invariant w.r.t. the left action of the group $\mathrm{SO}(n+1)$ and can therefore be represented in the form $d\mu_s = d\nu_s \otimes d\mu_f$, where the measure on the set $\mathbb{R}_+ = (0,\infty)$ $d\nu_s = r^{n-1}dr/(1+r^2)^n$ coincides with the one-particle measure and $d\mu_f$ is the measure on the space $\mathrm{SO}(n+1)/K$ that is left invariant w.r.t. the action of the group $\mathrm{SO}(n+1)$.

Each Lie group admits unique (up to a constant multiplier) left-invariant and right-invariant measures (the Haar measures [19]). For the groups $\mathrm{SO}(n+1)$ and $\mathrm{SO}(1,n)$ under consideration, such measures are two-side invariant. There hence exists a unique two-side-invariant measure $d\eta_s$ on the group $\mathrm{SO}(n+1)$ such that the integral of an integrable function f on the space $\mathrm{SO}(n+1)/K$ w.r.t. the measure $d\mu_f$ equals the integral of the function $\tilde f$ on the group $\mathrm{SO}(n+1)$ w.r.t. the measure $d\eta_s$.

Given a subgroup Γ_0 of a Lie group Γ, we let $\mathcal{L}^2(\Gamma, \Gamma_0, d\eta)$ denote the space of square-integrable functions on the group Γ (w.r.t. the measure $d\eta$ on Γ) that are invariant w.r.t. the right action of the subgroup Γ_0.

Theorem 1. *The free quantum Hamiltonian of the two-particle system on the sphere $\mathbb{S}^n$ is a self-adjoint differential operator (on the manifold $\tilde Q_s = \mathbb{R}_+ \times \mathrm{SO}(n+1)$) in the space $\mathcal{H}_s$,*

$$\widehat H_0^s = -\frac{(1+r^2)^n}{8mR^2 r^{n-1}}\frac{\partial}{\partial r}\left(\frac{r^{n-1}}{(1+r^2)^{n-2}}\frac{\partial}{\partial r}\right) - \Delta_f, \tag{13}$$

with the domain of definition

$$D_s := D_s^{(1)} \otimes D_s^{(2)} \subset \mathcal{H}_s := \mathcal{L}^2(\mathbb{R}_+, d\nu_s) \otimes \mathcal{L}^2\big(\mathrm{SO}(n+1), K, d\eta_s\big),$$

where

$$D_s^{(1)} := \big\{\phi \in \mathcal{L}^2(\mathbb{R}_+, d\nu_s) \mid \Delta_s^{(1)}\phi \in \mathcal{L}^2(\mathbb{R}_+, d\nu_s)\big\},$$

$$D_s^{(2)} := \big\{\phi \in \mathcal{L}^2\big(\mathrm{SO}(n+1), K, d\eta_s\big) \mid \Delta_f\phi \in \mathcal{L}^2\big(\mathrm{SO}(n+1), K, d\eta_s\big)\big\},$$

$$\Delta_s^{(1)} := -\frac{(1+r^2)^n}{r^{n-1}}\frac{\partial}{\partial r}\left(\frac{r^{n-1}}{(1+r^2)^{n-2}}\frac{\partial}{\partial r}\right),$$

the subgroup K is isomorphic to the group $\mathrm{SO}(n-1)$, and $d\eta_s$ is an unambiguously determined (up to a constant multiplier) two-side-invariant measure on the group $\mathrm{SO}(n+1)$. There hence exists an isometry of the initial space of functions $\mathcal{L}^2(Q_s, d\mu_s)$ on the space $\mathcal{H}_s$ that generates the isomorphism of Hamiltonians. The space D_s is everywhere dense in $\mathcal{H}_s$.

