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WONDERFUL VARIETIES OF RANK TWO

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Abstract. Let G be a complex connected reductive group. Well known wonderful G-varieties are those of rank zero, namely the generalized flag varieties G/P, those of rank one, classified in [A], and certain complete symmetric varieties described in [DP] such as the famous space of complete conics. Recently, there is a renewed interest in wonderful varieties of rank two since they were shown to hold a keystone position in the theory of spherical varieties, see [L], [BP], and [K].

The purpose of this paper is to give a classification of wonderful varieties of rank two. These are nonsingular complete G-varieties containing four orbits, a dense orbit and two orbits of codimension one whose closures D_1 and D_2 intersect transversally in the fourth orbit which is of codimension two. We have gathered our results in tables, including isotropy groups, explicit basis of Picard groups, and several combinatorial data in relation with the theory of spherical varieties.

1. Introduction

We start by defining wonderful G-varieties for a complex connected reductive group G, and we sum up a few of their remarkable properties. Then we shall introduce our results.

Recall that a *divisor with normal crossings* on a nonsingular variety is a divisor $D = \bigcup D_i$ such that each irreducible component D_i is nonsingular, and whenever r irreducible components D_i meet at a point p, then their intersection at p is transversal, i. e., the local equations f_i of the D_i form part of a regular system of parameters at p.

Definition 1.1. A *wonderful G-variety* is a nonsingular complete G-variety X having the following properties:

(1) The group G preserves a divisor with normal crossings $D = \bigcup_{i\in \mathrm{I}} D_i$.
(2) The intersection $\bigcap_{i\in \mathrm{I}} D_i$ of all the D_i is nonempty.
(3) Two points of X are in the same orbit if and only if they are in the same set of divisors D_i.

The *rank* of X is the number $r = \operatorname{card} \mathrm{I}$ of irreducible components D_i of D. Note that G has 2^r orbits in X and $\bigcap_{i\in \mathrm{I}} D_i$ is the unique closed one. The orbit closures are the $\bigcap_{i\in \mathrm{J}} D_i$ where J runs through all subsets of I, and each $\bigcap_{i\in \mathrm{J}} D_i$ is a wonderful variety of rank $r - \operatorname{card} \mathrm{J}$.

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It is shown in [L] that a Borel subgroup B in G has a dense orbit in X. When a normal G-variety has this property, then it is called *spherical.* Moreover the following condition is *necessary* for a spherical homogeneous variety G/H to have a wonderful completion, see [BP]:

(aut) *The automorphism group* $\mathrm{Aut}^G\, G/H = N_G(H)/H$ *is finite.*

If this condition is satisfied, then there is a unique candidate X for a wonderful completion of G/H, and it is wonderful if and only if it has no singularities. This is the case for example if $H = N_G(H)$, see [K]. Furthermore the following *universal* property is satisfied by this candidate X, see [LV]:

(uni) *Let Y be a normal complete G-variety containing only one closed orbit. Then any dominant G-equivariant morphism $G/H \to Y$ extends equivariantly to $X \to Y$.*

Wonderful varieties were shown to play a central role in the theory of spherical varieties, see [BP] and [K]. As in [B3] and [L], many questions about wonderful varieties may be handled by reducing them to the rank two case, namely the lowest rank for which $\bigcup_{i\in \mathrm{I}} D_i \neq \bigcap_{i\in \mathrm{I}} D_i$.

The goal of this paper is to classify wonderful G-varieties of rank two. An essential difficulty is that most isotropy groups are not reductive. So we made the following observation. The well-known weight decomposition on the Lie algebra $\mathfrak{b}^u$ of the unipotent radical of B naturally generalizes for an arbitrary parabolic subgroup P in G: Indeed, let L be a Levi subgroup in P and let M be an irreducible L-module in $\mathfrak{p}^u$. Denote by χ_M the character through which the radical of L acts on M.

Lemma 1.2. *The L-module $M \subset \mathfrak{p}^u$ is uniquely determined by χ_M.*

See Lemma 5.5 for a proof. Lemma 1.2 is a practical tool to deal with the radical of H. It enables us to determine the pairs $(\mathfrak{g}, \mathfrak{h})$ (where $G/H \hookrightarrow X$ is of rank two and $\mathfrak{h} = \mathrm{Lie}(H)$ is not semisimple) by choosing a nice parabolic subalgebra $\mathfrak{p}$ in $\mathfrak{g}$ containing $\mathfrak{h}$. The few remaining cases readily follow from the known classification of connected reductive spherical subgroups, see [Kr], [M], and [B]. Lemma 1.2 also gives us the eigenvectors of $\mathfrak{h}$ in the rational representations of G.

Our target is to obtain the following theorem. For a parabolic subgroup P in G, let $\hat{\underline{G}}$ be the universal covering of $\underline{G} = P/P^r$, where P^r denotes the radical of P. Split $\hat{\underline{G}}$ into simple components $\prod G_\sigma$ where σ runs through an index set, say Σ. Recall that if there exists an equivariant morphism $\phi : X \to G/P$, then there is an equivariant isomorphism $X \simeq G \times^P \phi^{-1}(P/P)$, see [Bi]. In Theorem 1.3 below, we refer to tables which lie in Sections 2 and 3.

Theorem 1.3. *Wonderful G-varieties of rank two are the $G \times^P X$, where $P \subset G$ is any subgroup which strictly contains B and X is any wonderful $\underline{G}$-variety $\hat{\underline{G}}$-equivariantly isomorphic to either one of the following varieties:*

(a) *A* G_0*-variety in* Tables A–G, *where* $G_0 = \prod_{\sigma\in\Sigma_0} G_\sigma$ *and* $\Sigma_0 \subset \Sigma$.
(b) *A* G_1*-variety times a* G_2*-variety both in* Table 1, *where* $G_i = \prod_{\sigma\in\Sigma_i} G_\sigma$ *and* $\Sigma_1 \amalg \Sigma_2 \subset \Sigma$.

See Lemma 2.2 and Theorem 7.8 for a proof. We have identified the G-orbits in $D_1 \cup D_2$ as follows. Fix a Borel subgroup B^- opposite to B and let $\{z\} = X^{B^-}$. It can be seen, thanks to Table 1, that the set Θ of weights *with multiplicities* (see for example case 8 in Table A and Theorem 5.6) of $T := B \cap B^-$ in the tangent space T_zX uniquely determines the group action on D_1 and D_2. These sets Θ also enable us to determine the normalizer $N_G(H)$.

Furthermore Θ almost determines X up to isomorphism. Indeed, let $\mathcal{X}(B)$ denote the group of characters of B, and let Ξ be the lattice of characters $\chi_f \in \mathcal{X}(B)$ of all rational B-eigenfunctions f on X. It is routine to identify two weights of Θ which yield a basis of Ξ. Let Δ be the set of colors, namely the B-stable prime divisors in X which are not G-stable. If v_D denotes the discrete valuation associated with $D \in \Delta$, then there is a natural map $\varrho : \Delta \to \mathrm{Hom}(\Xi, \mathbf{Z})$ which maps $D \mapsto (\chi_f \mapsto v_D(f))$. (Note that χ_f determines f up to a scalar.)

Theorem 1.4. *Fix* G *and let* X *be a wonderful* G*-variety of rank two. The sets* Θ *and* $\varrho(\Delta)$ *determine* X *up to isomorphism.*

Theorem 1.4, which yields a combinatorial description of wonderful varieties of rank two, follows from Theorem 1.3 and Tables A–G. Observe that it is *not* necessary to know how many colors yield a given point in $\varrho(\Delta)$.

Remark 1.5. The group action on any normal G-variety is locally projective rational, see [S, p. 8]. Because X has a unique closed orbit, X is projective, and for any equivariant embedding of X in a projective space $\mathbf{P}(V)$, and any one parameter subgroup in the center of G, the quasi-affine cone over X lies in a unique grading subspace of V. Therefore the action of the connected center of G in X is trivial. Without loss of generality, we assume that G is semi-simple and simply connected.

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Notation. The base field k is the field of complex numbers. Let $k^\star := k\backslash\{0\}$. We denote by T a maximal torus in G, and by B a Borel subgroup in G containing T. Let $\mathcal{X}(T) = \mathcal{X}(B)$ be the corresponding lattice of characters. We denote by B^- the Borel subgroup in G containing T and opposite to B. Let S be the basis of the root system R of (G,T) determined by the choice of B. Notations used for groups, roots and weights are those of [BN]. In a few formulas, B_1, C_1 and D_1 are used for brevity to mean type A_1, and D_2 to mean type $\mathrm{A}_1 \times \mathrm{A}_1$. For a root β, let $\mathrm{w} = \mathrm{s}_\beta$ be the reflection in the Weyl group $\mathrm{W} = N_G(T)/T$ of G canonically defined by β. Consider the quotient $N_G(T) \to \mathrm{W}$ and choose $\hat{\mathrm{w}}$ in the preimage of w. Let $\langle\hat{\mathrm{w}}\rangle \subset N_G(T)$ be the subgroup generated by $\hat{\mathrm{w}}$. For a dominant weight λ, let V_λ be the irreducible G-module with highest weight λ, and V_λ^* its dual. If S is an algebraic subgroup in G, let $C_G(S)$ denote its centralizer, S^0 its identity component and S^u its unipotent radical. Let C_S denote the center of S. The letter X is used to denote the (candidate) wonderful completion of a spherical homogeneous space G/H satisfying **(aut)**, and Δ is used for its colors, see Theorem 1.4. By $k(X)^{(B)}$ we denote the multiplicative group of B-eigenvectors in the field of rational functions on X. Consider the abelian group homomorphism $k(X)^{(B)} \to \mathcal{X}(T) = \mathcal{X}(B)$ which associates to each B-eigenfunction f its character χ_f. The image of this homomorphism is denoted by Ξ.

2. On wonderful varieties of rank one

This section has no pretention to originality. Its goal is to present known results of Akhiezer [A], see also [B1], in a simpler way when one aims at determining the wonderful subvarieties of rank one lying in a wonderful variety X of rank r.

Let $z \in X$ be the unique point fixed by B^-.

Definition 2.1. A *spherical root* γ of X is a weight of T in the T-module $T_zX/T_zG{\cdot}z$.

Let G_z be the isotropy subgroup of G at z. Then the set of spherical roots γ_i, $i \in \mathrm{I}$, of X yield a $\mathbf{Z}$-basis of Ξ. Indeed, the point z lies in a toric T-subvariety $W \subset X$ stable under the Levi subgroup of G_z containing T, such that the following map is an open immersion [BLV, p. 621]:

$$(G_z^-)^u \times W \to (G_z^-)^u W \subset X$$

where G_z^- contains B and is opposite to (the parabolic subgroup) G_z. Therefore, the spherical roots of X give a basis of the image of $k(W)^{(T)}$ in $\mathcal{X}(T)$, see [F, p. 29] and observe that dim $W =$ codim $G{\cdot}z = r$. So rk $\Xi = r$.

The lattice Ξ can be determined in the following way. Recall that the algebra of regular functions on G has a natural $G \times G$-module structure $k[G] \simeq \bigoplus V_\lambda \otimes V_\lambda^*$, where λ runs through all dominant weights in $\mathcal{X}(B)$. Let $\pi : G \to G/H$ denote the projection, where $G/H \hookrightarrow X$ is the open orbit. For each color $D \in \Delta$, choose an equation $f_D \in k[G]$ defining $\pi^{-1}(D)$. Since H is spherical in G and $\mathcal{X}(G)$ is trivial thanks to Remark 1.5, $f_D \in k[G]^{(B\times H)}$ and f_D is determined up to a scalar by $\chi_{f_D} \in \mathcal{X}(B)\times\mathcal{X}(H)$. Furthermore, $(\chi_{f_D})_{D\in\Delta}$ generates (with the scalar functions) the multiplicative semi-group $k[G]^{(B\times H)}$ since $k[G]$ is factorial, see Remark 1.5 and [KKLV, p. 74]. Thereby we can easily recover $k(G/H)^{(B)}$, and hence Ξ. In particular, card $\Delta \leq r+$ rk $\mathcal{X}(H)$.

Our goal is to find all possible spherical roots. The following lemma yields an important reduction for this calculation.

Lemma 2.2. *Let P be a parabolic subgroup in G such that $P^r \subset H \subset P$. Then $X = G \times^P \tilde{X}$, where $\tilde{X}$ is a wonderful P/P^r-variety of rank r with open orbit P/H.*

Proof. Consider the projection $G/H \to G/P$. By the property **(uni)**, see the introduction, it extends equivariantly to $\phi : X \to G/P$. Let $\tilde{X} := \phi^{-1}(P/P)$.

By [Bi], $G \times^P \tilde{X}$ is an algebraic G-variety, and it is G-equivariantly isomorphic to X. Now observe that P/H can be identified with the open P/P^r-orbit in $\tilde{X}$ and that $\tilde{X}$ is a wonderful P/P^r-variety of rank r. □

Definition 2.3. A *prime* group in G is a subgroup S having the following properties:

(ind) *If $P^r \subset S \subset P$ for a parabolic subgroup P in G, then $P = G$.*
(pro) *The only pairs of semi-simple groups (G_1, G_2) such that $G = G_1 \times G_2$ and $S = S_1 \times S_2$ with $S_i \subset G_i$ are $(G, 1)$ and $(1, G)$.*

Thanks to Lemma 2.2, it will be sufficient to focus on prime wonderful G-varieties, i. e., those for which H is prime in G. Even then, we will see further on that H is often not reductive. So we use the following decomposition. Let $U := H^u = (H^0)^u$. By [Mo, p. 200], there exists a connected reductive subgroup $K \subset H$ such that $H^0 = KU$ is a direct product. Moreover, there exists a parabolic subgroup $P \subset G$ containing $N_G(U)$ such that $U \subset P^u$ [BT, p. 102] (I have learned that this result was found independently by B. Weisfeiler.) Hence we can choose a minimal parabolic subgroup $Q \subset G$ containing H^0 such that $U \subset Q^u$. Let L be a Levi subgroup in Q containing K. From now on, we fix $B \subset Q$ and $T \subset L$.

Since all spherical roots come from the case r=1, we have listed in Table 1 the prime wonderful G-varieties of rank one.

How to read Table 1. According to [BN], types A_n, B_n, C_n, and D_n start respectively with $n = 1$, $n = 2$, $n = 2$, and $n = 3$. Also, bear in mind Remark 1.5 while reading column 2.

In column 1, labels were designed for the use of spherical roots which we will make in Section 3. Two labels such as 7B and 7C share the same number 7: this means that when types B and C match, i. e., when $n = 2$, then the two corresponding cases are G-isomorphic. Case 5 shares the same number 5 as case 5D: it expresses the fact that there is an *outer* automorphism of Spin_8 which maps the first Spin_7 on the second (for $n = 4$).

In column 3, the connected group K is given up to a finite covering, and if the Lie algebra $\mathfrak{u}$ of U is nontrivial, then we split it into irreducible K-modules. The group H is either KU, $KU\langle\hat{\mathrm{w}}\rangle$ if an element $\mathrm{w} \in \mathrm{W}$ is given in column 3 (then the pair (K, U) is found one case above), or KUC_G if the letter c appears in column 4. If $C_G \not\subset H$, then $HC_G/H \simeq \mathbf{Z}/2\mathbf{Z}$.

In column 4, we compute χ_{f_D} for each color $D \in \Delta$. The letters c and ε denote nontrivial characters in $\mathcal{X}(H)$ satisfying $c^2 = 1$ and $\varepsilon^2 = 1$. When a fundamental weight w_i is used as a character of H, then $H \subset Q$ and Q

is the largest parabolic subgroup in G containing B such that $w_i \in \mathcal{X}(Q)$. Case 5 corresponds to two distinct cases: $w_{3/4}$ means w_3, respectively w_4.

In column 5, we give the unique spherical root γ_1. (We have seen above that Ξ can be easily recovered from $(\chi_{f_D})_{D\in\Delta}$, therefore γ_1 is also determined since it is positive in $\mathcal{X}(B)$, see [B3, p. 130].) Because of the following remark, we label γ_1 as in column 1, and we give the dimension of X in column 7.

Finally, for column 6, we refer to the last paragraph of this section.

Remark 2.4. Reading through columns 4, 5 and 7 of Table 1, we immediately see that if X is a prime wonderful G-variety of rank one, then for a given group G, X is uniquely determined by γ_1 and another data such as card Δ or dim X. This shows that a (not necessarily prime) wonderful G-variety of rank one is uniquely determined by γ_1 and G_z, see Lemma 2.2.

Here is a beautiful interpretation of the expression of the spherical roots γ_i, $i \in$ I, in terms of the $\chi_{f_D}, D \in \Delta$. See also [L1]. Let $f_i \in k(X)^{(B)}$ be up to a scalar the unique function satisfying $\chi_{f_i} = -\gamma_i$. If X_i denotes the unique wonderful G-variety of rank one having γ_i as spherical root and G/G_z as closed orbit, then $\bigcap_{j\neq i} D_j \simeq X_i$ with $D_i = f_i^{-1}(0)$ [B3, p. 126]. Moreover Pic $X = \bigoplus_{D\in\Delta} \mathbf{Z}\cdot\{D\}$, where $\{D\}$ denotes the class of D in the Picard group Pic X [B2, p. 405]. Since we can express $\gamma_i = \sum_{D\in\Delta} n^i_D \chi_{f_D}$ with $n^i_D \in \mathbf{Z}$, we get $\{D_i\} = \sum_{D\in\Delta} n^i_D\{D\}$. In column 6 of Table 1, we give $n^1_D, D \in \Delta$, with Δ ordered as in column 4.

