ANALYTIC THEORY OF LINEAR q-DIFFERENCE EQUATIONS.

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§ 1. **Introduction.** The subject of this paper is to develop the analytic theory of a q-difference system

$$Y(q\,x) = A(x)\,Y(x), \qquad Y(x) = (y_{ij}(x)),$$
$$A(x) = (a_{ij}(x)), \quad |A(x)| \not\equiv 0 \qquad (i, j = 1, \dots n) \tag{1}$$

or, which is an essentially equivalent matter, the analytic theory of a single q-difference equation

$$L_n(y) \equiv y(q^n x) + a_1(x)\,y(q^{n-1}x) + \cdots + a_n(x)\,y(x) = 0$$
$$(a_n(x) \neq 0). \tag{1 a}$$

It is assumed that the coefficients $a(x)$ in (1) or (1 a) are analytic for $|x| \leqq \varrho$, being representable for these values of x as follows

[1] The author began this work while he was a National Research Fellow at Harvard University.

$$a_{ij}(x) = a_0^{i,j} + a_1^{i,j} x + a_2^{i,j} x^2 + \cdots \qquad (i; j = 1, \ldots n) \tag{1 b}$$
$$(|(a_0^{i,j})| \neq 0);$$

$$a_i(x) = a_0^i + a_1^i x + a_2^i x^2 + \cdots \qquad (i = 1, \ldots n), \tag{1 c}$$

where not all the a_0^i are zero. In the past the main contributions to the subject were made by Carmichael and by Birkhoff.[1] Carmichael successfully develops the theory under the supposition that the roots of the characteristic equation of (1 a),

$$E(\varrho) \equiv \varrho^n + a_0^1 \varrho^{n-1} + \cdots + a_0^{n-1} \varrho + a_0^n = 0, \tag{2}$$

are all finite and different from zero.[2] He demonstrates existence of a full set of analytic solutions which actually are the 'formal series solutions'. Birkhoff solves the corresponding Riemann problem. The proof of the existence result of Carmichael has been simplified by Adams[3], who also investigated the nature of the formal series solutions in the general case when no restrictions are made concerning the roots of (2). The purpose of the present paper is to develop the analytic theory when the roots of the characteristic equation are not restricted in any way. We then obviously may have several characteristic equations in place of the single one (2).

In view of the fact pointed out in the beginning of (C) and since the complete analytic theory of difference equations has been already developed in a paper by Birkhoff and myself[4] the present paper assumes added interest. Without any loss of generality it will be assumed that $|q| > 1$.[5] Moreover, in the greater part of the text the theory is developed for the vicinity of the origin. On the basis of such developments it will follow at once that a structurally identical theory would hold for the neighborhood of $x = \infty$ provided that the coefficients

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[1] R. D. Carmichael, *The general theory of linear q-difference equations*, Amer. Journ. Math., vol. 34 (1912), pp. 147—168. This paper will be referred to as (C). G. D. Birkhoff, *The Generalized Riemann Problem for Linear Differential Equations and the Allied Problems for Linear Difference and q-Difference Equations*, Proc. Am. Acad. Arts and Sciences, vol. 49 (1914), pp. 521—568. This paper will be referred to as (B).

[2] He supposes also that no root of (2) is equal to the product of another by an integral power of q. This restriction, however, is not fundamental.

[3] C. R. Adams, *On the Linear Ordinary q-Difference Equation*, Annals of Math., vol. 30, No. 2, April, 1929, pp. 195—205. This paper will be referred to as (A).

[4] G. D. Birkhoff and W. J. Trjitzinsky, *Analytic Theory og Singular Difference Equations*, Acta mathematica, 60: 1—2, pp. 1—89, hereafter referred to as (BT)

[5] The case when $|q| = 1$ has been satisfactorily treated in (C).

in (1) or (1 a) are representable (near $x=\infty$) by series in negative powers of x with, possibly, a few positive powers present. All the results of this paper will continue to hold, with no essential modifications, also when in (1 b), (1 c) x is replaced by $x^{\frac{1}{s}}$ (s, integer) and when a finite number of negative powers of x $\left(\text{or } x^{\frac{1}{s}}\right)$ is admitted in (1 b) and (1 c). A corresponding statement can be made concerning the vicinity of $x=\infty$.

