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EXACT LINEARIZATION OF THE SLIDING PROBLEM FOR A DILUTE GAS IN THE BHATNAGAR–GROSS–KROOK MODEL

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We consider the nonlinear Boltzmann equation in the Bhathnagar–Gross–Krook model for the gas flow in a half-space (the Kramers problem). The problem can be exactly linearized, and its solution can be reduced to a linear integral equation with an addition–difference kernel and a simple nonlinear relation.

1. Introduction

One of the main models in physical kinetics is the Bhatnagar–Gross–Krook (BGK) model of the Boltzmann equation. It has numerous applications in investigating the simplest or multicomponent gases in a half-space bounded by a solid wall (the Kramers problem) and the gas flow between two parallel walls (the Couette problem) [1, 2].

Applying the BGK model involves approximately linearizing the nonlinear integral-differential equation obtained. The locally Maxwell velocity partition function of the gas is then replaced by the sum of the first two terms of the power series with respect to the mean mass velocity U, where it is assumed that

$$|U| \ll \langle |V| \rangle \tag{1}$$

with V being the molecular velocity of the gas [1–3]. This model is called the linearized BGK model. For the Kramers problem, applying the linearized BGK model leads to a contradictory result: the function $U = U(x)$ (where x is the distance from the wall) determined from the linearized equation grows infinitely as $x \to +\infty$, which is inconsistent with the original assumption (1).

Our analysis of the Kramers and Couette problems shows that they can be exactly linearized and effectively solved within the BGK model. In what follows, we restrict ourselves to considering the Kramers problem.

2. The exact linearization of the problem

We let (x, y, z) be a Cartesian coordinate system in the space $\mathbb{R}^3$ and let a one-component gas fill the half-space $x > 0$ bounded by a solid wall coinciding with the plane $x = 0$. We consider the gas flow problem in the Oy-axis direction. We let $f(x, \vec{s})$ denote the sought gas partition function with respect to the velocities $\vec{s} = (s_1, s_2, s_3)$. We let $U(x)$ denote the gas mean mass velocity $\vec{U}(x) = (0, U, 0)$, where

$$U(x) = \frac{1}{n} \int s_2 f(x, \vec{s})\, d^3 s. \tag{2}$$

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In the BGK model, the Boltzmann equation has the form [1, 2–4]

$$s_1 \frac{\partial f(x, \vec{s})}{\partial x} = -\sigma f(x, \vec{s}) + \beta e^{-\alpha|\vec{s}-\vec{U}(x)|^2}, \tag{3}$$

where $\sigma = 1/\tau$, τ is the relaxation time, $\alpha = m/(2kT)$, $\beta = \sigma n(\alpha/\pi)^{3/2}$, m is the mass of the molecules, T is the gas temperature, and k is the Boltzmann constant.

We introduce the auxiliary functions

$$f^+(x, \vec{s}) = \begin{cases} f(x, s_1, s_2, s_3) & \text{if } s_1 \geq 0, \\ 0 & \text{if } s_1 < 0, \end{cases}$$
$$f^-(x, \vec{s}) = \begin{cases} 0 & \text{if } s_1 < 0, \\ f(x, -s_1, s_2, s_3) & \text{if } s_1 \geq 0. \end{cases} \tag{4}$$

Equation (3) is completed by the boundary condition near the wall,

$$f^+(0, \vec{s})\big|_{s_1>0} = q f_0^{\mathrm{M}}(\vec{s}) + (1-q) f^-(0, \vec{s})\big|_{s_1<0}, \tag{5}$$

where q is the accomodation coefficient that gives the fraction of the molecules that after interacting with the wall, leave it with the Maxwell velocity distribution

$$f_0^{\mathrm{M}}(\vec{s}) = \frac{\beta}{\sigma} e^{-\alpha s^2}.$$

The remaining part $(1-q)$ of the molecules is mirror reflected from the wall surface.

It follows from Eq. (3) that

$$f^+(x, \vec{s}) = C e^{-x\sigma/s_1} + \int_0^x e^{-(x-t)\sigma/s_1} \Psi(t, \vec{s}) \frac{dt}{s_1}, \tag{6}$$

$$f^-(x, \vec{s}) = \int_x^\infty e^{-(t-x)\sigma/s_1} \Psi(t, \vec{s}) \frac{dt}{s_1}, \tag{7}$$

where

$$\Psi(x, \vec{s}) = \beta e^{-\alpha|\vec{s}-\vec{U}(x)|^2} = \beta e^{-\alpha s_1^2} e^{-\alpha s_3^2} e^{-\alpha(s_2-U(x))^2}. \tag{8}$$

Taking Eq. (5) into account, we obtain

$$C = q f_0^{\mathrm{M}}(\vec{s}) + (1-q) \int_0^\infty e^{-t\sigma/s_1} \Psi(t, \vec{s}) \frac{dt}{s_1} \tag{9}$$

from Eqs. (6) and (7). Inserting Eqs. (6) and (7) in (2) and taking Eq. (9) into account, we obtain the mean mass velocity

$$U(x) = S(x) + Q(x), \tag{10}$$

where

$$S(x) = \frac{1}{n}\int_{-\infty}^{\infty} ds_3 \int_{-\infty}^{\infty} s_2\, ds_2 \int_0^{\infty} ds_1 \int_0^{\infty} e^{-|x-1|\sigma/s_1}\Psi(t,\vec{s})\,\frac{dt}{s_1}, \tag{11}$$

$$Q(x) = (1-q)\frac{1}{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s_2\, ds_2\, ds_3 \int_0^{\infty} ds_1 \int_0^{\infty} e^{-(x+t)\sigma/s_1}\Psi(t,\vec{s})\,\frac{dt}{s_1}. \tag{12}$$

Relation (10) together with Eqs. (8), (11), and (12) is a nonlinear integral equation for U. We now consider transforming this equation.

From Eqs. (8) and (11), we obtain

$$S(x) = \int_0^{\infty} K(|x-t|)q(t)\, dt, \tag{13}$$

where

$$K(x) = \sqrt{\frac{\alpha}{\pi}}\sigma \int_0^{\infty} e^{-x\sigma/s_1} e^{-\alpha s_1^2}\,\frac{ds_1}{s_1},$$

$$q(t) = \sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty} s_2 e^{-\alpha[s_2-U(t)]^2}\, ds_2.$$

Passing to the new integration variable $p = s_2 - U(t)$, we obtain

$$q(t) = \sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty} [p+U(t)]e^{-\alpha p^2}\, dp = U(t)$$

and therefore

$$S(x) = \int_0^{\infty} K(|x-t|)U(t)\, dt.$$

We obtain a similar expression for the function Q. As result, expression (10) becomes

$$U(x) = \int_0^{\infty} K(|x-t|)U(t)\, dt + \varepsilon\int_0^{\infty} K(x+t)U(t)\, dt, \quad \varepsilon = 1-q. \tag{14}$$

It can be easily verified that the conservation condition

$$\int_{-\infty}^{\infty} K(x)\, dx = 1$$

is then satisfied. With the function $U(x)$ found from Eq. (14), the sought velocity partition function is determined in accordance with Eqs. (6)–(8).

The problem in Eqs. (2) and (3) for the nonlinear integral-differential equation is thus exactly linearized. Its solution is reduced to linear conservative integral equation (14) and simple nonlinear relations (6)–(8). We also investigated the exact solution of conservative linear equation (14) in [5]. We note that the results given above can be generalized to multicomponent gas flows in the half-space.

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