3. Prime wonderful varieties of rank two

In this section, we gather our classification results in Tables A–G which contain the *prime* wonderful G-varieties of rank two, see Definition 2.3 (and Theorem 1.3). Proofs will be given in the following sections.

How to read Tables A-G. Labels in column 1 refer to Theorem 5.6, Proposition 5.7 and the propositions of Section 6. These labels were designed to help keeping track of the isomorphism between different types, such as in Tables B and C; cases 7 are the same for $n = 2$.

Columns 2 and 3 can be read as in Table 1. When useful, weights of $T \cap K$ are given for $\mathfrak{u}$, such as in case A8.

In column 4, we sometimes have $c^3 = 1$ instead of $c^2 = 1$. For Q, follow *How to read* Table 1 (with sometimes more than one w_i) except in the second cases of BC8 where Q is the same as one case above.

In column 5, a basis (γ_i) of Ξ is given, see Theorem 1.4. Each γ_i is a combination of the characters of column 4: the corresponding coefficients are given in column 6. For Tables A–G, we identify $\Xi\otimes_{\mathbf{Z}}\mathbf{Q}$ and $\mathrm{Hom}_{\mathbf{Z}}(\Xi, \mathbf{Q})$ by choosing the length of γ_1 to be $\sqrt{2}$, shorter than γ_2. In the last column, γ_1 and γ_2 are represented by arrows. The cone $\mathcal{V}$ dual to the cone generated

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Table 1

	G		Finite covering of K $\mathfrak{U}$ as a K-module	$\chi_{f_D} \in \mathcal{X}(B) \times \mathcal{X}(H)$ with $f_D \in k[G]^{(B\times H)}$	$\gamma_1^{(\text{col. } 1)}$	n_D^1	dim X
1A	SL_{n+1}	$n \geq 2$	GL_n	$(w_1, -w_1)$ (w_n, w_1)	$\alpha_1+\ldots+\alpha_n^{(1)}$	1 1	$2n$
2	SL_2		s_{α_1}	$(2w_1, \varepsilon)$	$2\alpha_1^{(2)}$	2	2
3	$\mathrm{SL}_2 \times \mathrm{SL}_2$		SL_2	$(w_1+w_1', 1)$	$\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_1'^{(3)}$	1	3
4	$\mathrm{SL}_2 \times \mathrm{SL}_2$		SL_2	(w_1+w_1', c)	$\alpha_1+\alpha_1'^{(4)}$	2	3
5A	SL_4		Sp_4	$(w_2, 1)$	$\frac{1}{2}\alpha_1+\alpha_2+\frac{1}{2}\alpha_3^{(5)}$	1	5
6A	SL_4		Sp_4	(w_2, c)	$\alpha_1+2\alpha_2+\alpha_3^{(6)}$	2	5
7B	Spin_{2n+1}	$n \geq 2$	Spin_{2n}	$(w_1, 1)$	$\alpha_1+\ldots+\alpha_n^{(7)}$	1	$2n$
8B	Spin_{2n+1}	$n \geq 2$	s_{α_n}	(w_1, ε)	$2\alpha_1+\ldots+2\alpha_n^{(8)}$	2	$2n$
9B	Spin_{2n+1}	$n \geq 2$	$\mathrm{SL}_n \times k^\star$ $\wedge^2 k^n$	$(w_1, 1)$ (w_n, w_n)	$\alpha_1+\ldots+\alpha_n^{(9)}$	1 0	$\frac{1}{2}n(n+3)$
10	Spin_7		G_2	$(w_3, 1)$	$\frac{1}{2}\alpha_1+\alpha_2+\frac{3}{2}\alpha_3^{(10)}$	1	7
11	Spin_7		G_2	(w_3, c)	$\alpha_1+2\alpha_2+3\alpha_3^{(11)}$	2	7
7C	Sp_{2n}	$n \geq 2$	$\mathrm{SL}_2 \times \mathrm{Sp}_{2n-2}$	$(w_2, 1)$	$\alpha_1+2\alpha_2+\ldots+2\alpha_{n-1}+\alpha_n^{(7)}$	1	$4n-4$
8C	Sp_4		s_{α_1}	(w_2, ε)	$2\alpha_1+2\alpha_2^{(8)}$	2	4
9C	Sp_{2n}	$n \geq 2$	$k^\star \times \mathrm{Sp}_{2n-2}$ k	$(w_2, 1)$ (w_1, w_1)	$\alpha_1+2\alpha_2+\ldots+2\alpha_{n-1}+\alpha_n^{(9)}$	1 0	$4n-3$
1D	Spin_6		GL_3	$(w_2, -w_2)$ (w_3, w_2)	$\alpha_1+\alpha_2+\alpha_3^{(1)}$	1 1	6
5D	Spin_{2n}	$n \geq 3$	Spin_{2n-1}	$(w_1, 1)$	$\alpha_1+\ldots+\alpha_{n-2}+\frac{1}{2}\alpha_{n-1}+\frac{1}{2}\alpha_n^{(5)}$	1	$2n-1$
5	Spin_8		Spin_7	$(w_{3/4}, 1)$	$\frac{1}{2}\alpha_1+\alpha_2+\alpha_{3/4}+\frac{1}{2}\alpha_{4/3}^{(5)}$	1	7
6D	Spin_{2n}	$n \geq 3$	Spin_{2n-1}	(w_1, c)	$2\alpha_1+\ldots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n^{(6)}$	2	$2n-1$
6	Spin_8		Spin_7	$(w_{3/4}, c)$	$\alpha_1+2\alpha_2+2\alpha_{3/4}+\alpha_{4/3}^{(6)}$	2	7
12	F_4		Spin_9	$(w_4, 1)$	$\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4^{(12)}$	1	16
13	G_2		SL_3	$(w_1, 1)$	$2\alpha_1+\alpha_2^{(13)}$	1	6
14	G_2		s_{α_1}	(w_1, ε)	$4\alpha_1+2\alpha_2^{(14)}$	2	6
15	G_2		$k^\star \times \mathrm{SL}_2$ $k \oplus k^2$	(w_2, w_1) (w_1, w_1)	$\alpha_1+\alpha_2^{(15)}$	1 1	7

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	G	Finite covering of K $\mathcal{U}$ as a K-module	$\chi_{f_D} \in \mathcal{X}(B)\times\mathcal{X}(H)$ with $f_D \in k[G]^{(B\times H)}$	$\gamma_1^{(\text{Table 1})}$ $\gamma_2^{(\text{Table 1})}$	n_D^1 n_D^2	$\varrho(\Delta), \mathcal{V}$

Table A

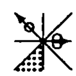
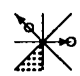
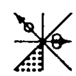
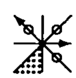
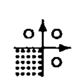
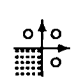
1	SL_3	SO_3	$(2w_1, 2c)(2w_2, c)$	$2\alpha_1^{(2)}$	2 -1	
				$2\alpha_2^{(2)}$	-1 2	
2	$SL_3\times SL_3$	SL_3	$(w_1+w_1', 2c)(w_2+w_2', c)$	$\alpha_1+\alpha_1'^{(4)}$	2 -1	
				$\alpha_2+\alpha_2'^{(4)}$	-1 2	
3	SL_6	Sp_6	$(w_2, 2c)(w_4, c)$	$\alpha_1+2\alpha_2+\alpha_3^{(6)}$	2 -1	
				$\alpha_3+2\alpha_4+\alpha_5^{(6)}$	-1 2	
4	SL_4	$SL_2\times k^\star\times SL_2$	$(w_1+w_3, 1)(w_2, -w_2)$	$\alpha_2^{(1)}$	-1 1 1	
			(w_2, w_2)	$\alpha_1+\alpha_3^{(4)}$	2 -1 -1	
4	SL_4	$s_{\alpha_2}\circ s_{\alpha_1+\alpha_2+\alpha_3}$	$(w_1+w_3, \varepsilon)(2w_2, 1)$	$\alpha_1+\alpha_3^{(4)}$	2 -1	
				$2\alpha_2^{(2)}$	-2 2	
4	SL_{n+1}	$SL_2\times k^\star\times SL_{n-1}$	$(w_1+w_n, 1)(w_2, -w_2)$	$\alpha_2+\ldots+\alpha_{n-1}^{(1)}$	-1 1 1	
	$n\geq 4$		(w_{n-1}, w_2)	$\alpha_1+\alpha_n^{(4)}$	2 -1 -1	
5	SL_4	$k^\star\times SL_2$	$(w_2, 1)(w_1+w_3, w_2)$	$\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_3^{(3)}$	0 1 -1	
		S^2k^2	(w_2, w_2)	$\alpha_2^{(1)}$	1 -1 1	
5	SL_4	$k^\star\times SL_2$	$(w_2, c)(w_1+w_3, w_2+c)$	$\alpha_2^{(1)}$	1 -1 1	
		S^2k^2	(w_2, w_2)	$\alpha_1+\alpha_3^{(4)}$	0 2 -2	
6	SL_3	$k^\star\times k^\star$	$(w_1, -w_1)(w_2, w_2-w_1)$	$\alpha_1^{(1)}$	1 -1 1 0	
		k	$(w_1, w_2)(w_2, w_1)$	$\alpha_2^{(1)}$	0 1 -1 1	
6	SL_{n+1}	$k^\star\times SL_{n-1}\times k^\star$	$(w_1, -w_1)(w_2, w_n-w_1)$	$\alpha_1^{(1)}$	1 -1 1 0	
	$n\geq 3$	k^{n-1}	$(w_1, w_n)(w_n, w_1)$	$\alpha_2+\ldots+\alpha_n^{(1)}$	0 1 -1 1	
6	SL_{n+1}	$k^\star\times k^\star\times SL_{n-1}$	$(w_1, -w_1)(w_n, w_2-w_1)$	$\alpha_1+\ldots+\alpha_{n-1}^{(1)}$	1 -1 1 0	
	$n\geq 3$	k^{n-1*}	$(w_{n-1}, w_2)(w_n, w_1)$	$\alpha_n^{(1)}$	0 1 -1 1	
6	SL_{n+1}	$k^\star\times SL_{n-l}\times k^\star\times SL_l$	$(w_1, -w_1)(w_{l+1}, w_{n-l+1}-w_1)$	$\alpha_1+\ldots+\alpha_l^{(1)}$	1 -1 1 0	
	$n\geq l+2\geq 4$	$k^{n-l}\otimes k^{l*}$	$(w_l, w_{n-l+1})(w_n, w_1)$	$\alpha_{l+1}+\ldots+\alpha_n^{(1)}$	0 1 -1 1	
7	SL_3	$k^\star$	(w_1+w_2, w_1+c)	$\alpha_1^{(1)}$	1 1 -2	
		$k_{\alpha_1\simeq\alpha_2}\oplus k$	$(w_1, w_2)(w_2, w_1)$	$\alpha_2^{(1)}$	1 -2 1	
7	SL_4	$k^\star\times SL_2$	(w_2, c)	$\alpha_1+\alpha_2^{(1)}$	1 1 -1	
		$k^2_{\alpha_1\simeq\alpha_3}\oplus k$	$(w_1, w_3)(w_3, w_1)$	$\alpha_2+\alpha_3^{(1)}$	1 -1 1	
8	$SL_2\times SL_2$	$k^\star$	(w_1+w_1', c)	$\alpha_1^{(1)}$	1 1 -1	
		$k_{\alpha_1\simeq\alpha_1'}$	$(w_1, w_1)(w_1', w_1')$	$\alpha_1'^{(1)}$	1 -1 1	

Table B

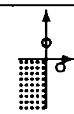
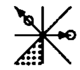
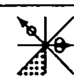
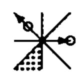
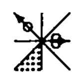
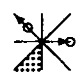
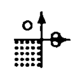
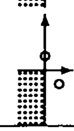
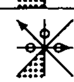
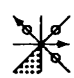
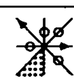
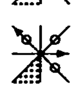
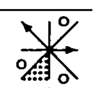
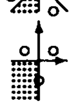
1	$\mathrm{Spin}_5 \times \mathrm{SL}_2$	$\mathrm{SL}_2 \times \mathrm{SL}_2$	$(w_1, 1)(w_2 + w_1', c)$	$\alpha_1 + \alpha_2^{(9)}$	1 0	
				$\alpha_2 + \alpha_1'^{(4)}$	-1 2	
2	$\mathrm{Spin}_5 \times \mathrm{Spin}_5$	Spin_5	$(w_1 + w_1', 1)(w_2 + w_2', 1)$	$\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_1'^{(3)}$	1 -1	
				$\alpha_2 + \alpha_2'^{(4)}$	-1 2	
2	$\mathrm{Spin}_5 \times \mathrm{Spin}_5$	Spin_5	$(w_1 + w_1', 1)(w_2 + w_2', c)$	$\alpha_2 + \alpha_2'^{(4)}$	-1 2	
				$\alpha_1 + \alpha_1'^{(4)}$	2 -2	
3	Spin_9	Spin_7	$(w_1, 1)(w_4, c)$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4^{(9)}$	1 0	
				$\alpha_2 + 2\alpha_3 + 3\alpha_4^{(11)}$	-1 2	
4	Spin_5	$k^\star \times \mathrm{SL}_2$	$(w_1, -w_1)(2w_2, 1)$	$\alpha_1^{(1)}$	1 -1 1	
			(w_1, w_1)	$2\alpha_2^{(2)}$	-1 2 -1	
4	Spin_5	$s_{\alpha_1 + \alpha_2}$	$(2w_1, 1)(2w_2, \varepsilon)$	$2\alpha_2^{(2)}$	-1 2	
				$2\alpha_1^{(2)}$	2 -2	
4	Spin_{2n+1}	$k^\star \times \mathrm{Spin}_{2n-1}$	$(w_1, -w_1)(w_2, 1)$	$\alpha_1^{(1)}$	1 -1 1	
	$n \geq 3$		(w_1, w_1)	$2\alpha_2 + \ldots + 2\alpha_n^{(8)}$	-1 2 -1	
4	Spin_{2n+1}	$s_{\alpha_1 + \ldots + \alpha_n}$	$(2w_1, 1)(w_2, \varepsilon)$	$2\alpha_2 + \ldots + 2\alpha_n^{(8)}$	-1 2	
	$n \geq 3$			$2\alpha_1^{(2)}$	2 -2	
5	Spin_5	$\mathrm{SL}_2 \times k^\star$	$(w_1, 1)(w_2, -w_2)$	$\alpha_2^{(1)}$	-1 1 1	
			(w_2, w_2)	$\alpha_1 + \alpha_2^{(9)}$	1 0 0	
5	Spin_5	$s_{\alpha_1 + 2\alpha_2}$	$(w_1, 1)(2w_2, \varepsilon)$	$\alpha_1 + \alpha_2^{(9)}$	1 0	
				$2\alpha_2^{(2)}$	-2 2	
6	Spin_{2n+1}	$\mathrm{SL}_p \times k^\star \times \mathrm{Spin}_{2n-2p}$	$(w_1, 1)(w_{p+1}, w_p + 1)$	$\alpha_{p+1} + \ldots + \alpha_n^{(7)}$	0 1 -1	
	$n \geq p+2 \geq 3$	$k^p \otimes k^{2n-2p} \oplus \wedge^2 k^p$	(w_p, w_p)	$\alpha_1 + \ldots + \alpha_p^{(1)}$	1 -1 1	
6	Spin_{2n+1}	s_{α_n}	$(w_1, \varepsilon)(w_{p+1}, w_p + \epsilon)$	$\alpha_1 + \ldots + \alpha_p^{(1)}$	1 -1 1	
	$n \geq p+2 \geq 3$		(w_p, w_p)	$2\alpha_{p+1} + \ldots + 2\alpha_n^{(8)}$	0 2 -2	
7	Spin_{2n+1}	$k^\star \times \mathrm{SL}_{n-1} \times k^\star$	$(w_{n-1}, 2w_n - w_1)(w_n, w_n - w_1)$	$\alpha_n^{(1)}$	-1 1 0 1	
	$n \geq 2$	$k^{n-1} \oplus k^{n-1} \oplus \wedge^2 k^{n-1}$	$(w_1, w_1)(w_n, w_n)$	$\alpha_1 + \ldots + \alpha_{n-1}^{(1)}$	1 0 1 -2	
8	Spin_{2n+1}	$\mathrm{SL}_{n-1} \times k^\star \times k^\star$	$(w_1, 1)(w_n, w_{n-1} - w_n)$	$\alpha_n^{(1)}$	0 1 -1 1	
	$n \geq 2$	$k^{n-1} \oplus k^{n-1} \oplus \wedge^2 k^{n-1}$	$(w_{n-1}, w_{n-1})(w_n, w_n)$	$\alpha_1 + \ldots + \alpha_{n-1}^{(1)}$	1 -1 1 -1	
8	Spin_{2n+1}	s_{α_n}	$(w_1, \varepsilon)(2w_n, w_{n-1} + \epsilon)$	$\alpha_1 + \ldots + \alpha_{n-1}^{(1)}$	1 -1 1	
	$n \geq 2$		(w_{n-1}, w_{n-1})	$2\alpha_n^{(2)}$	0 2 -2	
8	Spin_{2n+1}	$\mathrm{SL}_q \times k^\star \times \mathrm{SL}_{n-q} \times k^\star$	$(w_1, 1)(w_{q+1}, w_q)$	$\alpha_{q+1} + \ldots + \alpha_n^{(9)}$	0 1 -1 0	
	$n \geq q+2 \geq 3$	$k^q \otimes k^{n-q*} \oplus \wedge^2 k^{n-q}$	$(w_q, w_q)(w_n, w_n)$	$\alpha_1 + \ldots + \alpha_q^{(1)}$	1 -1 1 0	
		$\oplus k^q \otimes k^{n-q} \oplus \wedge^2 k^q$				
9	Spin_5	$k^\star$	$(w_1 + w_2, w_2 + w_1/2)$	$\alpha_2^{(1)}$	1 -2 1	
		$k_{\alpha_1 \simeq \alpha_2} \oplus k \oplus k$	$(w_1, w_1)(w_2, w_2)$	$\alpha_1^{(1)}$	1 1 -3	
9	Spin_7	$k^\star \times \mathrm{SL}_2$	(w_2, w_1)	$\alpha_2 + \alpha_3^{(9)}$	1 -1 0	
		$k^2_{\alpha_1 \simeq \alpha_3} \oplus k \oplus k \oplus k^2$	$(w_1, w_1)(w_3, w_3)$	$\alpha_1 + \alpha_2^{(1)}$	1 1 -2	