§ 2. **Setting of the Problem.** On the basis of (A) it can be asserted that (1 a; § 1) is satisfied by n linearly independent[1] formal solutions

$$q^{\mu_j \frac{t^2}{2}} x^{r_j} \sigma_j^1(x) \qquad \left(t=\frac{\log x}{\log q};\ j=1,\ldots n\right) \tag{1}$$

where $\log x$ is given a suitable determination and

$$\begin{gathered}\sigma_j^1(x)=\sigma_j^{1\,0}\left(x^{\frac{1}{s_j}}\right)+t\sigma_j^{1\,1}\left(x^{\frac{1}{s_j}}\right)+\cdots+t^{k_j}\sigma_j^{1\,k_j}\left(x^{\frac{1}{s_j}}\right),\\ \sigma_j^{1\,H}(u)=\sigma_{j,0}^{1\,H}+\sigma_{j,1}^{1\,H}u+\sigma_{j,2}^{1\,H}u^2+\cdots.\end{gathered} \tag{1 a}$$

We shall term a series $\sigma(x)$ of the type given by (1 a) *a σ-series. A series $s(t)$ such that $s(t)=\sigma(x)\left(t=\dfrac{\log x}{\log q}\right)$, where $\sigma(x)$ is a σ-series, will be also called a σ series.*

The series (1) we order so that

$$\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_n.^2 \tag{1 b}$$

Here $\mu_j=\dfrac{\eta_j}{s_j}$ ($s_j>0$), the fraction being written in its lowest terms. When $\mu_j=\mu_1$ we take $s_j=1$. In general, the μ_j form $\sigma(>1)$ groups

$$\begin{gathered}m_1=\mu_1=\mu_2=\cdots=\mu_{\Gamma_1}<m_2=\mu_{\Gamma_1+1}=\cdots=\mu_{\Gamma_2}<\cdots<m_\sigma=\mu_{\Gamma_{\sigma-1}+1}=\cdots=\mu_{\Gamma_\sigma}\\ (1\leqq\Gamma_1<\Gamma_2<\cdots<\Gamma_\sigma=n).\end{gathered} \tag{1 c}$$

Under the supposition that the series (1) are suitably ordered the following

[1] It will be said that $g_1(x), g_2(x), \ldots g_n(x)$ are linearly independent if the determinant $|(g_j(q^{i-1}x))|$ is not identically zero.

[2] The $\mu_j (j=1,\ldots n)$ are all equal if and only if the roots of (2; § 1) are all finite and different from zero.

can be said concerning the r_j and the integers k_j. Consider all the series belonging to a group of μ_j $\left(= m = \frac{\eta}{s}\right)$

(2) $$q^{\frac{m t^2}{2}} x^{r_{\alpha+w}} \sigma^1_{\alpha+w}(x) \qquad (w = 1, 2, \ldots \beta)$$

and write

(2 a) $$r_{\alpha+w} = -\frac{m}{2} + \varrho_{\alpha+w}.^1$$

In general, the $\varrho_{\alpha+w}$ $(w=1, \ldots \beta)$ can be partitioned into several groups. A group of $\varrho_{\alpha+w}$,

(3) $$\varrho_{\alpha_1+\omega} \qquad (\omega = 1, \ldots \beta_1),$$

is specified as follows:

(3 a) $$\varrho_{\alpha_1+\omega} = \varrho_{\alpha_1+1} - \frac{\delta_{\alpha_1+\omega}}{s}$$

$$(\omega = 2, \ldots \beta_1;\ 0 \leqq \delta_{\alpha_1+2} \leqq \cdots \leqq \delta_{\alpha_1+\beta_1};\ \text{the } \delta_{\alpha_1+\omega},\ \text{integers});$$