Proof. Expression (6) represents the Hamiltonian $\widehat{H}_0^s$ in the coordinate system in which Q_s is presented as the direct product $\mathbb{R}_+ \times \mathrm{SO}(n+1)/\mathrm{SO}(n-1)$ up to a zero-measure set $F_0^s \cup F_\infty^s$, which is inessential when studying functions that are integrable over this measure. Therefore,

$$\mathcal{L}^2(Q_s, d\mu_s) = \mathcal{L}^2(\mathbb{R}_+, d\nu_s) \otimes \mathcal{L}^2\big(\mathrm{SO}(n+1)/\mathrm{SO}(n-1), d\mu_f\big).$$

The isometry $\lambda\colon f \to \tilde{f}$ of the spaces $\mathcal{L}^2\big(\mathrm{SO}(n+1)/\mathrm{SO}(n-1), d\mu_f\big)$ and $\mathcal{L}^2\big(\mathrm{SO}(n+1), \mathrm{SO}(n-1), d\eta_s\big)$ generates the isometry $\mathrm{id}\otimes\lambda$ of the spaces $\mathcal{L}^2(\mathbb{R}_+, d\nu_s) \otimes \mathcal{L}^2\big(\mathrm{SO}(n+1)/\mathrm{SO}(n-1), d\mu_f\big)$ and $\mathcal{H}_s$. The calculations imply that the isometry $\mathrm{id}\otimes\lambda$ transforms operator (6) into operator (13); the space W_s then transforms into the space D_s.

Remark. In the case $n = 2$, this result can be obtained by treating the basis of left-invariant vector fields on the group $\mathrm{SO}(n+1) = \mathrm{SO}(3)$ as the movable reper on the layer F_r [14]. For $n > 2$, such a consideration is impossible because the action of the group $\mathrm{SO}(n+1)$ on the layer F_r is not free and the projections of left-invariant vector fields on the group $\mathrm{SO}(n+1)$ to the space $\mathrm{SO}(n+1)/\mathrm{SO}(n-1)$ are not uniquely determined. Lifting the Hamiltonian to the symmetry group, we express the Hamiltonian via the group generators.

3.2. The Hamiltonian on the hyperbolic space $\mathbb{H}^n$. The formal change $x_j \to ix_j$, $j = 1, \ldots, n$, $r \to ir$, $R \to iR$ (i is the imaginary unit) transforms objects on the sphere $\mathbb{S}^n$ into objects on the hyperbolic space $\mathbb{H}^n$ (see also [6]), and we thus obtain the expression for the two-particle free Hamiltonian on the space $\mathbb{H}^n$,

$$\begin{aligned}\widehat{H}_0^h = &-\frac{(1-r^2)^n}{8mR^2r^{n-1}}\frac{\partial}{\partial r}\left(\frac{r^{n-1}}{(1-r^2)^{n-2}}\frac{\partial}{\partial r}\right) - \frac{1}{a}(Y_1^{h,l})^2 - \\ &-\sum_{k=2}^{n}\left[A_h(X_k^{h,l})^2 - C_h(Y_k^{h,l})^2 + \frac{1}{4}B_h\{X_k^{h,l}, Y_k^{h,l}\}\right],\end{aligned} \tag{14}$$

where the vector fields $X_k^{h,l}$ and $Y_k^{h,l}$ correspond to fields (4) as the fields $X_k^{s,l}$ and $Y_k^{s,l}$ correspond to fields (2),

$$A_h(r) = \frac{1}{2R^2}\left(\frac{(1-r^2)^2}{8mr^2} + \frac{1-r^4}{8mr^2}\cosh\zeta - \frac{1-r^2}{4m_1m_2r}(m_1-m_2)\sinh\zeta\right),$$

$$B_h(r) = \frac{1}{2R^2}\left(\frac{m_2-m_1}{m_1m_2r}(1-r^2)\cosh\zeta + \frac{1-r^4}{2mr^2}\sinh\zeta\right),$$

$$C_h(r) = \frac{1}{2R^2}\left(\frac{(1-r^2)^2}{8mr^2} - \frac{1-r^4}{8mr^2}\cosh\zeta + \frac{1-r^2}{4m_1m_2r}(m_1-m_2)\sinh\zeta\right),$$

$$\zeta = 2\frac{m_1-m_2}{m_1+m_2}\operatorname{arctanh} r.$$

Theorem 2. *The free quantum two-particle Hamiltonian on the hyperbolic space $\mathbb{H}^n$ is a self-adjoint differential operator* (14) *(on the manifold $\widetilde{Q}_h = I \times \mathrm{SO}(1,n)$) in the space $\mathcal{H}_h$ with the domain of definition*

$$D_h := D_h^{(1)} \otimes D_h^{(2)} \subset \mathcal{H}_h := \mathcal{L}^2(\mathbb{R}_+, d\nu_h) \otimes \mathcal{L}^2(\mathrm{SO}(1,n), K, d\eta_h),$$