Table C

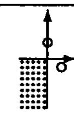
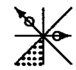
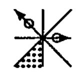
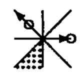
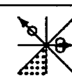
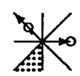
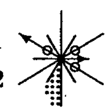
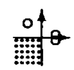
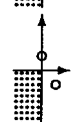
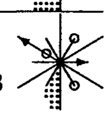
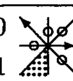
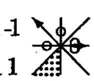
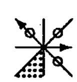
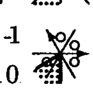
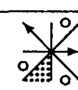
1	$Sp_4 \times SL_2$	$SL_2 \times SL_2$	$(w_1+w_1', c)(w_2, 1)$	$\alpha_1+\alpha_2^{(9)}$	0 1	
				$\alpha_1+\alpha_1'^{(4)}$	2 -1	
2	$Sp_4 \times Sp_4$	Sp_4	$(w_1+w_1', 1)(w_2+w_2', 1)$	$\frac{1}{2}\alpha_2+\frac{1}{2}\alpha_2'^{(3)}$	-1 1	
				$\alpha_1+\alpha_1'^{(4)}$	2 -1	
2	$Sp_4 \times Sp_4$	Sp_4	$(w_1+w_1', c)(w_2+w_2', 1)$	$\alpha_1+\alpha_1'^{(4)}$	2 -1	
				$\alpha_2+\alpha_2'^{(4)}$	-2 2	
3	Sp_{2n}	$Sp_4 \times Sp_{2n-4}$	$(w_2, 1)(w_4, 1)$	$\alpha_3+2\alpha_4+\ldots+\alpha_n^{(7)}$	-1 1	
	$n \geq 4$			$\alpha_1+2\alpha_2+\alpha_3^{(6)}$	2 -1	
3	Sp_8	$s_{\alpha_1+\alpha_2} \circ s_{\alpha_2+\alpha_3}$	$(w_2, \varepsilon)(w_4, 1)$	$\alpha_1+2\alpha_2+\alpha_3^{(6)}$	2 -1	
				$2\alpha_3+2\alpha_4^{(8)}$	-2 2	
4	Sp_4	$SL_2 \times k^\star$	$(2w_1, 1)(w_2, -w_2)$	$\alpha_2^{(1)}$	-1 1 1	
			(w_2, w_2)	$2\alpha_1^{(2)}$	2 -1 -1	
4	Sp_4	$s_{\alpha_1+\alpha_2}$	$(2w_1, \varepsilon)(2w_2, 1)$	$2\alpha_1^{(2)}$	2 -1	
				$2\alpha_2^{(2)}$	-2 2	
4′	Sp_{2n}	$SL_{n-1} \times k^\star \times SL_2$	$(w_1+w_n, w_{n-1})(w_2, 1)$	$\alpha_2+\ldots+\alpha_{n-1}^{(1)}$	-1 1 1	
	$n \geq 3$	$S^2 k^{n-1}$	(w_{n-1}, w_{n-1})	$\alpha_1+\alpha_n^{(4)}$	2 -1 -2	
5	Sp_{2n}	$k^\star \times Sp_{2n-2}$	$(w_1, -w_1)(w_2, 1)$	$\alpha_1^{(1)}$	1 -1 1	
	$n \geq 2$		(w_1, w_1)	$\alpha_1+2\alpha_2+\ldots+2\alpha_{n-1}+\alpha_n^{(9)}$	0 1 0	
5	Sp_{2n}	$s_{2\alpha_1+\ldots+2\alpha_{n-1}+\alpha_n}$	$(2w_1, \varepsilon)(w_2, 1)$	$\alpha_1+2\alpha_2+\ldots+\alpha_n^{(9)}$	0 1	
	$n \geq 2$			$2\alpha_1^{(2)}$	2 -2	
6	Sp_6	$SL_2 \times k^\star$	$(w_1+w_3, 3w_2/2)(w_2, w_2/2)$	$\alpha_2^{(1)}$	-1 1 1	
		$S^2k^2 \oplus S^2k^2$	(w_2, w_2)	$\alpha_1+\alpha_3^{(4)}$	2 0 -3	
6′	Sp_{2n}	$k^\star \times SL_2 \times Sp_{2n-4}$	$(w_2, 1)(w_3, w_1)$	$\alpha_1+\alpha_2^{(1)}$	1 -1 1	
	$n \geq 3$	$k^{2n-4} \oplus k$	(w_1, w_1)	$\alpha_2+2\alpha_3+\ldots+2\alpha_{n-1}+\alpha_n^{(9)}$	0 1 -1	
7	Sp_{2n}	$SL_{n-1} \times k^\star \times k^\star$	$(w_1, w_{n-1}-w_n)(w_n, 2w_{n-1}-w_n)$	$\alpha_1 \ldots \alpha_{n-1}^{(1)}$	1 -1 1 0	
	$n \geq 2$	$k^{n-1} \oplus S^2 k^{n-1}$	$(w_{n-1}, w_{n-1})(w_n, w_n)$	$\alpha_n^{(1)}$	0 1 -2 1	
8	Sp_4	$k^\star \times k^\star$	$(w_1, w_2-w_1)(w_2, 1)$	$\alpha_1^{(1)}$	1 0 1 -1	
		$k_{\alpha_2} \oplus k_{2\alpha_1+\alpha_2}$	$(w_1, w_1)(w_2, w_2)$	$\alpha_2^{(1)}$	-1 1 -1 1	
8	Sp_4	s_{α_1}	$(2w_1, w_2+\varepsilon)(w_2, \varepsilon)$	$\alpha_2^{(1)}$	-1 1 1	
			(w_2, w_2)	$2\alpha_1^{(2)}$	2 0 -2	
8′	Sp_{2n}	$SL_p \times k^\star \times k^\star \times Sp_{2n-2p-2}$	$(w_1, w_{p+1}-w_p)(w_{p+2}, w_p)$	$\alpha_1+\ldots+\alpha_p^{(1)}$	1 0 1 -1	
	$n \geq p+2 \geq 3$	$k^p \otimes k^{2n-2p-2} \oplus k$	$(w_p, w_p)(w_{p+1}, w_{p+1})$	$\alpha_{p+1}+2\alpha_{p+2} \ldots \alpha_n^{(9)}$	0 1 -1 0	
		$\oplus k^p \oplus S^2 k^p$				
9	Sp_4	$k^\star$	$(w_1+w_2, w_1+w_2/2)$	$\alpha_1^{(1)}$	1 1 -2	
		$k_{\alpha_1 \simeq \alpha_2} \oplus k \oplus k$	$(w_1, w_1)(w_2, w_2)$	$\alpha_2^{(1)}$	1 -3 1	

Table D

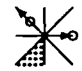
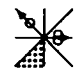
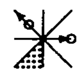
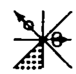
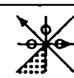
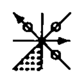
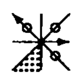
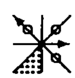
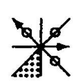
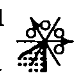
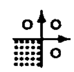
1	Spin_6	$k^\star\times\mathrm{SL}_2\times\mathrm{SL}_2$	$(w_1,-w_1)(w_2+w_3,1)$	$\alpha_1^{(1)}$	1-1 1	
			(w_1,w_1)	$\alpha_2+\alpha_3^{(4)}$	-1 2-1	
1	Spin_6	$s_{\alpha_1}\circ s_{\alpha_1+\alpha_2+\alpha_3}$	$(2w_1,1)(w_2+w_3,\varepsilon)$	$\alpha_2+\alpha_3^{(4)}$	-1 2	
				$2\alpha_1^{(2)}$	2-2	
1	Spin_{2n}	$k^\star\times\mathrm{Spin}_{2n-2}$	$(w_1,-w_1)(w_2,1)$	$\alpha_1^{(1)}$	1-1 1	
	$n\geq 4$		(w_1,w_1)	$2\alpha_2+\ldots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n^{(6)}$	-1 2-1	
1	Spin_{2n}	$s_{\alpha_1}\circ s_{\alpha_1+2\alpha_2+\ldots}$	$(2w_1,1)(w_2,\varepsilon)$	$2\alpha_2\ldots2\alpha_{n-2}+\alpha_{n-1}+\alpha_n^{(6)}$	-1 2	
	$n\geq 4$	$+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n$		$2\alpha_1^{(2)}$	2-2	
2	Spin_8	$\mathrm{SL}_4\times k^\star$	$(w_2,1)(w_{4/3},-w_{4/3})$	$\alpha_{4/3}^{(1)}$	-1 1 1	
			$(w_{4/3},w_{4/3})$	$\alpha_1+2\alpha_2+\alpha_{3/4}^{(6)}$	2-1-1	
2	Spin_8	$s_{\alpha_{4/3}}\circ s_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$	$(w_2,\varepsilon)(2w_{4/3},1)$	$\alpha_1+2\alpha_2+\alpha_{3/4}^{(6)}$	2-1	
				$2\alpha_{4/3}^{(2)}$	-2 2	
3	Spin_{10}	$\mathrm{SL}_5\times k^\star$	$(w_2,1)(w_4,-w_4)$	$\alpha_3+\alpha_4+\alpha_5^{(1)}$	-1 1 1	
			(w_5,w_4)	$\alpha_1+2\alpha_2+\alpha_3^{(6)}$	2-1-1	
4	Spin_{2n}	$\mathrm{SL}_{n-2}\times k^\star\times\mathrm{SL}_2$	$(w_1,1)(w_{n-1}+w_n,w_{n-2})$	$\frac{1}{2}\alpha_{n-1}+\frac{1}{2}\alpha_n^{(3)}$	0 1-1	
	$n\geq 3$	$k^{n-2}\otimes S^2k^2\oplus\wedge^2k^{n-2}$	(w_{n-2},w_{n-2})	$\alpha_1+\ldots+\alpha_{n-2}^{(1)}$	1-1 1	
4	Spin_{2n}	$\mathrm{SL}_{n-2}\times k^\star\times\mathrm{SL}_2$	$(w_1,c)(w_{n-1}+w_n,w_{n-2}+c)$	$\alpha_1+\ldots+\alpha_{n-2}^{(1)}$	1-1 1	
	$n\geq 3$	$k^{n-2}\otimes S^2k^2\oplus\wedge^2k^{n-2}$	(w_{n-2},w_{n-2})	$\alpha_{n-1}+\alpha_n^{(4)}$	0 2-2	
4	Spin_{2n}	$\mathrm{SL}_p\times k^\star\times\mathrm{Spin}_{2n-2p-1}$	$(w_1,1)(w_{p+1},w_p)$	$\alpha_{p+1}+\ldots\frac{1}{2}\alpha_{n-1}+\frac{1}{2}\alpha_n^{(5)}$	0 1-1	
	$n\geq p+3\geq 4$	$k^p\otimes k^{2n-2p-1}\oplus\wedge^2k^p$	(w_p,w_p)	$\alpha_1+\ldots+\alpha_p^{(1)}$	1-1 1	
4	Spin_{2n}	$\mathrm{SL}_p\times k^\star\times\mathrm{Spin}_{2n-2p-1}$	$(w_1,c)(w_{p+1},w_p+c)$	$\alpha_1+\ldots+\alpha_p^{(1)}$	1-1 1	
	$n\geq p+3\geq 4$	$k^p\otimes k^{2n-2p-1}\oplus\wedge^2k^p$	(w_p,w_p)	$2\alpha_{p+1}+\ldots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n^{(6)}$	0 2-2	
4	Spin_8	$\mathrm{SL}_2\times k^\star\times\mathrm{SL}_2$	$(w_{4/3},1)(w_1+w_{3/4},w_2)$	$\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_{3/4}^{(3)}$	0 1-1	
		$S^2k^2\otimes k^2\oplus k$	(w_2,w_2)	$\alpha_2+\alpha_{4/3}^{(1)}$	1-1 1	
4	Spin_8	$\mathrm{SL}_2\times k^\star\times\mathrm{SL}_2$	$(w_{4/3},c)(w_1+w_{3/4},w_2+c)$	$\alpha_2+\alpha_{4/3}^{(1)}$	1-1 1	
		$S^2k^2\otimes k^2\oplus k$	(w_2,w_2)	$\alpha_1+\alpha_{3/4}^{(4)}$	0 2-2	
5	Spin_8	$\mathrm{Sp}_4\times k^\star$	$(w_{4/3},1)(w_2,w_{4/3})$	$\frac{1}{2}\alpha_1+\alpha_2+\frac{1}{2}\alpha_{3/4}^{(5)}$	0 1-1	
		k^5	$(w_{4/3},w_{4/3})$	$\alpha_{4/3}^{(1)}$	1-1 1	
5	Spin_8	$\mathrm{Sp}_4\times k^\star$	$(w_{4/3},c)(w_2,w_{4/3}+c)$	$\alpha_{4/3}^{(1)}$	1-1 1	
		k^5	$(w_{4/3},w_{4/3})$	$\alpha_1+2\alpha_2+\alpha_{3/4}^{(6)}$	0 2-2	
6	Spin_6	$\mathrm{SL}_2\times k^\star\times k^\star$	$(w_1,w_{2/3}-w_{3/2})(w_{3/2},-w_{3/2})$	$\alpha_1+\alpha_{2/3}^{(1)}$	1 0 1-1	
		k^2	$(w_2,w_3)(w_3,w_2)$	$\alpha_{3/2}^{(1)}$	-1 1 0 1	
7	Spin_{2n}	$\mathrm{SL}_{n-1}\times k^\star$	(w_1,c)	$\alpha_1+\ldots+\alpha_{n-2}+\alpha_{n-1}^{(1)}$	1 1-1	
	$n\geq 3$	$k^{n-1}_{\alpha_{n-1}\simeq\alpha_n}\oplus\wedge^2k^{n-1}$	$(w_{n-1},w_i)(w_n,w_j)$	$\alpha_1+\ldots+\alpha_{n-2}+\alpha_n^{(1)}$	1-1 1	
			even $n=i+1=j$, odd $n=i=j+1$			

Table E

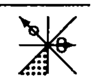
1	E_6	F_4	$(w_1,2c)(w_6,c)$	$2\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5^{(6)}$	2 -1	
				$\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6^{(6)}$	-1 2	
2	E_6	$k^\star\times\mathrm{Spin}_{10}$	$(w_1,-w_1)(w_2,1)$	$\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6^{(1)}$	1 -1 1	
			(w_6,w_1)	$2\alpha_2+\alpha_3+2\alpha_4+\alpha_5^{(6)}$	-1 2 -1	