(3 b) $$q^{\varrho_{\alpha+w}} \neq q^{\varrho_{\alpha_1+1} \pm \frac{\delta}{s}}$$

$$(\delta = 0, 1, 2, \ldots;\quad \alpha + w \neq \alpha_1 + 1, \alpha_1 + 2, \ldots \alpha_1 + \beta_1).$$

The k_j, occurring in the $\sigma_j(x)$ which correspond to the group (3), have the values

$$k_{\alpha_1+\omega} = \omega - 1 \qquad (\omega = 1, 2, \ldots \beta_1).$$

When $\mu_1 = \mu_2 = \cdots = \mu_n$ then, according to the existence result of Carmichael the power series factors in the formal solutions (1) will all converge for $|x| \leqq \varrho_1$ $(\varrho_1 > 0)$. The formal solutions are then also 'actual' solutions.[2] When not all the μ_j are equal the power series factors in the series (1), which belong to $m_1 (= \mu_1 = \cdots = \mu_{\Gamma_1})$, will converge for $|x| \leqq \varrho_1$[3]; accordingly, Γ_1 'actual' solutions will be known. The power series involved in (1), for $j > \Gamma_1$, may converge or they may diverge.

Write $q = |q| e^{\sqrt{-1}\,\bar{q}}$ and let $0 \leqq \bar{q} < 2\pi$. The transformations

(4) $$x = e^{t \log q} \qquad (x = |x| e^{\sqrt{-1}\,\alpha};\ t = u + \sqrt{-1}\,v)$$

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[1] The $q^{\varrho_{\alpha+w}} (w = 1, \ldots \beta)$ are the roots of the characteristic equation which belongs to the mentioned group of μ_j.

[2] Cf. (C).

[3] According to (A), $\varrho_1 = |q| \varrho$ at least when the coefficients are in positive powers of x.

will be now applied. The system (1; § 1) will assume the form

$$Z(t+1) = B(t)\,Z(t) \tag{5}$$

$$(A(e^{t\log q}) = B(t);\quad Y(e^{t\log q}) = Z(t))$$

and the equation (1 a; § 1) will be transformed into an ordinary difference equation

$$M_n(z) \equiv z(t+n) + b_1(t)\,z(t+n-1) + \cdots + b_n(t)\,z(t) = 0 \tag{5 a}$$

$$(a_j(e^{t\log q}) = b_j(t),\ (j = 1, \ldots n);\ y(e^{t\log q}) = z(t)).$$

The coefficients of the difference system (5) or of the equation (5 a) are analytic (except, possibly, at $t=\infty$) on and to the left of the line

$$\bar{q}\,v = \log|q|\,u - \log\varrho. \tag{6}$$

In this region R the coefficients are of period

$$\frac{2\pi}{|\log q|^2}(\bar{q} + \sqrt{-1}\log|q|) = \frac{2\pi\sqrt{-1}}{\log q}. \tag{6 a}$$

The region

$$|x| \leqq \varrho \qquad\qquad (0 \leqq \alpha \leqq 2\pi)$$

will correspond to a strip $S_{0,1}$, in the t-plane, with its boundary consisting of the part of the line (6) enclosed between the points

$$t_0(= u_0 + \sqrt{-1}\,v_0;\ x = \varrho),\ t_1(= u_1 + \sqrt{-1}\,v_1;\ x = \varrho e^{\sqrt{-1}\,2\pi}),$$

$$u_0 = \log\varrho\log|q|\,|\log q|^{-2}, \qquad v_0 = -\bar{q}\log\varrho\,|\log q|^{-2},$$

$$u_1 = (\log\varrho\log|q| + 2\pi\bar{q})\,|\log q|^{-2}, \quad v_1 = (2\pi\log|q| - \bar{q}\log\varrho)\,|\log q|^{-2},$$

and of the parts of the lines through t_0, t_1, lying in R and at right angles with the line (6).