where

$$D_h^{(1)} := \{\phi \in \mathcal{L}^2(\mathbb{R}_+, d\nu_h) \mid \Delta_h^{(1)}\phi \in \mathcal{L}^2(\mathbb{R}_+, d\nu_h)\},$$

$$D_h^{(2)} := \{\phi \in \mathcal{L}^2(\mathrm{SO}(1,n), K, d\eta_h) \mid \Delta_h\phi \in \mathcal{L}^2(\mathrm{SO}(1,n), K, d\eta_h)\},$$

$$\Delta_h^{(1)} := -\frac{(1-r^2)^n}{r^{n-1}}\frac{\partial}{\partial r}\left(\frac{r^{n-1}}{(1-r^2)^{n-2}}\frac{\partial}{\partial r}\right), \qquad d\nu_h = \frac{r^{n-1}dr}{(1-r^2)^n},$$

$$\Delta_h := -\frac{1}{a}(Y_1^{h,l})^2 - \sum_{k=2}^{n}\left[A_h(X_k^{h,l})^2 - C_h(Y_k^{h,l})^2 + \frac{1}{4}B_h\{X_k^{h,l}, Y_k^{h,l}\}\right],$$

and $d\eta_h$ is a unique (up to a constant multiplier) two-side-invariant measure on the group $\mathrm{SO}(1,n)$.

Proof. The proof is analogous to the proof of Theorem 1.

4. Reducing the Hamiltonian dynamic system on a cotangent fiber bundle of a homogeneous space

Let Γ be a Lie group with the algebra g and Γ_0 be a subgroup of the group Γ with the algebra $g_0 \subset g$ that acts on Γ from the right. We let $M = T^*\Gamma_1$ denote the cotangent fiber bundle of the homogeneous space $\Gamma_1 = \Gamma/\Gamma_0$ endowed with the standard symplectic structure. The standard left action of the group Γ on M is Poissonian [13]. We let $\Phi\colon M \to g^*$ denote the corresponding moment map and H be a Γ-invariant function on M. We now apply the Marsdain–Weinstein reduction method to the Hamiltonian dynamic system with the function H on the manifold M. It is well known [13] that for $\Gamma_0 = \{\mathrm{e}\}$, the reduced phase space is symplectomorphic to an orbit of the group Γ in the cotangent fiber bundle endowed with the Kirillov form. The construction below generalizes this statement.

Let O_{β_0} be the orbit of the cotangent action of the group Γ on g^* that contains the point $\beta_0 \in g^*$, and let $O'_{\beta_0} := \{\beta \in O_{\beta_0} \,|\, \beta|_{g_0} = 0\}$. Obviously, $\mathrm{Ad}^*_{\Gamma_0} O'_{\beta_0} = O'_{\beta_0}$. Let $\widetilde{O}_{\beta_0} = O'_{\beta_0}/\mathrm{Ad}^*_{\Gamma_0}$ and $\pi\colon O'_{\beta_0} \to \widetilde{O}_{\beta_0}$ be the canonical projection. Let ω be the restriction of the Kirillov form on O'_{β_0}. Therefore, for the elements $X, Y \in T_\beta O'_{\beta_0}$, $\beta \in O'_{\beta_0}$, of the form

$$X = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}^*_{\exp(tX')}\beta, \qquad Y = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}^*_{\exp(tY')}\beta, \quad X', Y' \in g,$$

we have $\omega(X,Y) = \beta([X',Y'])$. Because $\operatorname{Ad}^*_{\exp(tX')}\beta\big|_{g_0} = 0$, we have

$$\beta([X',Y'_0]) = \frac{d}{dt}\bigg|_{t=0} \operatorname{Ad}^*_{\exp(tX')}\beta(Y'_0) = 0$$

for any element $Y'_0 \in g_0$. The 2-form $\widetilde{\omega}$ is therefore correctly defined on $T\widetilde{O}_{\beta_0}$, and we have $\widetilde{\omega}(\widetilde{X},\widetilde{Y}) = \omega(d\pi^{-1}\widetilde{X}, d\pi^{-1}\widetilde{Y})$ for $\widetilde{X} \in T_{\pi\beta}\widetilde{O}_{\beta_0}$ and $\widetilde{Y} \in T_{\pi\beta}\widetilde{O}_{\beta_0}$.