Table F

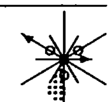
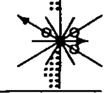
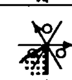
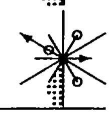
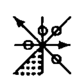
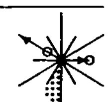
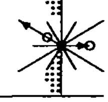
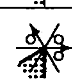
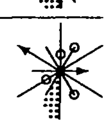
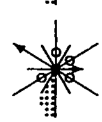
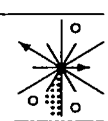
1	F_4	$SL_3\times k^\star\times SL_2$	$(w_1+w_4,w_3)(w_2,w_3)$	$\alpha_2+\alpha_3^{(9)}$	-1 1 0	
		$k^3\otimes S^2k^2\oplus k^2\oplus k^3$	(w_3,w_3)	$\alpha_1+\alpha_4^{(4)}$	2 -1 -1	
2	F_4	$\mathrm{Spin}_7\times k^\star$	$(w_3,w_4)(w_4,1)$	$\alpha_4^{(1)}$	-1 1 1	
		k^7	(w_4,w_4)	$\alpha_1+2\alpha_2+3\alpha_3^{(11)}$	2 -1 -2	
3	F_4	$\mathrm{Spin}_6\times k^\star$	$(w_1,w_4)(w_3,w_4)$	$\alpha_1+\alpha_2+\alpha_3^{(9)}$	1 0 -1	
		$k^4\oplus k^7$	(w_4,w_4)	$\alpha_2+2\alpha_3+\alpha_4^{(9)}$	-1 1 0	
4	F_4	$G_2\times k^\star$	$(w_3,3w_4/2)(w_4,w_4/2)$	$\alpha_4^{(1)}$	-1 1 1	
		$k^7\oplus k^7$	(w_4,w_4)	$\alpha_1+2\alpha_2+3\alpha_3^{(11)}$	2 0 -3	
5	F_4	$SL_3\times k^\star\times k^\star$	$(w_1,w_4)(w_4,w_3-w_4)$	$\alpha_1+\alpha_2+\alpha_3^{(9)}$	1 0 0 -1	
		$k^3\oplus k^{3*}\oplus k^{3*}\oplus k$	$(w_3,w_3)(w_4,w_4)$	$\alpha_4^{(1)}$	0 1 -1 1	
		$\oplus k^{3*}\oplus k\oplus k^3$				
6	F_4	$SL_2\times k^\star\times k^\star\times SL_2$	$(w_1,2w_3-w_2)(w_4,w_2-w_3)$	$\alpha_3+\alpha_4^{(1)}$	0 1 -1 1	
		$k^2\otimes k^{2*}\oplus k^2\otimes S^2k^{2*}$	$(w_2,w_2)(w_3,w_3)$	$\alpha_1+\alpha_2^{(1)}$	1 0 1 -2	
		$\oplus S^2k^{2*}\oplus k^{2*}\oplus k\oplus k^2$				

Table G

1	G_2	$SL_2\times SL_2$	$(2w_1,1)(2w_2,1)$	$2\alpha_1^{(2)}$	2 -1	
				$2\alpha_2^{(2)}$	-3 2	
2	$G_2\times G_2$	G_2	$(w_1+w_1',1)(w_2+w_2',1)$	$\alpha_1+\alpha_1'^{(4)}$	2 -1	
				$\alpha_2+\alpha_2'^{(4)}$	-3 2	
3	G_2	$k^\star\times SL_2$	$(w_1,1)(w_2,w_1)$	$\alpha_1^{(1)}$	1 -1 1	
		k^2	(w_1,w_1)	$\alpha_1+\alpha_2^{(15)}$	0 1 -1	
4	G_2	$k^\star\times k^\star$	$(w_1,w_2-w_1)(w_2,3w_1-w_2)$	$\alpha_1^{(1)}$	1 0 1 -1	
		$k_{\alpha_1+\alpha_2}\oplus k_{2\alpha_1+\alpha_2}$	$(w_1,w_1)(w_2,w_2)$	$\alpha_2^{(1)}$	0 1 -3 1	
		$\oplus k_{3\alpha_1+\alpha_2}\oplus k_{3\alpha_1+2\alpha_2}$				
4	G_2	$k^\star\times k^\star$	$(w_1,w_2-w_1)(w_2,w_1)$	$\alpha_1^{(1)}$	1 0 1 -1	
		$k_{\alpha_2}\oplus k_{2\alpha_1+\alpha_2}$	$(w_1,w_1)(w_2,w_2)$	$\alpha_2^{(1)}$	-1 1 -2 1	
		$\oplus k_{3\alpha_1+\alpha_2}\oplus k_{3\alpha_1+2\alpha_2}$				
5	G_2	$k^\star$	$(w_1+w_2,7w_1/3)$	$\alpha_1^{(1)}$	1 1 -2	
		$k_{\alpha_1-\alpha_2}\oplus k\oplus k\oplus k\oplus k$	$(w_1,w_1)(w_2,w_2)$	$\alpha_2^{(1)}$	1 -4 1	

by $(-\gamma_i)$ is shadowed. The circles represent the image of Δ by ϱ (see Theorem 1.4). Their precise coordinates in the basis (γ_1, γ_2) are given by the coefficients of column 6, as in Table 1.

Remark 3.1. Note that the combinatorial data given in Tables A–G are precisely the data needed to classify all equivariant normal embeddings of G/H, using [LV] or [K1].

Remark 3.2. One can define the *little Weyl group* $\mathrm{W}_X \subset \mathrm{W}$ of a wonderful G-variety X to be the reflection group of $\mathcal{Q}$ having $\mathcal{V}$ as a chamber, see [B3]. Namely, W_X is the group generated by the reflections s_i about the hyperplanes $\{\gamma_i = 0\}$: if γ_i is not a multiple of a root of (G,T), then it is easily seen that there exists a multiple of γ_i which can be written as a sum of two orthogonal roots β and β'. Therefore s_i is induced by $\mathrm{s}_\beta \circ \mathrm{s}_{\beta'} \in \mathrm{W}$.

4. The triplets (L, K, K_x)

The purpose of this section is to describe the homogeneous space L/K (We keep the setting $KU = H^0 \subset Q = LQ^u$, $U \subset Q^u$ and $K \subset L$.) One of the main features of L/K is that it is L-spherical. In fact, to claim that H is spherical in G is equivalent to saying that K has an open orbit $K{\cdot}x \subset Q^u/U$ and that $L/K_x \subset Q/H^0$ is open and L-spherical [B, p. 191].

This decomposition yields the following basic lemma. For any G-variety Y (not necessarily spherical), let

$$\mathrm{rk}_G\, Y := \min_{y \in Y} \mathrm{codim}_Y B^u{\cdot}y.$$

Lemma 4.1. *Let X be a wonderful G-variety of rank r. Then*

$$\mathrm{rk}_L\, L/K_x = \mathrm{rk}_G\, X = r.$$

Proof. Let $B_\natural$ be a Borel subgroup in G such that $B_\natural H$ is open in G. Because $B_\natural Q$ is open in G, there exists a Levi subgroup L_Q of Q such that $B_\natural \cap Q$ is a Borel subgroup in L_Q. Then $\mathrm{codim}_G\, B_\natural^u H^0 = \mathrm{codim}_Q\, (B_\natural^u \cap Q)H^0$. Hence $\mathrm{rk}_G\, G/H^0 = \mathrm{rk}_{L_Q}\, Q/H^0$. Finally note that L/K_x is the open L-orbit in Q/H^0 and that $\mathrm{rk}_G\, G/H^0 = \mathrm{rk}_G\, X = \mathrm{rk}_T\, W = r$, see the beginning of Section 2. □

Lemma 4.2. *A wonderful G-variety X is prime if and only if H^0 satisfies the property* **(pro)** *and the following condition:*

(ind)$^\sharp$ *If $P^u \subset H^0 \subset P$ for a parabolic subgroup P in G, then $P = G$.*

Proof. Assume that H satisfies the property **(ind)** and that $P^r \subset H^0 \subset P$ for a parabolic subgroup P in G. Recall that there exists, see Section 2, a parabolic subgroup $P_\sharp$ in G such that $U \subset P_\sharp^u$ and $H \subset P_\sharp$. Since $P^u \subset U \subset P_\sharp^u$, it follows that $P_\sharp \subset P$. Hence $P^r \subset H \subset P$ and $P = G$. Therefore H^0 satisfies the property **(ind)**.

It remains to prove that if H^0 satisfies the property **(ind)**, then it satisfies the condition **(ind)**$^\sharp$. So assume that $P^u \subset H^0 \subset P$ for a parabolic subgroup P in G. Then the radical of a Levi subgroup in P normalizes H^0 hence is in H^0 thanks to the condition **(aut)** (and the fact that $N_G(H^0) = N_G(H)$, see [BP, p. 283]). Hence $P^r \subset H^0 \subset P$. □

Thus our main goal will be (until the end of Section 6) to determine the set of *prime* connected spherical subgroups $S \subset G$ which satisfy the conditions **(aut)** and $\mathrm{rk}_G\ G/S = 2$. Indeed by Lemma 4.2 this set is one-to-one with the set of G/H^0 of all *prime* wonderful G-varieties of rank two.

Proposition 4.3. *Assume that X is prime of rank two. Then*

$$\mathrm{rk}_L\ L/K + \mathrm{rk}_K\ Q^u/U = 2.$$

Proof. Let B_L be a Borel subgroup in L such that $B_L K_x$ is open in L. Then we have

$$\mathrm{rk}_L\ L/K_x = \mathrm{codim}_L\ B_L^u K_x = \mathrm{codim}_L\ B_L^u K + \mathrm{codim}_K\ (B_L^u \cap K)K_x$$

and $\mathrm{rk}_K\ K/K_x \leq \mathrm{codim}_K\ (B_L^u \cap K)K_x$. So $\mathrm{rk}_L\ L/K +\ \mathrm{rk}_K\ Q^u/U \leq 2$.

Conversely, assume that $K \neq L$, and $Q \neq G$. Because K is reductive, K is not parabolic in L. Hence $\mathrm{rk}_L\ L/K \geq 1$. Now observe that $U \neq Q^u$, otherwise $Q^u \subset H^0 \subset Q$, see the condition **(ind)**$^\sharp$. Therefore $\mathrm{rk}_K\ Q^u/U \geq 1$ since Q^u/U is affine. So $\mathrm{rk}_L L/K +\ \mathrm{rk}_K\ Q^u/U \geq 2$. □

Corollary 4.4. *Assume that X is prime of rank two and $Q \neq G$. Then either $K = L$, $C_L^0 \subset K \neq L$ or there exists a character $\chi \in \mathcal{X}(L)$ such that $K = (\ker \chi)^0$.*

Proof. Assume that $C_L^0 \not\subset K$. Since K is connected, there exists $\chi \in \mathcal{X}(L)$ such that $K \subset K^+ := (\ker \chi)^0$. We have

$$\mathrm{rk}_L\ L/K = \mathrm{codim}_L\ B_L^u K = \mathrm{codim}_L\ B_L^u K^+ + \mathrm{codim}_{K^+}\ (B_L^u \cap K^+)K$$

Since $U \neq Q^u$, see the condition **(ind)**$^\sharp$, we get $\mathrm{rk}_L\ L/K = 1$. Moreover, $\mathrm{codim}_L\ B_L^u K^+ = \mathrm{rk}_L\ L/K^+ = 1$. Therefore $\mathrm{rk}_{K^+}\ K^+/K = 0$, i.e., K is parabolic in K^+. Now recall that K is reductive, so $K = K^+$. □

Let (L, L) denote the derived group of L.

Corollary 4.5. *Assume that X is prime of rank two, $Q \neq G$, and $C_L^0 \subset K \neq L$. Then $K = SC_L^0$ where $S := K \cap (L, L)$ is semisimple. Moreover $(L, L)/S$ is, up to a finite covering, the open orbit of the wonderful (L, L)-variety of rank one corresponding to cases* 3, 5, 7 *or* 10 *in* Table 1.

Proof. By [BP, p. 283], $N_L(K)/K$ is a diagonalizable group and it is easily seen that $\mathrm{rk}_L\ L/N_L(K)+\dim N_L(K)/K = \mathrm{rk}_L\ L/K = 1$. Hence $N_L(K)/K$ is finite. Otherwise $\mathrm{rk}_L\ L/N_L(K) = 0$, i.e., $N_L(K)$ is parabolic in L, and since we have chosen Q minimal such that $H^0 \subset Q$ and $U \subset Q^u$, no strict parabolic subgroup in L contains K: so $N_L(K) = L$. But then K would be a reductive group of codimension one in L with $C_L^0 \subset K$. Contradiction.

Therefore L/K is up to a finite covering isomorphic to the open orbit of a wonderful L-variety of rank one, and clearly K is prime in L. We can forget about cases 4, 6, and 11 in Table 1 thanks to cases 3, 5, and 10 respectively. Since K is connected, cases 2, 8 and 14 are ruled out. Cases 1, 9 and 15 cannot occur because no strict parabolic subgroup in L contains K. Finally, cases 12 and 13 are ruled out as well since $L \neq G$. □

5. The L-module structure on $\mathfrak{q}^u$

We keep the setting $KU = H^0 \subset Q = LQ^u$ as we have done so far. In this section, we estimate the rank of $\mathcal{X}(Q)$ (for arbitrary r) and thereby the rank of Pic X. Then, by investigating the L-module structure on $\mathfrak{q}^u =$ Lie (Q^u), we describe the K-module $\mathfrak{q}^u/\mathfrak{u}$ when r=2. We start with the following basic lemma. Denote by S the basis of the root system R of (G,T) corresponding to B (recall that $B \subset Q$ and $T \subset L$, see Section 2). For any positive root $\beta = \sum_{\alpha\in \mathrm{S}} \lambda_\alpha^\beta \alpha$, define the *support* of β by

$$\mathrm{S}^\beta := \{\alpha \in \mathrm{S} \mid \lambda_\alpha^\beta > 0\}.$$

Lemma 5.1. *Assume that $\beta \notin$ S. Fix $\alpha_p, \alpha_q \in \mathrm{S}^\beta$ such that $\lambda_{\alpha_p}^\beta \geq 2$ if $\alpha_p = \alpha_q$. Then there exists a pair of positive roots $\beta_\triangleright, \beta_\triangleleft \in$ R satisfying the following conditions:*

(1) $\alpha_p \in \mathrm{S}^{\beta_\triangleright}$ and $\alpha_q \in \mathrm{S}^{\beta_\triangleleft}$.
(2) $\beta = \beta_\triangleright + \beta_\triangleleft$.

Proof. Assume that the lemma is not true for some positive roots. Choose β' of minimal height among them. Let $\alpha \in \mathrm{S}\backslash\{\alpha_p,\alpha_q\}$ such that $\beta = \beta' - \alpha$ is a positive root. Fix a decomposition $\beta = \beta_\triangleright + \beta_\triangleleft$ according to properties (1) and (2). It is easily seen that either $\alpha + \beta_\triangleright$ or $\alpha + \beta_\triangleleft$ is a root (use the Jacobi identity). Contradiction. □

Let $\mathfrak{g}_\beta$ denote the weight space in $\mathfrak{g}$ corresponding to β. For a Lie subalgebra $\mathfrak{n} \subset \mathfrak{b}^u$, let $\mathrm{R}(\mathfrak{n}) := \{\beta \in \mathrm{R} \mid \mathfrak{g}_\beta\backslash\{0\} \subset \mathfrak{b}^u\backslash\mathfrak{n}\}$ and $\mathrm{S}(\mathfrak{n}) := \cup_{\beta\in\mathrm{R}(\mathfrak{n})}\mathrm{S}^\beta$.

Lemma 5.2. *For any Lie subalgebra $\mathfrak{n} \subset \mathfrak{b}^u$, $\sum_{\beta\in\mathrm{R}(\mathfrak{n})} \mathbf{Z}{\cdot}\beta = \bigoplus_{\alpha\in\mathrm{S}(\mathfrak{n})} \mathbf{Z}{\cdot}\alpha$.*

Proof. Observe that if card $\mathrm{R}(\mathfrak{n}) \leq 1$ then $\mathrm{R}(\mathfrak{n}) = \mathrm{S}(\mathfrak{n})$. So let us assume that card $\mathrm{R}(\mathfrak{n}) \geq 2$ and let $\beta_1 \in \mathrm{R}(\mathfrak{n}) \cap \mathrm{S}$. Clearly, there exists $\beta_2 \in \mathrm{R}(\mathfrak{n})$ such that $\sum_{\alpha\neq\beta_1} \lambda_\alpha^{\beta_2} = 1$ (apply Lemma 5.1 with $\alpha_p, \alpha_q \in \mathrm{S}^\beta\backslash\{\beta_1\}$). Thus $\mathbf{Z}{\cdot}\beta_1 + \mathbf{Z}{\cdot}\beta_2 = \mathbf{Z}{\cdot}\beta_1 \oplus \mathbf{Z}{\cdot}\alpha$ where $\{\alpha\} = \mathrm{S}^{\beta_2}\backslash\{\beta_1\}$.

Now assume that there is an integer $i \geq 2$ and that there are distinct elements $\beta_1, ..., \beta_i \in \mathrm{R}(\mathfrak{n})$ such that $\mathbf{Z}\cdot\beta_1 + ... + \mathbf{Z}\cdot\beta_i = \bigoplus \mathbf{Z}\cdot\alpha$ where α runs through all elements of $\mathrm{S}_i := \mathrm{S}^{\beta_1} \cup ... \cup \mathrm{S}^{\beta_i}$. If card $\mathrm{R}(\mathfrak{n}) \neq i$, then thanks to Lemma 5.1 there exists a root $\beta_{i+1} \in \mathrm{R}(\mathfrak{n})\backslash\{\beta_1, ..., \beta_i\}$ such that our assumption remains true for $i+1$ instead of i, i. e., $\sum_{\alpha\in \mathrm{S}\backslash \mathrm{S}_i} \lambda_\alpha^{\beta_{i+1}} < 2$. □

It turns out that Lemma 5.2 implies a key bounding statement for the rank of $\mathcal{X}(Q)$. Let S_Q denote the subset of S associated to Q, *i.e.*, the subset satisfying $L = C_G((\bigcap_{\alpha\in \mathrm{S}_Q} \ker \alpha)^0)$.