The formal series solutions of (5 a) will be

$$e^{Q_j(t)}\,e^{t r_j\log q}\,\sigma_j(t), \quad Q_j(t) = \frac{t^2\mu_j\log q}{2}, \tag{7}$$

$$\sigma_j(t) = \sigma_j^0(t) + t\sigma_j^1(t) + \cdots + t^{k_j}\sigma_j^{k_j}(t),$$

$$\sigma_j^m(t) = \sigma_j'^m\left(e^{t\frac{\log q}{s_j}}\right) \qquad (m=0, 1, \ldots k_j;\ j=1, \ldots n).$$

The equation (5 a) is seen to be simply related to the system[1]

$$(8)\qquad Y(t+1)=D(t)\,Y(t),$$

$$D(t)=\begin{pmatrix} 0, & 1, & 0, & \dots & 0 \\ 0, & 0, & 1, & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -b_n, & -b_{n-1}, & \dots & & -b_1 \end{pmatrix}=(d_{ij}(t)),$$

which is satisfied by a formal matrix

$$(8\text{ a})\qquad S(t)\equiv\left(e^{Q_j(t+i-1)}\,e^{(t+i-1)r_j\log q}\,\sigma_j(t+i-1)\right)$$
$$=\left(e^{Q_j(t)}\,e^{t\delta_{ij}}\,\sigma_{ij}(t)\right),\qquad \delta_{ij}=[(i-1)\mu_j+r_j]\log q.$$

Here

$$(8\text{ b})\qquad \sigma_{ij}(t)=\sigma_{ij}^0(t)+t\sigma_{ij}^1(t)+\cdots+t^{k_j}\sigma_{ij}^{k_j}(t),$$

$$\sigma_{ij}^m(t)=\sigma_{ij;0}^m+\sigma_{ij;1}^m e^{t\frac{\log q}{s_j}}+\cdots+\sigma_{ij;k}^m e^{kt\frac{\log q}{s_j}}+\cdots\qquad (m=0,\dots k_j;\ j=1,\dots n).$$

Now, the series $\sigma_j^m(t)$ $(m=0,1,\dots k_j;\ j=1,\dots\Gamma_1)$, occurring in (7), as well as the series

$$(9)\qquad \sigma_{ij}^m(t)\qquad (m=0,1,\dots k_j;\ j=1,\dots\Gamma_1)$$

converge on and to the left of a line D parallel to the line (6). Accordingly, the first Γ_1 columns of $S(t)$ constitute Γ_1 solutions of (8) whose elements are analytic (except possibly at $t=\infty$) on and to the left of a line D.

Related to (8) there is the associated difference system of order $C_k^n\,(1<k\leqq n)$[2]

$$(10)\qquad Y_k(t+1)=D_k(t)\,Y_k(t)$$

where

$$(10\text{ a})\qquad D_k(t)=\left(d_{i_1\dots i_k;j_1\dots j_k}(t)\right)$$

$$(i_1,\dots i_k,j_1,\dots j_k=1,\dots n;\ i_1<i_2<\cdots<i_k;\ j_1<j_2<\cdots<j_k).\text{[3]}$$

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[1] Cf. (BT; § 1; (6), (6 a)).

[2] Cf. (BT; § 3; (9), (9 a), (10)). For $k=1$ we have the system (8).

[3] In agreement with (BT) we let $h_{i_1\dots i_k;j_1\dots j_k}$ denote the determinant $\left|(h_{i_r,j_c})\right|$ where in the matrix (h_{i_r,j_c}) $(r,c=1,\dots k)$ the term displayed is in the r-th row and c-th column. On the other hand, $(h_{i_1\dots i_k;j_1\dots j_k})$ will denote a matrix of order C_k^n in which the set of subscripts $(i_1\dots i_k)$ refers to a row and the set of subscripts $(j_1\dots j_k)$ refers to a column. It is clear that $|D_k(t)|\not\equiv 0$.