Theorem 3. *The reduced phase space $\widetilde{M}_{\beta_0}$ that corresponds to the value β_0 of the moment map is symplectomorphically equivalent to the symplectic space $(\widetilde{O}_{\beta_0}, \widetilde{\omega})$.*

Proof. We treat a point $x \in M_{\beta_0} := \Phi^{-1}(\beta_0) \subset M$ as an orbit $O_{x'}$ of a point $x' = (\gamma, p) \in T^*\Gamma$, $\gamma \in \Gamma$, $p \in T^*_\gamma\Gamma$, w.r.t. the right action of the subgroup Γ_0 on $T^*\Gamma$. To avoid cumbersome notation, we preserve the symbols L_{γ_1} and R_{γ_1} for the respective left, $(\gamma,p) \to (\gamma_1\gamma, L^*_{\gamma_1^{-1}}p)$, and right, $(\gamma,p) \to (\gamma\gamma_1, R^*_{\gamma_1^{-1}}p)$, actions of the element $\gamma_1 \in \Gamma$ on $T^*\Gamma$. Following the definition of the moment map, if

$$X = \frac{d}{dt}\bigg|_{t=0} L_{\exp(tX')}\gamma, \quad X' \in g, \quad X \in T_\gamma\Gamma,$$

then $p(X) = \beta_0(X')$, i.e., $p = R^*_{\gamma^{-1}}\beta_0$. If $X' \in \operatorname{Ad}_\gamma g_0$, then $X \in d\pi_1(T_{x'}O_{x'})$, where $\pi_1 \colon T^*\Gamma \to \Gamma$ is the standard projection, and $p(X) = 0$. Hence, $\operatorname{Ad}^*_\gamma \beta_0\big|_{g_0} = 0$. We set $O = \{x' = (\gamma,p) \in T^*\Gamma \,|\, \operatorname{Ad}^*_\gamma\beta_0\big|_{g_0} = 0, p = R^*_{\gamma^{-1}}\beta_0\}$. Let $\tau \colon O \to g^* = T^*_e\Gamma$ be the mapping $\tau(\gamma,p) = L^*_\gamma p$. The diagram

$$\begin{array}{ccc} T^*\Gamma & \xrightarrow{L_{\gamma^{-1}}} & T^*\Gamma \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} \\ g^* & \xrightarrow{\operatorname{Ad}^*_\gamma} & g^* \end{array}$$

is commutative [13], and the mapping τ sends an orbit of the stabilizing subgroup Γ_{β_0} on $T^*\Gamma$ to a single point. By the definition of the set O, we have $\tau(O) = O'_{\beta_0}$, and the map τ sends the element (γ,p) to $\operatorname{Ad}^*_\gamma\beta_0$, and the element $R_{\gamma_0}(\gamma,p)$ is therefore sent to $\operatorname{Ad}^*_{\gamma\gamma_0}\beta_0 = \operatorname{Ad}^*_{\gamma_0} \circ \operatorname{Ad}^*_\gamma\beta_0$. Orbits of the right action of the group Γ_0 on O are therefore transformed into orbits of the coadjoint action of the group Γ_0 on O'_{β_0}. The mapping τ therefore induces the diffeomorphism

$$\phi \colon \widetilde{M}_{\beta_0} = \Gamma_{\beta_0}\backslash M_{\beta_0} = \Gamma_{\beta_0}\backslash(O/\Gamma_0) \to O'_{\beta_0}/\operatorname{Ad}^*_{\Gamma_0} = \widetilde{O}_{\beta_0}.$$

The remaining fact that the symplectic form $\widehat{\omega}$ on $\widetilde{M}_{\beta_0}$ is transformed by the mapping ϕ into the form $-\widetilde{\omega}$ follows from its validity for the case $\Gamma_0 = \{e\}$, the possibility to represent tangent vectors on the space $\widetilde{M}_{\beta_0}$ via tangent vectors on O, and the commutativity of the diagram

$$\begin{array}{ccc} O & \xrightarrow{R_{\gamma_0}} & O \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ O'_{\beta_0} & \xrightarrow{R_{\gamma_0}} & O'_{\beta_0} \end{array}$$

for any $\gamma_0 \in \Gamma_0$.

Because the form $\widehat{\omega}$ is symplectic, we obtain the following corollary.