Theorem 5.3. *Assume that X is prime of rank r. Then* $\mathrm{rk}\mathcal{X}(Q) \leq r$.

Proof. Let B_L be a Borel subgroup of L such that $\dim B_L^u K_x$ is maximal. Since $\mathrm{rk}\ \mathcal{X}(Q) = \dim C_L^0$ and $r = \mathrm{codim}_L\ B_L^u K_x$ (see Lemma 4.1) it suffices to prove that $\Gamma := C_L^0 \cap B_L^u K_x = C_L^0 \cap K_x$ is finite.

Observe that $C_L^0 \subset \bigcap_{\alpha\in \mathrm{S}_Q} \ker \alpha$ and that $K_x \subset \bigcap \ker \beta$ where β runs through all roots such that $\mathfrak{g}_\beta\backslash\{0\} \subset \mathfrak{q}^u\backslash\mathfrak{u}$. In particular, $\Gamma \subset \bigcap_{\beta\in \mathrm{R}(\mathfrak{u})} \ker \beta$. So by Lemma 5.2, $\Gamma \subset \bigcap_{\alpha\in \mathrm{S}(\mathfrak{u})} \ker \alpha$. Now note that if P is the parabolic subgroup in G containing B associated to $\mathrm{S}(\mathfrak{u})$, then $P^u \subset U$ and $Q \subset P$. Hence $P^u \subset H^0 \subset P$. So $P = G$ by the condition $(\mathbf{ind})^\sharp$. Therefore $\mathrm{S}(\mathfrak{u}) = \mathrm{S}$ and Γ is finite. □

Corollary 5.4. *Let P be a minimal parabolic subgroup in G such that $P^r \subset H \subset P$. Then $r \leq \mathrm{rk}\ \mathrm{Pic}\ X - \mathrm{rk}\ \mathcal{X}(P) \leq 2r$.*

Proof. Let $\tilde{H} = H/P^r \subset \tilde{G} = P/P^r$ and let $\tilde{X}$ be as in Lemma 2.2. Consider the setting $\tilde{K}\tilde{U} = \tilde{H}^0 \subset \tilde{Q} = \tilde{L}\tilde{Q}^u$ as we have done so far for H. By Theorem 5.3, we have $\mathrm{rk}\ \mathcal{X}(\tilde{L}) = \mathrm{rk}\ \mathcal{X}(\tilde{Q}) \leq r$. Moreover, $\tilde{K}$ is reductive spherical in $\tilde{L}$ and no strict parabolic subgroup in $\tilde{L}$ contains $\tilde{K}$. In particular, $\mathrm{rk}\ \mathcal{X}(\tilde{K}) \leq \mathrm{rk}\ \mathcal{X}(\tilde{L})$, see for example [Kr, p. 149], [B, p. 190]. Hence $\mathrm{rk}\ \mathcal{X}(\tilde{H}) = \mathrm{rk}\ \mathcal{X}(\tilde{K}) \leq r$. Since $\mathrm{rk}\ \mathrm{Pic}\ X = \mathrm{card}\ \Delta \leq r +\ \mathrm{rk}\ \mathcal{X}(H)$, see Section 2, and $\mathrm{rk}\ \mathcal{X}(H) = \mathrm{rk}\ \mathcal{X}(\tilde{H}) +\ \mathrm{rk}\ \mathcal{X}(P)$, we get $\mathrm{rk}\ \mathrm{Pic}\ X - \mathrm{rk}\ \mathcal{X}(P) \leq 2r$. Finally, the remaining inequality means that $\mathrm{card}\ \Delta \geq r +\ \mathrm{rk}\ \mathcal{X}(P)$ and this is clear (see Section 2). □

The following lemma is just a generalization of the well-known T-module decomposition on $\mathfrak{b}^u$ (and it implies Lemma 1.2). For a positive root β such that $\mathfrak{g}_\beta \subset \mathfrak{q}^u$, let $\langle L \cdot \mathfrak{g}_\beta\rangle \subset \mathfrak{q}^u$ denote the irreducible L-module generated by $\mathfrak{g}_\beta$.

Lemma 5.5. *Let P be a parabolic subgroup in G containing B, and choose $\mathfrak{g}_\beta \subset \mathfrak{p}^u$. Let $L_P = C_G((\bigcap_{\alpha\in \mathrm{S}_P} \ker \alpha)^0)$. Then $V := \langle L_P \cdot \mathfrak{g}_\beta\rangle$ is uniquely determined by $(\lambda_\alpha^\beta)_{\alpha\in \mathrm{S}\backslash \mathrm{S}_P}$. Moreover these integers do not depend on the choice of $\mathfrak{g}_\beta \subset V$.*

Proof. Let $\Lambda_P \subset \Lambda$ denote the root lattices of $((L_P, L_P), T_P) \subset (G, T)$ where $T_P = T \cap (L_P, L_P)$. Then the weights of T in V are in $\beta + \Lambda_P$. Moreover we know from representation theory that the convex hull of these weights (which looks like the convex hull of the corresponding (L_P, L_P)-irreducible module) should meet the convex hull of the weights of any irreducible L-submodule $V' \subset \mathfrak{p}^u$ whose weights lie in $\beta + \Lambda_P$. So V and V' should share a T-eigenspace. Therefore $V = V' \subset \mathfrak{p}^u$.

Finally note that V is an irreducible (L_P, L_P)-module. So the last statement of the lemma is clear. □

Now we apply our previous results to start the classification of the groups H^0 for X prime of rank two.

Theorem 5.6. *Assume that X is prime of rank two and $Q \neq G$. If G is not simple, then $G = \mathrm{SL}_2 \times \mathrm{SL}_2$ and H^0 is a Borel group in the diagonal. (See case* (A8) *in the tables of Section* 3, *i.e., case* 8 *in* Table A.)

Proof. Split G into simple components $G_1 \times ... \times G_l$, $l \geq 2$. Then $\mathrm{S} = \mathrm{S}_1 \cup ... \cup \mathrm{S}_l$ and $Q = Q_1 \times ... \times Q_l$, with Levi decompositions $Q_i = L_i Q_i^u$. First note that thanks to the condition $(\mathbf{ind})^\sharp$, $Q_i^u \not\subset U$ for all i.

We claim that $C_L^0 \not\subset K$. For otherwise, $M \not\simeq N$ for any pair of irreducible K-modules M, N such that $M \subset \mathfrak{q}_i^u, N \subset \mathfrak{q}_j^u$ with $i \neq j$ (a consequence of Lemma 1.2) and therefore $U = U_1 \times ... \times U_l$. So $K \neq L$ thanks to the property **(pro)**. Hence, by Proposition 4.3, $\mathfrak{q}^u/\mathfrak{u}$ is a K-module of rank one. In particular $\mathfrak{q}^u/\mathfrak{u}$ is an irreducible K-module and $U_i = Q_i^u$ for all i except one. Contradiction. This proves the claim.

Thanks to Proposition 4.3 and Corollary 4.4, $K = (\ker \chi)^0$ with $\chi \in \mathcal{X}(L)$, and $\mathfrak{q}^u/\mathfrak{u}$ is an irreducible K-module of rank one. In particular, $\mathfrak{q}^u/\mathfrak{u}$ is isomorphic to a K-module lying in $\mathfrak{q}_i^u$ for each i since $Q_i^u \not\subset U$. This shows that $l = 2$ because $\operatorname{codim}_T K \cap T = 1$. Besides $(L, L) \subset K$. Hence there exists a pair of simple roots $\alpha \in \mathrm{S}_1 \backslash \mathrm{S}_Q$, $\alpha' \in \mathrm{S}_2 \backslash \mathrm{S}_Q$ such that $\mathfrak{g}_\alpha \not\subset \mathfrak{u}$ and $\mathfrak{g}_{\alpha'} \not\subset \mathfrak{u}$. Therefore $K = \ker(\alpha - \alpha')^0$ and $\mathfrak{q}^u/\mathfrak{u} \simeq \langle K \cdot \mathfrak{g}_\alpha \rangle \simeq \langle K \cdot \mathfrak{g}_{\alpha'} \rangle$.

Thanks to Theorem 5.3, $\mathrm{S} \backslash \mathrm{S}_Q = \{\alpha, \alpha'\}$. It follows that $\mathrm{S}_1 = \{\alpha\}$ and $\mathrm{S}_2 = \{\alpha'\}$ since $\langle K \cdot \mathfrak{g}_\alpha \rangle \simeq \langle K \cdot \mathfrak{g}_{\alpha'} \rangle$ and $(L, L) \subset K$. Hence G is of type $A_1 \times A_1$ and K is the diagonal torus. □

A nice consequence of this result is that the group G is simple for most prime wonderful varieties of rank two. Indeed, if $Q = G$, then a computation of ranks, using [Kr, p. 149] and [B, p. 190–191] leads us to the following list. The labels we use below refer to Tables A–G, for instance label BC1 refers to Tables B and C, cases 1.

Proposition 5.7. *Assume that X is prime of rank two and $Q = G$. Then G/H^0 is isomorphic to either one of the following homogeneous spaces:*

(A1) $\mathrm{SL}_3/\mathrm{SO}_3$.

(A3) $\mathrm{SL}_6/\mathrm{Sp}_6$.

(BC1) $\mathrm{Sp}_4 \times \mathrm{SL}_2/\mathrm{SL}_2 \times \mathrm{SL}_2$.

(B3) $\mathrm{Spin}_9/\mathrm{Spin}_7$.

(C3) $\mathrm{Sp}_{2n}/\mathrm{Sp}_4 \times \mathrm{Sp}_{2n-4}$ *for* $n \geq 4$.

(E1) E_6/F_4.
(G1) $G_2/SL_2 \times SL_2$.
(ABCG2) $\hat{G} \times \hat{G}/\hat{G}$ *where* $\hat{G} = SL_3$, Sp_4 *or* G_2.

It remains to consider the cases where G is simple and $Q \neq G$. Thanks to Theorem 5.3, we know that either card $S\backslash S_Q = 1$ or card $S\backslash S_Q = 2$. In the following two theorems, β will denote a positive root satisfying $\mathfrak{g}_\beta \subset \mathfrak{q}^u$.

Theorem 5.8. *Assume that X is prime of rank two. If G is simple and $S\backslash S_Q = \{\alpha_p\}$, then either one of the following possibilities occurs:*
(a) *$K = L$, $\mathfrak{q}^u/\mathfrak{u} \simeq \langle L \cdot \mathfrak{g}_{\alpha_p}\rangle$ is of rank two.*
(b) *$K = L$, $\mathfrak{q}^u/\mathfrak{u} \simeq \langle L \cdot \mathfrak{g}_{\alpha_p}\rangle \oplus \langle L \cdot \mathfrak{g}_\beta\rangle$ is of rank two, and $\lambda^\beta_{\alpha_p} \geq 2$.*
(c) *$\mathrm{rk}_L\ L/K = 1$, $C^0_L \subset K$, $\mathfrak{q}^u/\mathfrak{u}$ is of rank one, and the projection $\mathfrak{q}^u \to \mathfrak{q}^u/\mathfrak{u}$ factors through the L-equivariant projection $\mathfrak{q}^u \to \langle L \cdot \mathfrak{g}_{\alpha_p}\rangle$.*

Proof. We claim that $\langle L \cdot \mathfrak{g}_{\alpha_p}\rangle \not\subset \mathfrak{u}$. Indeed, if $\langle L \cdot \mathfrak{g}_{\alpha_p}\rangle \subset \mathfrak{u}$, then fix β (with minimal height) such that $\mathfrak{g}_\beta\backslash\{0\} \subset \mathfrak{q}^u\backslash\mathfrak{u}$, see the condition **(ind)**$^\sharp$, Section 4. Since $\mathfrak{g}_\beta \not\subset \langle L \cdot \mathfrak{g}_{\alpha_p}\rangle$, we can apply Lemma 5.1 (with $\alpha_p = \alpha_q$). This yields $\beta = \beta_\triangleright + \beta_\triangleleft$ with $\mathfrak{g}_{\beta_\triangleright}, \mathfrak{g}_{\beta_\triangleleft} \subset \mathfrak{u}$. Hence $\mathfrak{g}_\beta \subset \mathfrak{u}$ since $\mathfrak{u}$ is a Lie subalgebra in $\mathfrak{b}^u$. Contradiction. This proves the claim. Now we go through the three cases of Corollary 4.4.

If $K = L$, then by Proposition 4.3, $\mathfrak{q}^u/\mathfrak{u}$ is a K-module of rank two. If it is irreducible, then we get case (a); if it is not, then we get case (b).

If $C^0_L \subset K \neq L$, then by Proposition 4.3, $\mathrm{rk}_K\ \mathfrak{q}^u/\mathfrak{u} = 1$. Hence $\mathfrak{q}^u/\mathfrak{u}$ is an irreducible K-module. Moreover, by Lemma 5.5, $\langle L \cdot \mathfrak{g}_{\alpha_p}\rangle$ is the only L-module in $\mathfrak{q}^u$ on which C^0_L acts with weight α_p. In particular, $\langle L\cdot\mathfrak{g}_\beta\rangle \subset \mathfrak{u}$ whenever $\lambda^\beta_{\alpha_p} \geq 2$, see Lemma 5.5. This yields (c).

Finally, assume that $K = (\ker \chi)^0$ with $\chi \in \mathcal{X}(L)$. Then $(L, L) \subset K$, and by Proposition 4.3, $\mathrm{rk}_K\ \mathfrak{q}^u/\mathfrak{u} = 1$. Hence $\mathfrak{q}^u/\mathfrak{u}$ is an irreducible (L, L)-module isomorphic to $\langle L \cdot \mathfrak{g}_{\alpha_p}\rangle$. Thanks to the condition **(aut)**, $\mathfrak{u}$ is embedded diagonally in $\mathfrak{q}^u$, *i.e.*, there exists (at least one) irreducible L-module $V\backslash\{0\} \subset \mathfrak{q}^u\backslash\mathfrak{u}$ such that $V \not\simeq \langle L\cdot\mathfrak{g}_{\alpha_p}\rangle$ as L-modules but $V \simeq \langle L\cdot \mathfrak{g}_{\alpha_p}\rangle$ as K-modules. Fix β with maximal height such that $\mathfrak{g}_\beta\backslash\{0\} \subset \mathfrak{q}^u\backslash\mathfrak{u}$. Lemma 5.5 yields $\lambda^\beta_{\alpha_p} \geq 2$. So we apply Lemma 5.1 (with $\alpha_p = \alpha_q$) as many times as necessary to find a set of β_i such that $\beta = \sum \beta_i$ and $\mathfrak{g}_{\beta_i} \subset \langle L\cdot\mathfrak{g}_{\alpha_p}\rangle$, i.e., $\lambda^{\beta_i}_{\alpha_p} = 1$. Then for each i, there exists a $K \cap T$-eigenvector $v_i \in \mathfrak{u}$ such that its component (with respect to the T-decomposition on $\mathfrak{b}^u$) on $\mathfrak{g}_{\beta_i}$ is non zero. In particular, these vectors v_i generate in $\mathfrak{u}$ a Lie subalgebra containing a vector v having a non zero component on $\mathfrak{g}_\beta$. Now note that all the remaining components of v are in $\mathfrak{u}$ since the height of β was chosen maximal. Hence $\mathfrak{g}_\beta \in \mathfrak{u}$. Contradiction. Therefore this last case is ruled out. $\square$

Theorem 5.9. *Assume that X is prime of rank two. If G is simple and $S\backslash S_Q = \{\alpha_p, \alpha_q\}$ with $p \neq q$, then either of the following possibilities ocurs:*

(a) *$K = L$, $\mathfrak{q}^u/\mathfrak{u} \simeq \langle L \cdot \mathfrak{g}_{\alpha_p}\rangle \oplus \langle L \cdot \mathfrak{g}_\beta\rangle$ is of rank two, and $\lambda^\beta_{\alpha_q} = 1$.*
(b) *$K = \ker(\alpha_p - \alpha_q)^0$ and $\mathfrak{q}^u/\mathfrak{u} \simeq \langle K \cdot \mathfrak{g}_{\alpha_p}\rangle \simeq \langle K \cdot \mathfrak{g}_{\alpha_q}\rangle$ is of rank one.*

Proof. By Lemmas 5.1 and 5.5, there is an irreducible L-module in $\mathfrak{q}^u/\mathfrak{u}$ with lowest weight a simple root. So there exists $\alpha \in \mathrm{S}\backslash\mathrm{S}_Q$ such that $\langle L \cdot \mathfrak{g}_\alpha\rangle \not\subset \mathfrak{u}$, say $\alpha = \alpha_p$. If P is the parabolic subgroup in G containing B associated to $\mathrm{S}_Q \cup \{\alpha_p\}$, then $P^u \not\subset U$ thanks to the condition **(ind)**$^\sharp$. Hence there exists a root β such that $\mathfrak{g}_\beta\backslash\{0\} \subset \mathfrak{p}^u\backslash\mathfrak{u}$. In particular $\alpha_q \in \mathrm{S}^\beta$ and $\mathfrak{g}_\beta\backslash\{0\} \subset \mathfrak{q}^u\backslash\mathfrak{u}$. Choose β (respectively β') with maximal (respectively minimal) height. (Note that Lemma 5.1 yields $\lambda^{\beta'}_{\alpha_q} = 1$.) Again, we go through the cases of Corollary 4.4.