System (10) will be satisfied by the formal matrix

$$(10\,\mathrm{b})\qquad S_k(t)=\left(e^{Q_{j_1}(t)+\cdots+Q_{j_k}(t)}\,e^{t(r_{j_1}+\cdots+r_{j_k})\log q}\,\sigma^k_{i_1\ldots i_k;\,j_1\ldots j_k}(t)\right)$$

where

$$(10\,\mathrm{c})\qquad \sigma^k_{i_1\ldots i_k;\,j_1\ldots j_k}(t)=\left|\left(e^{t(i_r-1)\eta_{j_c}\frac{\log q}{s_{j_c}}}\,\sigma_{i_r,\,j_c}(t)\right)\right| \qquad (r,\,c=1,\ldots k).$$

The $\sigma^k_{i_1\ldots i_k;\,j_1\ldots j_k}(t)$ are of the same nature as the $\sigma_{ij}(t)$ since $(i_r-1)\eta_{j_c}$ is a non negative integer; here t enters as a factor to at most the $(k_{j_1}+k_{j_2}+\cdots+k_{j_k})$-th power. In fact,

$$(10\,\mathrm{d})\qquad \sigma^k_{i_1\ldots i_k;\,j_1\ldots j_k}(t)=\sigma^0(t)+t\sigma^1(t)+\cdots+t^l\sigma^l(t)$$

$$\left[\sigma^m(t)=\sigma^m_{i_1\ldots i_k;\,j_1\ldots j_k}(t);\ m=0,\,1,\ldots l\,(=k_{j_1}+\cdots+k_{j_k})\right],$$

$$(10\,\mathrm{e})\qquad \sigma^m(t)=\sigma^m_0+\sigma^m_1\,e^{t\frac{\log q}{s}}+\sigma^m_2\,e^{2t\frac{\log q}{s}}+\cdots$$

$$\left(\sigma^m_\nu=\sigma^m_{i_1\ldots i_k;\,j_1\ldots j_k};\ \nu=0,\,1,\,2,\ldots\right)$$

where $s(=s_{j_1\ldots j_k})$ is the lowest common multiple of $s_{j_1}, s_{j_2}, \ldots s_{j_k}$. In view of the fact that to the system (10) there corresponds a single equation of order C^n_k (cf. BT; § 1; (1), (2), (3), (4)) we conclude that the series $\sigma^m_{i_1\ldots i_k;\,j_1\ldots j_k}(t)$, occurring in the columns of $S_k(t)$ for which $\mu_{j_1}+\mu_{j_2}+\cdots+\mu_{j_k}$ is smallest, all converge to the left of a line D parallel to the line (6). The sets of subscripts $(j_1, j_2, \ldots j_k)$ will be supposed to be so ordered that the values

$$\mu=\mu_{j_1}+\mu_{j_2}+\cdots+\mu_{j_k}$$

as they occur in the successive columns of $S_k(t)$, are in ascending order of magnitude. Accordingly, it may and will be assumed that in the first column of $S_k(t)$ we have $(j_1,\ldots j_k)=(1,\,2,\ldots k)$.

Throughout the paper W will denote the part of the t-plane bounded above by a portion of a line with the slope

$$(11)\qquad a_2=\frac{-\bar{q}-|\log q|}{\log|q|}\qquad(<0)$$

and bounded below by a portion of a line with the slope

$$a_1 = \frac{-\bar{q} + |\log q|}{\log |q|}. \qquad (> 0) \tag{11 a}$$

The slopes (11), (11 a) are those of the two lines T_2, T_1 given by the equation