Corollary. *The form $\widetilde{\omega}$ is symplectic (i.e., nondegenerate and closed) on $\widetilde{O}_{\beta_0}$.*

5. Reducing the classical two-body dynamic system

It was noted in [6] that a two-body problem in the spaces $\mathbb{H}^n$ and $\mathbb{S}^n$, $n \geq 3$, already becomes generous for $n = 3$ because the motion of two elements from the space $T^*\mathbb{H}^n$ or $T^*\mathbb{S}^n$ is always restricted to a subspace $T^*\mathbb{H}^3 \subset T^*\mathbb{H}^n$ or $T^*\mathbb{S}^3 \subset T^*\mathbb{S}^n$. Two material points with central interaction are therefore always bound to a subspace $\mathbb{H}^3$ or $\mathbb{S}^3$, and we consider the case $n = 3$ in what follows.

5.1. Two-particle problem on the sphere $\mathbb{S}^3$. We endow the space $M = T^*Q_s$ with the standard symplectic structure. Following Sec. 3.1, we can represent the manifold M in the form

$$M = T^*\mathbb{R}_+ \times T^*\big(\mathrm{SO}(4)/\mathrm{SO}(2)\big) \tag{15}$$

up to a zero-measure set corresponding to the values $r = 0, \infty$. The symmetry group SO(4) acts only on the second multiplier in product (15), and the construction in Sec. 4 is easily generalized to this case. After the reduction, we obtain the space

$$\widetilde{M}_{\beta_0} = T^*\mathbb{R}_+ \times \widetilde{Q}_{\beta_0},$$

instead of (15), where $\widetilde{Q}_{\beta_0}$ is constructed for the spaces $\Gamma = \mathrm{SO}(4)$ and $\Gamma_0 = \mathrm{SO}(2)$ as in Sec. 4.

We introduce actual coordinates on the space $\widetilde{M}_{\beta_0}$ and express the Hamilton function through these coordinates using formula (13). In the case $n = 3$, Killing vector fields (2) are X_{12}^s, X_{31}^s, X_{23}^s, Y_1^s, Y_2^s, and Y_3^s. For simplicity, we use the same notation for the basis of the algebra so(4) (omitting the superscript s) in accordance with (7). Let X^{12}, X^{31}, X^{23}, Y^1, Y^2, and Y^3 be the dual basis. We also introduce another basis of the algebra so(4),

$$L_1 = \frac{1}{2}(X_{23}^s + Y_1^s), \qquad L_2 = \frac{1}{2}(X_{31}^s + Y_2^s), \qquad L_3 = \frac{1}{2}(X_{12}^s + Y_3^s),$$

$$G_1 = \frac{1}{2}(X_{23}^s - Y_1^s), \qquad G_2 = \frac{1}{2}(X_{31}^s - Y_2^s), \qquad G_3 = \frac{1}{2}(X_{12}^s - Y_3^s).$$

In this basis, we have

$$[L_i, L_j] = \sum_{k=1}^{3} \varepsilon_{ijk} L_k, \qquad [G_i, G_j] = \sum_{k=1}^{3} \varepsilon_{ijk} G_k, \qquad [L_i, G_j] = 0, \quad i, j = 1, 2, 3, \tag{16}$$

where ε_{ijk} is the totally antisymmetric tensor and $\varepsilon_{123} = 1$. This basis corresponds to the decomposition $\mathrm{so}(4) = \mathrm{so}(3) \oplus \mathrm{so}(3)$. Let

$$L^1 = X^{23} + Y^1, \qquad L^2 = X^{31} + Y^2, \qquad L^3 = X^{12} + Y^3,$$

$$G^1 = X^{23} - Y^1, \qquad G^2 = X^{31} - Y^2, \qquad G^3 = X^{12} - Y^3$$

be the dual basis. We let

$$\mathbf{p} = p_1 X^{23} + p_2 X^{31} + p_3 X^{12} + p_4 Y^1 + p_5 Y^2 + p_6 Y^3 = \sum_{i=1}^{3} (u_i L^i + v_i G^i) \tag{17}$$

denote an arbitrary element of the space $\mathrm{so}^*(4)$. The correspondence between the classical and quantum Hamiltonians and formulas (11) and (13) imply the classical Hamilton function