If $K = L$, then $\mathfrak{q}^u/\mathfrak{u}$ is of rank two by Proposition 4.3. Thanks to the condition **(ind)**$^\sharp$, $\mathfrak{q}^u/\mathfrak{u}$ is not irreducible. Moreover the rank of the sum of m irreducible L-modules is at least m. So $\mathfrak{q}^u/\mathfrak{u} \simeq \langle L \cdot \mathfrak{g}_{\alpha_p}\rangle \oplus \langle L \cdot \mathfrak{g}_\beta\rangle$ and $\langle L \cdot \mathfrak{g}_\beta\rangle = \langle L \cdot \mathfrak{g}_{\beta'}\rangle$. This yields case (a).

Assume that $C^0_L \subset K \neq L$. Then there are two irreducible K-modules $M \subset \langle L \cdot \mathfrak{g}_{\alpha_p}\rangle$ and $N \subset \langle L \cdot \mathfrak{g}_\beta\rangle$ such that the two corresponding classes of irreducible K-modules occur in $\mathfrak{q}^u/\mathfrak{u}$. Moreover, $M \not\simeq N$ thanks to Lemma 5.5. Therefore $\mathrm{rk}_K\ \mathfrak{q}^u/\mathfrak{u} \geq \mathrm{rk}_K\ M \oplus N \geq 2$. This contradicts Proposition 4.3.

If $K = (\ker \chi)^0$ with $\chi \in \mathcal{X}(L)$, then $\mathrm{rk}_K\ \mathfrak{q}^u/\mathfrak{u} = 1$. So there are K-module isomorphisms $\mathfrak{q}^u/\mathfrak{u} \simeq \langle K \cdot \mathfrak{g}_{\alpha_p}\rangle \simeq \langle K \cdot \mathfrak{g}_\beta\rangle \simeq \langle K \cdot \mathfrak{g}_{\beta'}\rangle$. Moreover $\lambda^\beta_{\alpha_p} + \lambda^\beta_{\alpha_q} \geq 2$ never occurs. Indeed, for otherwise apply Lemma 5.1 as many times as necessary to find a set of β_i such that $\beta = \sum \beta_i$ and $\lambda^{\beta_i}_{\alpha_p} + \lambda^{\beta_i}_{\alpha_q} = 1$ for all i, and then follow the end of the proof of Theorem 5.8 to obtain a contradiction. Therefore $\lambda^\beta_{\alpha_p} + \lambda^\beta_{\alpha_q} = 1$. Since $\alpha_q \in \mathrm{S}^\beta$, we get $\lambda^\beta_{\alpha_q} = 1$ and $\lambda^\beta_{\alpha_p} = 0$. So $\langle L \cdot \mathfrak{g}_\beta\rangle = \langle L \cdot \mathfrak{g}_{\alpha_q}\rangle$ and $\beta' = \alpha_q$. Thus $K = \ker(\alpha_p - \alpha_q)^0$ and this yields (b). □

6. The pairs $(\mathfrak{g}, \mathfrak{h})$

The purpose of this section is to make a list containing all pairs $(\mathfrak{g}, \mathfrak{h})$ for X prime of rank two. Thanks to Theorem 5.6 and Proposition 5.7, we shall assume that G is simple and $Q \neq G$. We say that $(\mathfrak{g}_1, \mathfrak{h}_1)$ is *isomorphic* to $(\mathfrak{g}_2, \mathfrak{h}_2)$ if H^0_1 is conjugate to H^0_2 in $G_1 = \mathrm{G}_2$, see Remark 1.5. Our goal is to determine the pairs $(\mathfrak{g}, \mathfrak{h})$ up to isomorphism.

Remark 6.1. The pairs $(\mathfrak{g}, \mathfrak{h})$ that we give in this section are pairwise non isomorphic (unless otherwise stated). This might not be clear at once. But it will follow from the combinatorial data computed in Section 7 (and gathered in columns 5 and 7 of Tables A–G). Similarly, the condition **(aut)** will follow from these combinatorial data together with the eigenvectors of H^0 in the representations of G (gathered in column 4 of Tables A–G) except for the pairs $(\mathfrak{g}, \mathfrak{h})$ corresponding to Theorem 5.6 and Proposition 6.8 for which the condition **(aut)** can be easily checked on the Lie algebra level. (Note that $N_G(S^0) = N_G(S)$ for a spherical subgroup S in G [BP, p. 283].)

First of all, let us state the following routine lemma when G is of type $\mathrm{Y} = \mathrm{A}_n$, B_n, C_n or D_n. Let P_{rs} be a parabolic subgroup in G such that $\mathrm{S}_{P_{rs}} = \mathrm{S}\backslash\{\alpha_r, \alpha_s\}$ with $r \leq s$. Define $M := \mathrm{Hom}\,(V, k^{s-r}) \oplus \mathrm{Hom}\,(k^{s-r}, k^r) \oplus \mathrm{Hom}\,(V, k^r)$, where V denotes the irreducible representation of highest weight w_1 for Y_{n-s}.

Lemma 6.2. *Assume that* $\mathrm{Y} = \mathrm{A}_n$, B_n, C_n *or* D_n. *Then the semisimple type of Levi subgroups in* P_{rs} *is* $\mathrm{A}_{r-1} \times \mathrm{A}_{s-r-1} \times \mathrm{Y}_{n-s}$. *Moreover, for*

$$
\begin{aligned}
\mathrm{Y} = \mathrm{A}_n : &\quad \mathfrak{p}^u_{rs} \simeq M,\\
\mathrm{Y} = \mathrm{B}_n : &\quad \mathfrak{p}^u_{rs} \simeq M \oplus k^r \otimes k^{s-r} \oplus \wedge^2 k^r \oplus \wedge^2 k^{s-r},\\
\mathrm{Y} = \mathrm{C}_n : &\quad \mathfrak{p}^u_{rs} \simeq M \oplus k^r \otimes k^{s-r} \oplus S^2 k^r \oplus S^2 k^{s-r},\\
\mathrm{Y} = \mathrm{D}_n : &\quad \mathfrak{p}^u_{rs} \simeq M \oplus k^r \otimes k^{s-r} \oplus \wedge^2 k^r \oplus \wedge^2 k^{s-r} \quad \textit{when } s \neq n-1.
\end{aligned}
$$

Remark 6.3. One can easily compute similar formulas for the exceptional groups. For instance, if G is of type E_6, then the semisimple type of Levi subgroups of P_{12} is A_4 and $\mathfrak{p}^u_{12} \simeq k^{5*} \oplus \wedge^2 k^{5*} \oplus \wedge^2 k^5 \oplus k$. In the proofs of this section, we shall freely use these decompositions.

Now we go through the five cases of Theorems 5.8 and 5.9. In the following propositions, we check that K has an open orbit in Q^u/U and that Q/H^0 is L-spherical of rank two, see the notations of Section 4. Recall that the rank of an L-variety Y is the minimal codimension of B^u_L-orbits in Y, where B_L is a Borel subgroup in L. The labels below refer to Tables A–G, for instance label BC4 refers to Tables B and C, cases 4.

Proposition 6.4. *Assume that* $\mathrm{S}\backslash\mathrm{S}_Q = \{\alpha_p\}$, $K = L$, *and that* $\mathfrak{q}^u/\mathfrak{u} \simeq \langle L \cdot \mathfrak{g}_{\alpha_p} \rangle$ *is of rank two. Then up to isomorphism,* (Y, p) *is either one of the following pairs:*

(A4) $(\mathrm{A}_n, 2)$ *for* $n \geq 3$.
(BC4) $(\mathrm{B}_n, 1)$ *for* $n \geq 2$.
(C4′) $(\mathrm{C}_n, n-1)$ *for* $n \geq 3$.
(D1) $(\mathrm{D}_n, 1)$ *for* $n \geq 3$.
(D2) $(\mathrm{D}_4, 3(\text{respectively } 4))$.
(D3) $(\mathrm{D}_5, 4)$.
(E2) $(\mathrm{E}_6, 1)$.
(F1) $(\mathrm{F}_4, 3)$.
(F2) $(\mathrm{F}_4, 4)$.

Furthermore, cases (A4) *and* (D1) *yield isomorphic pairs* $(\mathfrak{g}, \mathfrak{h})$ *when* $n = 3$.

Proof. If $\mathrm{Y} = \mathrm{A}_n$, then by Lemma 6.2, we have $\mathfrak{q}^u = \mathfrak{p}^u_{pp} \simeq \mathrm{Hom}(k^{n-p+1}, k^p)$ and (L, L) is up to a finite covering $\mathrm{SL}_p \times \mathrm{SL}_{n-p+1}$. Hence $\mathfrak{u} = \{0\}$. Moreover, $\mathfrak{p}^u_{pp}$ is L-spherical and $\mathrm{rk}_L\, \mathfrak{p}^u_{pp} = \min\,(p, n-p+1)$. Therefore $n \geq 3$, and $p = 2$ or $p = n-1$. Now note that $(\mathrm{A}_n, 2) \simeq_{\mathrm{w}} (\mathrm{A}_n, n-1)$ where w$=$ $s_{\alpha_1+\ldots+\alpha_{n-1}} \circ s_{\alpha_2+\ldots+\alpha_n}$, i.e., the corresponding groups H^0 are conjugate in G by $\hat{w} \in N_G(T)$ (see the introduction). Since $\mathfrak{q}^u/\mathfrak{u}$ is obviously spherical, this gives (D1) for $n = 3$ and (A4) for all n.

When $\mathrm{Y}{=}\mathrm{D}_n$, $n \geq 4$, we have $\mathfrak{q}^u \simeq M \oplus \wedge^2 k^p$ with $M = \mathrm{Hom}(k^{2(n-p)}, k^p)$ (this is only true for $p \neq n-1$, but this restriction can easily be overcome thanks to the involution of the Dynkin diagram of D_n). If $p = n$, then $M = \{0\}$, and $\mathrm{rk}_L\ \wedge^2 k^p = 2$ if and only if $n = 4$ or $n = 5$. Moreover, for $n = 5$ we have $(\mathrm{D}_5, 4) \simeq_{\mathrm{w}} (\mathrm{D}_5, 5)$ where w$=$ $s_{\alpha_3+\alpha_4} \circ s_{\alpha_3+\alpha_5}$. This gives (D2) and (D3). If $p \leq n-2$, then $\mathfrak{q}^u/\mathfrak{u} \simeq M$, and $\mathrm{rk}_L\ M = 2$ if and only if $p = 1$. This gives (D1) for $n \geq 4$.

If $\mathrm{Y} = \mathrm{B}_n$, $n \geq 2$, then we find that $\mathfrak{q}^u = M \oplus \wedge^2 k^p$ with $\mathfrak{q}^u/\mathfrak{u} \simeq M = \mathrm{Hom}\ (k^{2(n-p)+1}, k^p)$. Now $\mathrm{rk}_L\ M = 2$ if and only if $p = 1$. This gives (BC4).

For $\mathrm{Y} = \mathrm{C}_n$, $n \geq 3$, $\mathfrak{q}^u \simeq M \oplus S^2 k^p$ with $M = \mathrm{Hom}\ (k^{2(n-p)}, k^p)$. If $p = n$ then $M = 0$ and $\mathrm{rk}_L\ S^2 k^p > 2$. Therefore $p < n$ and $\mathfrak{q}^u/\mathfrak{u} \simeq M$. Since $\mathrm{rk}_L\ M = 2$ if and only if $p = n-1$, this gives (C4′).

If $\mathrm{Y} = \mathrm{E}_n$, $n = 6$, 7 or 8, then $\mathrm{rk}_L\ \langle L \cdot \mathfrak{g}_{\alpha_p} \rangle = 2$ if and only if $n = 6$ and $p = 1$ or 6. Then (L, L) is of type D_5 and $\langle L \cdot \mathfrak{g}_{\alpha_p} \rangle \simeq k^{16}$ is one of the two (nonequivalent) half spin representations of D_5. Moreover, $(\mathrm{E}_6, 1) \simeq_{\mathrm{w}} (\mathrm{E}_6, 6)$ where w$=$ $s_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6} \circ s_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6}$. This gives (E2).

If $\mathrm{Y} = \mathrm{F}_4$, then $\mathrm{rk}_L\ \langle L \cdot \mathfrak{g}_{\alpha_p} \rangle = 2$ if and only if $p = 3$ or $p = 4$. When $p = 3$, (L, L) is of type $\mathrm{A}_2 \times \mathrm{A}_1$ and $\langle L \cdot \mathfrak{g}_{\alpha_p} \rangle \simeq k^3 \otimes k^2$. When $p = 4$, (L, L) is of type B_3 and $\langle L \cdot \mathfrak{g}_{\alpha_p} \rangle \simeq k^8$ is the spin representation of B_3. This gives (F1) and (F2).

Finally the case $\mathrm{Y} = \mathrm{G}_2$ is ruled out since we get $\mathrm{rk}_L\ \langle L \cdot \mathfrak{g}_{\alpha_1} \rangle = 1$ and $\mathrm{rk}_L\ \langle L \cdot \mathfrak{g}_{\alpha_2} \rangle = 3$. □

Proposition 6.5. *Assume that* $\mathrm{S} \backslash \mathrm{S}_Q = \{\alpha_p\}$, $K = L$ *and that* $\mathfrak{q}^u/\mathfrak{u} \simeq \langle L \cdot \mathfrak{g}_{\alpha_p} \rangle \oplus \langle L \cdot \mathfrak{g}_\beta \rangle$ *is of rank two, with* $\lambda^\beta_{\alpha_p} \geq 2$. *Then up to isomorphism,* $(\mathrm{Y}, p, \mathfrak{u})$ *is either one of the following triplets:*

(BC5) $(\mathrm{C}_n, 1, \{0\})$ *for* $n \geq 2$.

(G3) $(\mathrm{G}_2, 1, \mathrm{k}^2)$.

Proof. First make the observation that a *necessary* condition for $\mathrm{rk}_L\ \mathfrak{q}^u/\mathfrak{u} = 2$ is that $\mathrm{rk}_L\ \langle L \cdot \mathfrak{g}_{\alpha_p} \rangle = \mathrm{rk}_L\ \langle L \cdot \mathfrak{g}_\beta \rangle = 1$.

When $\mathrm{Y} = \mathrm{A}_n$, $\mathfrak{q}^u$ is irreducible. So this case is obviously ruled out.

If $\mathrm{Y} = \mathrm{C}_n$, $n \geq 2$, then we need to have $p < n$ and $\mathfrak{u} = 0$ in order to get a non irreducible $\mathfrak{q}^u/\mathfrak{u}$. Moreover, $\mathrm{rk}_L\ S^2 k^p = 1$ if and only if $p = 1$. So $p = 1$ and thus $\mathfrak{q}^u/\mathfrak{u}$ is obviously spherical. This gives (BC5).

If $\mathrm{Y} = \mathrm{B}_n$, $n \geq 3$, then $\mathrm{rk}_L\ \mathrm{Hom}(k^{2(n-p)+1}, k^p) = 1$ if and only if $p = n$. Therefore we need $p = n$. Now $\mathrm{rk}_L\ \wedge^2 k^p = 1$ if and only if $n = 3$. Thus $\mathfrak{q}^u \simeq \mathrm{Hom}(k, k^3) \oplus \wedge^2 k^3 = k^3 \oplus k^{3*}$. Therefore $\mathfrak{q}^u$ is of rank 3. So this case is ruled out.

The case $\mathrm{Y} = \mathrm{D}_n$, $n \geq 4$ is also ruled out. Indeed, $p \leq n-2$ (otherwise $\mathfrak{q}^u$ is irreducible) and hence $\mathrm{rk}_L\ \mathrm{Hom}(k^{2(n-p)}, k^p) \geq 2$.

If $\mathrm{Y} = \mathrm{E}_6$, E_7, E_8 or F_4, then $\mathrm{rk}_L\ \langle L \cdot \mathfrak{g}_{\alpha_p} \rangle \geq 2$ for all p.

Finally when $\mathrm{Y} = \mathrm{G}_2$, $\mathrm{rk}_L \langle L \cdot \mathfrak{g}_{\alpha_2} \rangle = 3$. Therefore $p = 1$ and this gives (G3). $\square$

A subgroup $S \subset G$ is called *horospherical* if it contains a maximal unipotent subgroup in G. By [P], this is equivalent to S being spherical in G and $\dim N_G(S)/S = \mathrm{rk}_G\, G/S$. Let Y_L and Y_K denote the semisimple types of L and K respectively.

Proposition 6.6. *Assume that* $\mathrm{S}\backslash\mathrm{S}_Q = \{\alpha_p\}$, $C_L^0 \subset K$, $\mathrm{rk}_L\, L/K = 1$, *that* $\mathfrak{q}^u/\mathfrak{u}$ *is a* K*-module of rank one and that there is a surjective* K*-homomorphism* $\langle L \cdot \mathfrak{g}_{\alpha_p} \rangle \twoheadrightarrow \mathfrak{q}^u/\mathfrak{u}$. *Then* $(\mathrm{Y}, p, \mathrm{Y}_L, \mathrm{Y}_K, \mathfrak{q}^u/\mathfrak{u})$ *is, up to isomorphism, either one of the following data:*

(A5) $(\mathrm{A}_3, 2, \mathrm{A}_1 \times \mathrm{A}_1, \mathrm{A}_1, k)$.
(B6) $(\mathrm{B}_n, p, \mathrm{A}_{p-1} \times \mathrm{B}_{n-p}, \mathrm{A}_{p-1} \times \mathrm{D}_{n-p}, k^p)$ *for* $n - p \geq 2$.
(C6) $(\mathrm{C}_3, 2, \mathrm{A}_1 \times \mathrm{A}_1, \mathrm{A}_1, k)$.
(C6′) $(\mathrm{C}_n, 1, \mathrm{C}_{n-1}, \mathrm{A}_1 \times \mathrm{C}_{n-2}, k^2)$ *for* $n \geq 3$.
(D4) $(\mathrm{D}_n, p, \mathrm{A}_{p-1} \times \mathrm{D}_{n-p}, \mathrm{A}_1 \times \mathrm{B}_{n-p-1}, k^p)$ *for* $n - p \geq 2$.
(D5) $(\mathrm{D}_4, 3(\text{respectively } 4), \mathrm{A}_3, \mathrm{C}_2, k)$.
(F3) $(\mathrm{F}_4, 4, \mathrm{B}_3, \mathrm{D}_3, k^4)$.
(F4) $(\mathrm{F}_4, 4, \mathrm{B}_3, \mathrm{G}_2, \mathrm{k})$.

Furthermore, cases (A5) *and* (D4) *for* $n = 3$ *yield isomorphic pairs* $(\mathfrak{g}, \mathfrak{h})$.

Proof. Recall that Corollary 4.5 gives us the candidates for L/K. If $\mathrm{Y} = \mathrm{A}_n$, then case 3 in Table 1 yields $(\mathrm{A}_3, 2, \mathrm{A}_1 \times \mathrm{A}_1, \mathrm{A}_1, k)$. This gives (A5) and (D4) for $n = 3$, since L/K_x (see the notations of Section 4) is in this case up to a finite covering $\mathrm{SL}_2 \times k^\star \times \mathrm{SL}_2/S$ where $S \simeq \mathrm{SL}_2$ is embedded diagonally. Case 5A yields $(\mathrm{A}_4, 1, \mathrm{A}_3, \mathrm{C}_2, k^4)$ and $(\mathrm{A}_4, 4, \mathrm{A}_3, \mathrm{C}_2, k^4)$. Indeed, $\langle L \cdot \mathfrak{g}_{\alpha_p} \rangle \simeq k^p \otimes k^4$ (respectively $k^4 \otimes k^{n-3}$) is an irreducible $\mathrm{SL}_p \times \mathrm{Sp}_4$ (respectively $\mathrm{Sp}_4 \times \mathrm{SL}_{n-3}$)-module and $\mathrm{rk}_K\, k^p \otimes k^4$ (respectively $k^4 \otimes k^{n-3}$)$= 1$ if and only if $p = 1$ (respectively $n = 4$). Now note that in both situations $\mathrm{rk}_L\, L/K_x \geq \dim L/K +\dim \mathfrak{q}^u/\mathfrak{u} - \dim B_L^u$. So $\mathrm{rk}_L\, L/K_x \geq 5 + 4 - 6 = 3$. Hence case 5A is ruled out.

If $\mathrm{Y} = \mathrm{B}_n$, $n \geq 2$, then case 3 in Table 1 yields $(\mathrm{B}_3, 2, \mathrm{A}_1 \times \mathrm{A}_1, \mathrm{A}_1, k^2)$ because we have $\langle L \cdot \mathfrak{g}_{\alpha_2} \rangle \simeq S^3 k^2 \oplus k^2$ (as K-modules) and $\mathrm{rk}_K\, S^3 k^2 = 3$. But this case is not good since $\dim L/K = 3$, $\dim \mathfrak{q}^u/\mathfrak{u} = 2$ and $\dim B_L^u = 2$. Case 5A yields $(\mathrm{B}_4, 4, \mathrm{A}_3, \mathrm{C}_2, k^4)$. Indeed, $\langle L \cdot \mathfrak{g}_{\alpha_4} \rangle \simeq k^4 \otimes k^{2(n-4)+1}$ is an irreducible K-module, and $\mathrm{rk}_K\, k^4 \otimes k^{2(n-4)+1} = 1$ if and only if $n = 4$. But when $n = 4$, $\dim L/K_x = 9$ and $\dim B_L^u = 6$. Thus case 5A is again ruled out. Case 7B yields $(\mathrm{B}_n, p, \mathrm{A}_{p-1} \times \mathrm{B}_{n-p}, \mathrm{A}_{p-1} \times \mathrm{D}_{n-p}, k^p)$ for $n - p \geq 2$. For there is a K-module isomorphism $\langle L \cdot \mathfrak{g}_{\alpha_p} \rangle \simeq k^p \otimes k^{2(n-p)} \oplus k^p$ and clearly $\mathrm{rk}_K\, k^p \otimes k^{2(n-p)} > 1$ for $n - p \geq 2$. This gives (B6) since L/K_x is then up to a finite covering $\mathrm{SL}_p \times k^\star \times \mathrm{Spin}_{2(n-p)+1}/S \times \mathrm{Spin}_{2(n-p)}$ where S is a generic isotropy group of $\mathrm{SL}_p \times k^\star$ in k^p. Finally case 10 is ruled out since $\langle L \cdot \mathfrak{g}_{\alpha_p}\rangle \simeq k^p \otimes k^7$ is an irreducible $SL_p \times G_2$-module and $\mathrm{rk}_{G_2}\ k^7 \geq \mathrm{rk}_{\mathrm{Spin}_7}\ k^7 = 2$.

If $Y = C_n$, $n \geq 3$, then case 3 gives clearly (C6) (see case A5)). Case 5A yields $(C_4, 4, A_3, C_2, S^2k^4)$ because if $n > 4$, then $\langle L \cdot \mathfrak{g}_{\alpha_4}\rangle \simeq k^4 \otimes k^{2(n-4)}$ is an irreducible K-module and $\mathrm{rk}_K\ k^4 \otimes k^{2(n-4)} > 1$. But since $\mathrm{rk}_K\ S^2k^4 > 1$, this case is ruled out. It remains to look at case 7C which yields $(C_n, p, A_{p-1} \times C_{n-p}, A_{p-1} \times A_1 \times C_{n-p-1}, k^p \otimes k^2$ (respectively $k^p \otimes k^{2(n-p-1)}$)) for $n - p \geq 2$. Indeed, $\langle L \cdot \mathfrak{g}_{\alpha_p}\rangle \simeq k^p \otimes k^{2(n-p)} \simeq k^p \otimes k^2 \oplus k^p \otimes k^{2(n-p-1)}$ as K-modules. Moreover $\mathrm{rk}_K\ k^p \otimes k^2$ (respectively $k^p \otimes k^{2(n-p-1)}$) $= 1$ if and only if $p = 1$. So we get $(C_n, 1, C_{n-1}, A_1 \times C_{n-2}, k^2$ (respectively k^{2n-4})). Furthermore, when $n = 3$, s_{α_2} identifies (see the proof of Proposition 6.4) the two candidates for $(\mathfrak{g}, \mathfrak{h})$. We claim that $\mathfrak{q}^u/\mathfrak{u} \neq k^{2n-4}$ when $n \geq 4$. Let P be a parabolic subgroup in Sp_{2n-4} such that $Sp_{2n-4}/P \simeq \mathbf{P}(k^{2n-4})$. Then $Sp_{2n-2}/SL_2 \times P$ is not spherical of rank one: otherwise, since it is prime and not in Table 1, it would be horospherical. Therefore L/K_x cannot be spherical of rank two when $\mathfrak{q}^u/\mathfrak{u} = k^{2n-4}$ since, up to a finite covering, L/K_x would be $Sp_{2n-2} \times k^\star/SL_2 \times S$ where S is a generic isotropy group of $Sp_{2n-4} \times k^\star$ in k^{2n-4}. This proves our claim. So $\mathfrak{q}^u/\mathfrak{u} = k^2$. Then L/K_x is up to a finite covering $k^\star \times Sp_{2n-2}/S \times Sp_{2n-4}$ where S is a generic isotropy group of $k^\star \times SL_2$ in k^2. Hence L/K_x is spherical of rank two, see 7C and 9C in Table 1. This gives (C6′).

If $Y = D_n$, $n \geq 4$, then case 3 yields $(D_n, n-2, A_{n-3} \times A_1 \times A_1, A_{n-3} \times A_1, k^{n-2})$ since we have necessarily $p = n-2$ and $\langle L \cdot \mathfrak{g}_{\alpha_{n-2}}\rangle \simeq k^{n-2} \otimes k^2 \otimes k^2 \simeq k^{n-2} \otimes S^2k^2 \oplus k^{n-2} \otimes k$ as K-modules. This gives clearly (D4) for $n - p = 2$ (see case (A5)). Cases 5A–5D yield $(D_4, 4$ (respectively $3), A_3, C_2, k)$, $(D_n, p, A_{p-1} \times D_{n-p}, A_{p-1} \times B_{n-p-1}, k^p)$ for $n - p \geq 3$, and $(D_n, n-4, A_{n-5} \times D_4, A_{n-5} \times B_3, k^{n-4} \times k^8)$ where k^8 is the spin representation of B_3. Indeed, $\mathrm{rk}_{\mathrm{Spin}_{2l}}\ k^{2l} = 2$, hence either $n = 4$, $p = 4$ (respectively 3) and $\langle L \cdot \mathfrak{g}_{\alpha_p}\rangle \simeq \wedge^2 k^4 \simeq k^5 \oplus k$ as C_2-modules, or $n - p \geq 3$ and $\langle L \cdot \mathfrak{g}_{\alpha_p}\rangle \simeq k^p \otimes k^{2(n-p)} \simeq k^p \otimes k^{2(n-p-1)+1} \oplus k^p \otimes k$ as $A_{p-1} \times B_{n-p-1}$-module (observe that $C_2 = B_2$ and $\mathrm{rk}_{\mathrm{Spin}_{2l+1}}\ k^{2l+1} = 2$). This gives clearly (D5), and also (D4) for $n - p \geq 3$ (see case (B6)). But the last candidate is ruled out since $\mathrm{rk}_K\ k^{n-4} \otimes k^8 > 1$.

If $Y = E_6$, E_7 or E_8, then case 5A yields $p = 4$ for E_7 (respectively $p = 5$ for E_8) with $\langle L \cdot \mathfrak{g}_{\alpha_p}\rangle \simeq k^3 \otimes k^2 \otimes k^4$ (respectively $\wedge^2 k^5 \otimes k^4$) as an $SL_3 \times SL_2 \times Sp_4$ (respectively $SL_5 \times Sp_4$)-module. But then $\mathrm{rk}_K\ \langle L \cdot \mathfrak{g}_{\alpha_p}\rangle > 1$. Case 5D ($n \geq 4$) yields $p = 1$ or 6 (for the three groups) with $\langle L \cdot \mathfrak{g}_{\alpha_p}\rangle \simeq k^{2^l} \otimes k^m$ as a $D_{l+1} \times A_{m-1}$-module, where k^{2^l} is one of the two spin representations of D_{l+1} (l=4, 5 or 6). Moreover k^{2^l} is also the spin representation of B_l, hence $k^{2^l} \otimes k^m$ is an irreducible K-module. But $\mathrm{rk}_{\mathrm{Spin}_{2l+1}}\ k^{2^l} = 1$ if and only if $l = 2$. So type E is ruled out.

If $Y = F_4$, then case 7B requires that $p = 4$ and this yields $(F_4, 4, B_3, D_3, k^4)$

$\simeq_{\mathrm{w}} (\mathrm{F}_4, 4, \mathrm{B}_3, \mathrm{D}_3, k^{4*})$ where $\mathrm{w}= \mathrm{s}_{\alpha_1+\alpha_2+\alpha_3} \circ \mathrm{s}_{\alpha_2+\alpha_3} \circ \mathrm{s}_{\alpha_3}$ and $\langle L \cdot \mathfrak{g}_{\alpha_4}\rangle \simeq k^8 \simeq k^4 \oplus k^{4*}$ as a Spin_6-module. Then L/K_x is up to a finite covering $\mathrm{Spin}_7 \times k^*$ where S is a generic isotropy group of $\mathrm{Spin}_6 \times k^*$ in k^4. Hence L/K_x is spherical of rank two (see 9B for n=3, Table 1) and this gives (F3). Case 7C yields $(\mathrm{F}_4, 1, \mathrm{C}_3, \mathrm{C}_2 \times \mathrm{A}_1, k^4)$. Indeed, $\langle L \cdot \mathfrak{g}_{\alpha_1}\rangle \simeq k^{14} \simeq k^5 \otimes k^2 \oplus k^4 \otimes k$ as $\mathrm{C}_2 \times \mathrm{A}_1$-modules and $\mathrm{rk}_K\ k^5 \otimes k^2 > 1$. Now observe that if P denotes a parabolic subgroup in Sp_4 such that $\mathrm{Sp}_4/P \simeq \mathbf{P}(k^4)$, then $\mathrm{Sp}_6/P \times \mathrm{SL}_2$ is not spherical of rank one: otherwise, since it is prime and not in Table 1, it would be horospherical. Therefore L/K_x cannot be spherical of rank two because it is up to a finite covering isomorphic to $k^* \times \mathrm{Sp}_6/S \times \mathrm{SL}_2$ where S is a generic isotropy group of $k^* \times \mathrm{Sp}_4$ in k^4. Case 10 yields $(\mathrm{F}_4, 4, \mathrm{B}_3, \mathrm{G}_2, \mathrm{k})$ since $\langle L \cdot \mathfrak{g}_{\alpha_4}\rangle \simeq k^8 \simeq k^7 \oplus k$ as a G_2-module and $\mathrm{rk}_{\mathrm{G}_2}\ k^7 \geq \mathrm{rk}_{\mathrm{Spin}_7}\ k^7 = 2$. This gives (F4).

Finally, $\mathrm{Y} = \mathrm{G}_2$ is obviously ruled out. □

Now we go through the cases of Theorem 5.9. In the following propositions, β is chosen to be the lowest weight in the L-module $\langle L \cdot \mathfrak{g}_\beta\rangle$. Let $r \leq s$ be such that $\{r, s\} = \{p, q\}$.

Proposition 6.7. *Assume that* $\mathrm{S}\backslash\mathrm{S}_Q = \{\alpha_p, \alpha_q\}$ *with* $p \neq q$, $K = L$ *and that* $\mathfrak{q}^u/\mathfrak{u} \simeq \langle L \cdot \mathfrak{g}_{\alpha_p}\rangle \oplus \langle L \cdot \mathfrak{g}_\beta\rangle$ *is of rank two with* $\lambda^\beta_{\alpha_q} = 1$. *Then up to isomorphism,* $(\mathrm{Y}, r, s, \alpha_p, \beta)$ *is either one of the following data:*

(AD6)	$(\mathrm{A}_n, 1, q, \alpha_1, \alpha_1 + ... + \alpha_q)$	*for* $n \geq q \geq 2$.
(B7)	$(\mathrm{B}_n, 1, n, \alpha_1, \alpha_1 + ... + \alpha_n)$	*for* $n \geq 2$.
(BC8)	$(\mathrm{B}_n, q, n, \alpha_n, \alpha_q + ... + \alpha_n)$	*for* $n \geq q + 1 \geq 2$.
(C7)	$(\mathrm{C}_n, n-1, n, \alpha_n, \alpha_{n-1} + \alpha_n)$	*for* $n \geq 2$.
(C8′)	$(\mathrm{C}_n, p, p+1, \alpha_p, \alpha_{p+1})$	*for* $n \geq p + 2 \geq 3$.
(F5)	$(\mathrm{F}_4, 3, 4, \alpha_3, \alpha_4)$.	
(F6)	$(\mathrm{F}_4, 2, 3, \alpha_2, \alpha_3)$.	
(G4)	$(\mathrm{G}_2, 1, 2, \alpha_1, \alpha_2$ (respectively $\alpha_1 + \alpha_2$)).	

Furthermore, cases (B7) *and* (C7) *yield isomorphic pairs* $(\mathfrak{g}, \mathfrak{h})$ *when* $n = 2$.