$$H_s = \frac{(1+r^2)^2}{8mR^2} p_r^2 + \frac{1}{a} p_4^2 + A_s(p_2^2 + p_3^2) + C_s(p_5^2 + p_6^2) + \frac{1}{2} B_s(p_3 p_5 - p_2 p_6) + U(r),$$

where p_r is the momentum conjugate to the coordinate r. Substituting $p_i = u_i + v_i$ and $p_{3+i} = u_i - v_i$, $i = 1, 2, 3$, we obtain

$$H_s = \frac{(1+r^2)^2}{8mR^2} p_r^2 + \frac{1}{a}(u_1 - v_1)^2 + A_s\big((u_2 + v_2)^2 + (u_3 + v_3)^2\big) +$$
$$+ C_s\big((u_2 - v_2)^2 + (u_3 - v_3)^2\big) + B_s(u_2 v_3 - v_2 u_3) + U(r).$$

We now construct the canonically conjugate coordinates on the space $\widetilde{O}_{\beta_0}$. Because of the special choice of the point $\mathbf{x}_0$ on the layer F_r in Sec. 3.1, its stabilizer SO(2) is generated by the element X_{23}. It is well known that orbits of the coadjoint action of the group SO(3) are spheres and their Kirillov form is the sphere area. The orbit O_{β_0} can therefore be represented as the set of elements of form (17) such that the coordinates u_i and v_i, $i = 1, 2, 3$, satisfy the relations

$$u_1^2 + u_2^2 + u_3^2 = \mu^2, \qquad v_1^2 + v_2^2 + v_3^2 = \nu^2, \tag{18}$$

where μ and ν are nonnegative real numbers. The subset $O'_{\beta_0} \subset O_{\beta_0}$ comprises those elements of O_{β_0} that are annihilated by the elements X_{23}, and we must therefore add the condition $p_1 = u_1 + v_1 = 0$ to Eqs. (18) in order to describe the set O'_{β_0}.

We first consider the case $\mu, \nu > 0$. Let u, ψ, and χ be the coordinates on the space O'_{β_0} determined by the equations

$$u_1 = -v_1 = u, \qquad u_2 = \sqrt{\mu^2 - u^2}\sin\psi, \qquad u_3 = \sqrt{\mu^2 - u^2}\cos\psi,$$
$$v_2 = \sqrt{\nu^2 - u^2}\sin\chi, \qquad v_3 = \sqrt{\nu^2 - u^2}\cos\chi, \qquad -\min\{\mu,\nu\} \le u \le \min\{\mu,\nu\}.$$

Restricting the Kirillov form from O_{β_0} to O'_{β_0}, we obtain

$$\omega = \frac{1}{\mu^2}(u_1\, du_2 \wedge du_3 + u_2\, du_3 \wedge du_1 + u_3\, du_1 \wedge du_2) +$$
$$+ \frac{1}{\nu^2}(v_1\, dv_2 \wedge dv_3 + v_2\, dv_3 \wedge v_1 + v_3\, dv_1 \wedge dv_2) = du \wedge d(\psi - \chi). \tag{19}$$

The coadjoint action of the one-parameter group corresponding to the element X_{23} on O'_{β_0} is $u \to u$, $\psi \to \psi + \xi$, $\chi \to \chi + \xi$, $0 \le \xi < 2\pi$; the coordinates $\phi = \psi - \chi$ and $p_\phi = u$ on $\widetilde{O}_{\beta_0}$ are therefore canonically conjugate. The space $\widetilde{O}_{\beta_0}$ is actually diffeomorphic to the two-dimensional sphere. The coordinate system p_ϕ, ϕ is singular at the points $p_\phi = \pm\min\{\mu,\nu\}$. It differs from the coordinate system on the reduced space used in [6]. The reduced Hamilton function is

$$\widetilde{H}_s = \frac{(1+r^2)^2}{8mR^2}p_r^2 + \frac{4p_\phi^2}{a} + A_s\Big(\mu^2 + \nu^2 - 2p_\phi^2 + 2\sqrt{\mu^2 - p_\phi^2}\sqrt{\nu^2 - p_\phi^2}\cos\phi\Big) +$$
$$+ C_s\Big(\mu^2 + \nu^2 - 2p_\phi^2 - 2\sqrt{\mu^2 - p_\phi^2}\sqrt{\nu^2 - p_\phi^2}\cos\phi\Big) + B_s\sqrt{\mu^2 - p_\phi^2}\sqrt{\nu^2 - p_\phi^2}\sin\phi + U(r).$$