Proof. First recall that if M_l denotes a representation for SL_l isomorphic to k^l or k^{l*}, then $\mathrm{rk}_{\mathrm{SL}_l \times \mathrm{SL}_m}\ M_l \otimes M_m = \min(k, l)$. Recall also (see the proof of Proposition 6.5) that $\mathrm{rk}_L\ \langle L \cdot \mathfrak{g}_{\alpha_p}\rangle = \mathrm{rk}_L\ \langle L \cdot \mathfrak{g}_\beta\rangle = 1$ and have in mind the fact that $\mathfrak{u}$ is a Lie algebra.

If $\mathrm{Y} = \mathrm{A}_n$, then $\mathfrak{q}^u \simeq k^r \otimes k^{s-r*} \oplus k^{s-r} \otimes k^{n-s+1*} \oplus k^r \otimes k^{n-s+1*}$. Hence r=1 or $s-r$=1 or $n-s+1$=1. So we get $(\mathrm{A}_n, 1, q, \alpha_1, \alpha_1+...+\alpha_q) \simeq_{\mathrm{w}} (\mathrm{A}_n, l, l+1, \alpha_l, \alpha_{l+1}) \simeq_{\mathrm{w}'} (\mathrm{A}_n, m, n, \alpha_n, \alpha_m + ... + \alpha_n)$ for $n \geq q = l + 1 = m + 1 \geq 2$, where $\mathrm{w}= \mathrm{s}_{\alpha_{q-1}} \circ ... \circ \mathrm{s}_{\alpha_1}$ and $\mathrm{w}' = \mathrm{s}_{\alpha_n} \circ ... \circ \mathrm{s}_{\alpha_{l+1}}$. This gives (AD6).

If $\mathrm{Y} = \mathrm{B}_n$, $n \geq 2$ (observe that for $l \geq 1$, $\mathrm{rk}_{\mathrm{Spin}_{2l+1}}\ k^{2l+1} = 2$ and $\mathrm{rk}_{\mathrm{SL}_l}\ k^l \oplus k^{l*} > 2$) then $(\mathrm{B}_n, q, n, \alpha_n, \alpha_q + ... + \alpha_n)$ for $n \geq p + 1 \geq 2$ and $(\mathrm{B}_n, n-1, n, \alpha_n, \alpha_{n-1}+2\alpha_n) \simeq_{\mathrm{w}} (\mathrm{B}_n, n-1, n, \alpha_{n-1}, \alpha_n) \simeq_{\mathrm{w}'} (\mathrm{B}_n, 1, n, \alpha_1, \alpha_1 +$

$... + \alpha_n)$ for $n \geq 2$, where $\mathrm{w} = s_{\alpha_n}$ and $\mathrm{w}' = s_{\alpha_1+...+\alpha_{n-1}}$. This gives (BC8), (B7) and also (C7) for $n = 2$.

If $\mathrm{Y} = \mathrm{C}_n$, $n \geq 3$ (note that for $l \geq 1$, $\mathrm{rk}_{\mathrm{Sp}_{2l}}\ k^{2l} = 1$ and $\mathrm{rk}_{\mathrm{SL}_l}\ S^2k^l > 1$) then we find $(\mathrm{C}_n, n-1, n, \alpha_n, \alpha_{n-1}+\alpha_n) \simeq_{\mathrm{w}} (\mathrm{C}_n, n-1, n, \alpha_{n-1}, \alpha_n) \simeq_{\mathrm{w}'} (\mathrm{C}_n, 1, n, \alpha_1, 2\alpha_1+...+2\alpha_{n-1}+\alpha_n)$ for $n \geq 3$ and $(\mathrm{C}_n, p, p+1, \alpha_p, \alpha_{p+1}) \simeq_{\mathrm{w}''} (\mathrm{C}_n, 1, l, \alpha_1, \alpha_1 + ... + \alpha_l)$ for $n \geq p + 2 = l + 1 \geq 3$, where the isomorphisms are given by $\mathrm{w} = s_{\alpha_n}$, $\mathrm{w}' = s_{\alpha_1+...+\alpha_{n-1}}$ and $\mathrm{w}'' = s_{\alpha_1+...+\alpha_p}$. This gives (C7) for $n \geq 3$, and (C8′).

It is elementary to check that the cases $\mathrm{Y} = \mathrm{D}_n$ for $n \geq 4$, E_6, E_7 and E_8 are ruled out, using the fact that $\mathrm{rk}_{\mathrm{Spin}_{2l}}\ k^{2l} = 2$ for $l \geq 1$.

If $\mathrm{Y} = \mathrm{F}_4$, then we find $(\mathrm{F}_4, 1, 3, \alpha_1, \alpha_1 + \alpha_2 + \alpha_3) \simeq_{\mathrm{w}} (\mathrm{F}_4, 2, 3, \alpha_2, \alpha_3) \simeq_{\mathrm{w}'} (\mathrm{F}_4, 2, 4, \alpha_4, \alpha_2 + 2\alpha_3 + 2\alpha_4)$ and $(\mathrm{F}_4, 3, 4, \alpha_3, \alpha_4) \simeq_{\mathrm{w}''} (\mathrm{F}_4, 3, 4, \alpha_4, \alpha_3 + \alpha_4)$, where $\mathrm{w} = s_{\alpha_1+\alpha_2}$, $\mathrm{w}' = s_{\alpha_3+\alpha_4}$ and $\mathrm{w}'' = s_{\alpha_4}$. This gives (F6) and (F5).

Finally, if $\mathrm{Y} = \mathrm{G}_2$ then we get $(\mathrm{G}_2, 1, 2, \alpha_1, 2\alpha_1+\alpha_2) \simeq_{\mathrm{w}} (\mathrm{G}_2, 1, 2, \alpha_1, \alpha_1 + \alpha_2)$, where $\mathrm{w} = s_{\alpha_1}$, and also $(\mathrm{G}_2, 1, 2, \alpha_1, \alpha_2)$. This gives (G4). □

Proposition 6.8. *Assume that* $\mathrm{S}\backslash\mathrm{S}_Q = \{\alpha_p, \alpha_q\}$ *with* $p \neq q$, $K = \ker(\alpha_p - \alpha_q)^0$ *and that* $\mathfrak{q}^u/\mathfrak{u} \simeq \langle K \cdot \mathfrak{g}_{\alpha_p}\rangle \simeq \langle K \cdot \mathfrak{g}_{\alpha_q}\rangle$ *is of rank one. Then up to isomorphism,* (Y, p, q) *is either one of the following triplets:*

(AD7) $(\mathrm{D}_n, n-1, n)$ *for* $n \geq 2$.

(BC9) $(\mathrm{B}_{2(\text{respectively } 3)}, 1, 2 \text{ (respectively } 3))$.

(G5) $(\mathrm{G}_2, 1, 2)$.

Proof. Let $\alpha \in \mathrm{S}_Q$. Since $\langle K \cdot \mathfrak{g}_{\alpha_p}\rangle \simeq \langle K \cdot \mathfrak{g}_{\alpha_q}\rangle$, $\langle \alpha, \alpha_p\rangle < 0$ (i.e., α is a neighbour of α_p in the Dynkin diagram of S) if and only if $\langle \alpha, \alpha_q\rangle < 0$. Therefore α_p and α_q are ends of the Dynkin diagram of S, and they are not be separated by more than one simple root. Hence $\mathrm{Y} = \mathrm{A}_n$ ($n = 2$ or 3), B_n ($n = 2$ or 3), C_n ($n = 2$ or 3), D_n ($n \geq 3$) or G_2. Observe that in each case $\langle K \cdot \mathfrak{g}_{\alpha_p}\rangle \simeq \langle K \cdot \mathfrak{g}_{\alpha_q}\rangle$ is of rank one except for $(\mathrm{C}_3, 1, 3)$ where $\langle K \cdot \mathfrak{g}_{\alpha_1}\rangle \simeq k^2$ and $\langle K \cdot \mathfrak{g}_{\alpha_3}\rangle \simeq S^2k^2$. Now recall that $\mathrm{D}_3 = \mathrm{A}_3$ and $\mathrm{C}_2 = \mathrm{B}_2$. □

7. Colors and normalizers

The goal of this section is to compute the pairs (G, H) for X prime of rank two. The group H is determined both geometrically and by means of its eigenvectors in the rational representations of G (see Tables A–G). The group action on X is also determined, and the irreducible components of the normal crossing divisor are expressed in Pic X, see Section 2.

So far we have determined a set Φ of candidate pairs (G, H^0) for X prime of rank two. If one sets aside in Tables A–G all cases where an element $\mathrm{w} \in \mathrm{W}$ is given in column 3, second cases of A5, B2, C2, D4, D5, fourth case of D4, and sixth case of D4, then the set Ψ of all triplets $(G, K, \mathfrak{u})$ is one-to-one with Φ, the correspondence being $(G, K, \mathfrak{u}) \mapsto (G, KU)$. Let Δ_0 denote the set of B-orbits of codimension one in G/H^0. For each $D_0 \in \Delta_0$, let f_{D_0} be an equation defining $\pi_0^{-1}(D_0)$ where π_0 denotes the projection $G \to G/H^0$.

(Note that $f_{D_0} \in k[G]^{(B\times H^0)}$ is determined up to a scalar by its character $\chi_{f_{D_0}} \in \mathcal{X}(B)\times\mathcal{X}(H^0)$.) Observe that $c \in \mathcal{X}(H)$ is trivial on H^0, see column 4 and Section 3.

Lemma 7.1. *If $(G, H^0) \in \Phi$, then $(\chi_{f_{D_0}})_{D_0\in\Delta_0}$ is the set of characters given in column 4 for the corresponding $(G, K, \mathfrak{u}) \in \Psi$.*

Proof. By Remark 1.5, if $f \in k[G]^{(B\times H^0)}$ is nonconstant, then there exists $D_0 \in \Delta_0$ such that $\chi_f = \chi_{f_{D_0}}$ if and only if $f \neq f_1 \times f_2$ with both $f_i \in k[G]^{(B\times H^0)}$ nonconstant. Thereby, it is easy to check that if we take $c = 1$, then the characters coresponding to Ψ in column 4 are the $\chi_{f_{D_0}}$, $D_0 \in \Delta_0$. Indeed, if the weight in $\mathcal{X}(B)$ of a candidate f_{D_0} in the tables is a sum $w_r + w_s$ of fundamental weights (with possibly $r = s$), then f_{D_0} cannot be split into a pair of eigenfunctions with weight w_r and w_s in $\mathcal{X}(B)$ respectively (thanks to card $\Delta \leq 2+$ rk $\mathcal{X}(K)$, see Section 2), except in the following cases: first A7, A8, first B9, C9, G5. For these five cases, we notice that in the irreducible G-module V_{w_1}, respectively V_{w_2}, the highest weight line is the only one fixed by U. □

Lemma 7.2. *Each $(G, H^0) \in \Phi$ satisfies the condition* **(aut)**.

Proof. We shall check the condition **(aut)** for all cases except those mentioned in Remark 6.1. Let $\Xi_0 = k(G/H^0)^{(B)}/k^\star$. If $\mathcal{V}_0$ denotes the cone of invariant valuations in $\mathcal{Q}_0 = \mathrm{Hom}_{\mathbf{Z}}(\Xi_0, \mathbf{Q})$, see for example [K1, p. 231 and 242], then we have dim $\mathcal{V}_0 \cap (-\mathcal{V}_0) =$ dim $N_G(H^0)/H^0$. Since H^0 is clearly not horospherical, see Section 6, we have $\mathcal{V}_0 \neq \mathcal{Q}_0$ (see [P]). So it remains to show that dim $\mathcal{V}_0 \cap (-\mathcal{V}_0)$ is not one-dimensional. If it is, then H^0 would be contained in a parabolic subgroup P in G such that card $\Delta_0 = \mathcal{X}(P) + 1$ (see [K1, p. 239]) which is clearly not the case. □

By Lemma 7.2, for each group lying between H^0 and $N_G(H^0)$, the corresponding homogeneous space G/H has a (unique normal) candidate wonderful completion X, see the condition **(aut)** and Remark 6.1. In Tables A–G, we have gathered some of these candidates X.

It is easily seen that the set of characters in $\mathcal{X}(B)\times\mathcal{X}(H)$ given in column 4 correspond to $(\chi_{f_D})_{D\in\Delta}$, see Section 2 for this notation and the proof of Lemma 7.1. This yields Ξ and the closed orbit $G{\cdot}z \subset X$ since G_z^- is the stabilizer of the line through $\prod_{D\in\Delta} f_D \in k[G]^{(B)}$, see Section 2 and for example [K1, p. 244].

Proposition 7.3. *The candidates X given in* Tables A–G *are wonderful.*

Proof. Note that a candidate X always contains two wonderful subvarieties of rank one, since normal singularities do not occur in codimension one. Let γ_1, γ_2 be the corresponding spherical roots. Recall from the introduction that X is wonderful if and only if X is nonsingular, i. e., γ_1, γ_2 generate Ξ, see Section 2. Since $G{\cdot}z$ is known and γ_1, γ_2 are linearly independent, it follows that γ_1 and γ_2 are uniquely determined thanks to Table 1, except in cases mentioned in Examples 7.4, 7.5 and 7.6 below. Therefore the proof of this proposition will be completed by the end of Example 7.6.

Example 7.4. (first cases of B5 and C5 for $n = 2$) Consider the projection $G/H \to G/\hat{w}H$ where $w \in W$ is given in the tables one case below (namely second cases of B5 and C5). By the property **(uni)**, see the introduction, we get an equivariant morphism $X \to X_w$, where X and X_w are the candidate wonderful completions of G/H and $G/\hat{w}H$ respectively. In particular, the (well defined, see above) spherical roots of X and X_w lie on the same half-lines in $\Xi \otimes_{\mathbf{Z}} \mathbf{Q}$. Since the spherical roots for X_w are known thanks to Table 1, we get those of X.

Example 7.5. (case G3) We claim that in this case the pair of spherical roots of X is not (α_1, α_2). For otherwise, consider the projection $G/H \to G/H_1$ where (G, H_1) is given in case 15, Table 1. By [K1, p. 239], there would be a quotient of $Q = \mathrm{Hom}_{\mathbf{Z}}(\Xi, \mathbf{Q})$ by a one-dimensional colored vector space V yielding $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}(\alpha_1 + \alpha_2), \mathbf{Q})$, and by [K1, p. 238], V should be generated as a cone by elements of $\varrho(\Delta)$ (see Theorem 1.4 for this notation) and by elements of the cone $\mathcal{V}$ dual to the cone generated by $-\alpha_1$ and $-\alpha_2$. But it is easily checked that this is not the case (note that $V \cap \mathcal{V} = \{0\}$). Contradiction.

Example 7.6. (cases where $H \subset B$) Consider the projection $G/H \to G/B$. By the property **(uni)**, we get an equivariant morphism $X \to G/B$. In particular, codimension one G-orbits in X have solvable isotropy groups. Therefore the spherical roots of X are simple roots, see Table 1. □

Remark 7.7. In cases A1, A2, A3, A7, A8, B1, B3, C1, D7, and E1 of Tables A–G, the candidate wonderful completion X_0 of G/H^0 is singular. Indeed, the spherical roots of X_0 (see the proof of Proposition 7.3) are in these cases the same as those of X and in particular they do not generate the lattice Ξ_0 corresponding to X_0. Note that in all other cases X_0 is wonderful thanks to Proposition 7.3.

By Lemma 2.2, the following theorem yields Theorem 1.3 while Theorem 1.4 can be deduced from Tables A–G.

Theorem 7.8. *The prime wonderful varieties of rank two are the varieties given in* Tables A–G.

Proof. Observe that for each pair $(G, H^0) \in \Phi$, there is at most two corresponding subgroups $H_\sharp$ and H in Tables A–G, always ordered by inclusion, with $H/H_\sharp$, respectively $H_\sharp/H^0$, containing at most two elements. So by Proposition 7.3 and Remark 7.7, it suffices to prove that $H = N_G(H)$, i. e., that there are no pairs of spherical roots other than those given in the tables, see Remark 6.1 and Introduction. This point is clear thanks to Table 1 (see the proof of Proposition 7.3) except in cases A5, first A6, first A7, A8, B6 for $p = 1$, B7, first and second B8 for $n = 2$, first B9, C7 for $n = 2$, C8, C9, first and second D4 for $n = 3$, third and fourth D4 for $p = 1$, D5, G4 and G5. For these, assume that $N_G(H) \neq H$. Note that γ_1 (see column 5) for G/H is a simple root α of (G,T) and that 2α should be a spherical root for $G/N_G(H)$. By [L1, 3.2] there should be a unique color of $G/N_G(H)$, represented by a pair $(\chi,\chi') \in \mathcal{X}(B) \times \mathcal{X}(N_G(H))$ such that the following property is satisfied: $\chi(\alpha^\vee) \neq 0$. Moreover, for G/H, there are two colors D^+, D^- (with equations f^+, respectively $f^- \in k[G]^{(B\times H)}$) having this property. Therefore the product $f^+f^- \in k[G]^{(B\times N_G(H))}$. So $\gamma_2 \notin \Xi_N \otimes_{\mathbf{Z}} \mathbf{Q} = \Xi \otimes_{\mathbf{Z}} \mathbf{Q}$ where $\Xi_N \simeq k(G/N_G(H))^{(B)}/k^\star$. Contradiction. □

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