In the case $\mu = 0$ and $\nu > 0$ (or $\nu = 0$ and $\mu > 0$), we obtain the conditions $u_1 = u_2 = u_3 = v_1 = 0$ for O'_{β_0}; therefore, $O'_{\beta_0} = \mathbb{S}^1$ and $\widetilde{O}_{\beta_0} = \text{pt}$. The reduced phase space is $T^*\mathbb{R}_+$ with the Hamilton function

$$\widetilde{H}_s = \frac{(1+r^2)^2}{8mR^2}\left(p_r^2 + \frac{\nu^2}{r^2}\right).$$

In the case $\mu = \nu = 0$, we obtain

$$\widetilde{O}_{\beta_0} = O'_{\beta_0} = \text{pt}, \qquad M = T^*\mathbb{R}_+, \qquad \widetilde{H}_s = \frac{(1+r^2)^2}{8mR^2}p_r^2.$$

5.2. Two-particle problem in the space $\mathbb{H}^3$. Because the Lie algebra so$(1,3)$ is simple, we cannot represent orbits of the adjoint action of the group SO$(1,3)$ in the direct product form similarly to Sec. 5.1. However, dynamic systems on the sphere $\mathbb{S}^3$ and in the space $\mathbb{H}^3$ are related by the formal substitution (see Sec. 3.2 and [6]), and we can use the following construction.

Let $L_1 = X_{23}$, $L_2 = X_{31}$, $L_3 = X_{12}$, Y_1, Y_2, and Y_3 be the basis in the algebra so$(1,3)$ that corresponds to Killing vector fields (4) and L^1, L^2, L^3, Y^1, Y^2, and Y^3 comprise the dual basis. Let $\mathbf{p} = p_1L^1 + p_2L^2 + p_3L^3 + p_4Y^1 + p_5Y^2 + p_6Y^3$ be an arbitrary element from so$^*(1,3)$. Direct calculation shows that the expressions

$$I_1 = p_1^2 + p_2^2 + p_3^2 - p_4^2 - p_5^2 - p_6^2, \qquad I_2 = p_1p_4 + p_2p_5 + p_3p_6$$

are invariant w.r.t. the adjoint action of the group SO$(1,3)$. Similarly to Sec. 5.1, we express the Hamilton function through the coordinates on so$^*(1,3)$,

$$H_h = \frac{(1-r^2)^2}{8mR^2}p_r^2 + \frac{1}{a}p_4^2 + A_h(p_2^2 + p_3^2) - C_h(p_5^2 + p_6^2) + \frac{1}{2}B_h(p_3p_5 - p_2p_6) + U(r). \tag{20}$$

Let O_{β_0} be the orbit of the coadjoint action of the group SO$(1,3)$ determined by the conditions $I_1 = \mu$ and $I_2 = \nu$, $\mu, \nu \in \mathbb{R}$. The stabilizing subgroup of the point $\mathbf{x}_0 \in F_r$ is generated by the element L_1. This subgroup acts by simultaneous rotation in the planes (p_2, p_3) and (p_5, p_6). The submanifold O'_{β_0} is determined by the equations $I_1 = \mu$, $I_2 = \nu$, and $p_1 = 0$. The coordinates p_4, ψ, and χ on this submanifold are

$$\begin{aligned} p_2 &= u\cosh\psi\cos\chi + v\sinh\psi\sin\chi, & p_3 &= v\sinh\psi\cos\chi - u\cosh\psi\sin\chi, \\ p_5 &= v\cosh\psi\cos\chi - u\sinh\psi\sin\chi, & p_6 &= -u\sinh\psi\cos\chi - v\cosh\psi\sin\chi, \end{aligned} \tag{21}$$

where $p_4, \psi \in \mathbb{R}$, $\chi \in \mathbb{R}$ (mod 2π), and u and v are determined by the equations

$$u^2 - v^2 = \mu + p_4^2, \qquad uv = \nu. \tag{22}$$

Two solutions of Eqs. (22) differ in sign, and it suffices to choose either of them. The action of the stationary subgroup SO(2) is the rotation $\chi \to \chi + \xi$. The reduced phase space $\widetilde{O}_{\beta_0}$ is obtained from O'_{β_0} if we "forget" the coordinate χ. The space $\widetilde{O}_{\beta_0}$ is diffeomophic to $\mathbb{R}^2$.

We use the degenerate Poisson brackets on so$^*(1,3)$ that correspond to the Kirillov form to construct the canonically conjugate coordinates on the space $\widetilde{O}_{\beta_0}$. These brackets can be constructed for an arbitrary Lie algebra g as follows [20].

Let $\{e_i\}_{i=1}^n$ be a basis of an algebra g, $[e_i, e_j] = c_{ij}^k e_k$, and $\{x_i\}_{i=1}^n$ be the coordinates on g^* that correspond to the dual basis $\{e^i\}_{i=1}^n$. Let f_1 and f_2 be smooth functions on g^*. Their Poisson brackets are then

$$\{f_1, f_2\} = -\sum_{i,j,k=1}^{n} c_{ij}^k x_k \frac{\partial f_1}{\partial x_i}\frac{\partial f_2}{\partial x_j}.$$

The restriction of these brackets to the coadjoint action orbit is nondegenerate. In the problem under consideration, direct calculations with the formulas

$$\psi = \frac{1}{4}\log\left(\frac{(p_2 - p_6)^2 + (p_5 + p_3)^2}{(p_2 + p_6)^2 + (p_5 - p_3)^2}\right),$$

$$\chi = \frac{1}{2}\left(\arctan\left(\frac{p_5 - p_3}{p_2 + p_6}\right) - \arctan\left(\frac{p_5 + p_3}{p_2 - p_6}\right)\right),$$

$$[L_i, L_j] = \sum_{k=1}^{3}\varepsilon_{ijk}L_k, \qquad [Y_i, Y_j] = -\sum_{k=1}^{3}\varepsilon_{ijk}L_k, \qquad [L_i, Y_j] = \sum_{k=1}^{3}\varepsilon_{ijk}Y_k,$$

yield the relations

$$\{p_4, \psi\} = 1, \qquad \{p_4, \chi\} = 0, \qquad \{\psi, \chi\} = 0.$$

The symplectic structure on $\widetilde{O}_{\beta_0}$ is then $dp_4 \wedge d\psi$. By virtue of (21), we obtain

$$p_2^2 + p_3^2 = \frac{1}{2}\Big(\mu + p_4^2 + \sqrt{(\mu + p_4^2)^2 + 4\nu^2}\,\cosh 2\psi\Big),$$

$$p_5^2 + p_6^2 = \frac{1}{2}\Big(-\mu - p_4^2 + \sqrt{(\mu + p_4^2)^2 + 4\nu^2}\,\cosh 2\psi\Big),$$

$$p_3 p_5 - p_2 p_6 = \frac{1}{2}\sqrt{(\mu + p_4^2)^2 + 4\nu^2}\,\sinh 2\psi.$$

Introducing the new canonically conjugate coordinates $p_\phi = p_4/2$ and $\phi = 2\psi$, we obtain the final expression for the reduced Hamilton function from (20):

$$\widetilde{H}_h = \frac{(1-r^2)^2}{8mR^2}p_r^2 + \frac{4p_\phi^2}{a} + A_h\left(\frac{\mu}{2} + 2p_\phi^2 + 2\sqrt{\left(\frac{\mu}{4} + p_\phi^2\right)^2 + \frac{\nu^2}{4}}\,\cosh\phi\right) +$$
$$+ C_h\left(\frac{\mu}{2} + 2p_\phi^2 - 2\sqrt{\left(\frac{\mu}{4} + p_\phi^2\right)^2 + \frac{\nu^2}{4}}\,\cosh\phi\right) + B_h\sqrt{\left(\frac{\mu}{4} + p_\phi^2\right)^2 + \frac{\nu^2}{4}}\,\sinh\phi + U(r).$$

6. Conclusion

We have constructed the representation of the quantum mechanical Hamiltonian of a system of two particles in the spaces $\mathbb{S}^n$ and $\mathbb{H}^n$ that explicitly takes the symmetries of the problem into account. We will use this expression elsewhere to establish that the corresponding spectral problem is quasi-exactly solvable for some potentials. The reduced Hamilton function explicitly expressed in canonical coordinates in [6] using analytic simulations was used there to prove the absence of particle collisions. In the present paper, we have derived the explicit form of the reduced Hamilton function and clarified its relation to the quantum mechanical Hamiltonian.

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