$$\Re\, Q(t) = 0. \qquad (\mu \neq 0) \tag{11 b}$$

The exact nature of the boundary of W near $t=0$ is not essential. A part of W on and above a line, below the axis of reals and parallel to this axis, will be denoted by $W(u)$. More explicitly, we shall write $W(u) = W(u; -c)$ if on the lower boundary of $W(u)$ $\Im\, t = c$. Similarly, a part of W on and below a line, situated above the axis of reals and parallel to the axis, will be denoted by $W(l)$ (or by $W(l; c)$ if $\Im t = c$ on the upper boundary of $W(l)$). Moreover, we shall let $W^{\varepsilon}(u)$ (or $W^{\varepsilon}(u; -c)$) denote a subregion of $W(u)$ with the right boundary, sufficiently far from $t=0$, consisting of a portion of a line with the slope $a_2 + \varepsilon\, (\varepsilon > 0)$. On the other hand, $W^{\varepsilon}(l)$ (or $W^{\varepsilon}(l; c)$) will denote a subregion of $W(l)$ bounded on the right, for $|t|$ sufficiently great, by a portion of a line with slope $a_1 - \varepsilon\, (\varepsilon > 0)$. Throughout, it will be possible to take ε arbitrarily small. The combined region $W^{\varepsilon}(u) + W^{\varepsilon}(l)$ will be denoted by W^{ε}. Similarly, to the right of a line with the slope $\frac{\log |q|}{q}$ we define, with respect to the slopes a_1 and a_2, corresponding regions V, $V(u)$ (above a line parallel to the axis of reals), $V(l)$ (below a line parallel to the axis of reals), $V^{\varepsilon}(u)$, $V^{\varepsilon}(l)$ and $V^{\varepsilon}\,(= V^{\varepsilon}(u) + V^{\varepsilon}(l))$.

The results of this paper stated for $W^{\varepsilon}(u)$, for instance, could be extended without modification to more general regions. For instance, we might require that, for $|t|$ sufficiently great, the right boundary of $W^{\varepsilon}(u)$ should consist of a portion of a curve C such that, when t' is on C and t $(\Im\, t' = \Im\, t)$ is on T_2,

$$t' = t - c\,(\Im\, t)^{\varepsilon}$$

$$(c > 0;\ \varepsilon > 0;\ \varepsilon \text{ arbitrarily small}).$$

It is clear that, with $\varepsilon < 1$, the limiting direction of C at $t = \infty$ is a_2. Corresponding extensions can be also made for $W^{\varepsilon}(l)$, $V^{\varepsilon}(u)$, $V^{\varepsilon}(l)$.

§ 3. **Formal q-Summation.** In view of the purposes at hand it is essential to solve the formal equation

$$y(qx) - y(x) = V(x), \tag{1}$$

(1 a) $$V(x) = e^{\frac{\mu}{2} t^2 \log q} x^r \left[\left(v_0^0 + v_1^0 x^{\frac{1}{s}} + \cdots\right) + t\left(v_0^1 + v_1^1 x^{\frac{1}{s}} + \cdots\right) + \cdots + t^k\left(v_0^k + v_1^k x^{\frac{1}{s}} + \cdots\right)\right]$$

$$\left(\mu = \frac{\eta}{s};\ \eta \geq 0,\ s \geqq 1;\ \eta,\ s,\ \text{integers};\ t = \frac{\log x}{\log q}\right),$$

where not all the $v_0^i\ (i=0, 1, \ldots k)$ are zero and the series are, in general, divergent. For an equivalent form of (1) we shall take

(1 b) $$y(x) = \sum_{\Gamma = x} V(\Gamma).$$

The operation implied in the second member of (1 b) will be termed q-summation.

Suppose $\eta \geqq 1$. Let

(2) $$y(x) = e^{\frac{\mu}{2} t^2 \log q} x^r \left[\left(s_0^0 + s_1^0 x^{\frac{1}{s}} + \cdots\right) + t\left(s_0^1 + s_1^1 x^{\frac{1}{s}} + \cdots\right) + \cdots + t^k\left(s_0^k + s_1^k x^{\frac{1}{s}} + \cdots\right)\right].$$

Substituting (2) in (1), by a laborious though straightforward computation, we get

(2 a) $$s_j^i = -v_j^i, \qquad (j = 0, 1, \ldots \eta - 1;\ i = 1, \ldots k)$$

(2 b) $$g q^{\frac{m}{s}} \left(C_i^i s_m^i + C_i^{i+1} s_m^{i+1} + \cdots + C_i^k s_m^k\right) - s_{\eta+m}^i = v_{\eta+m}^i$$

$$\left(m = 0, 1, \ldots;\ i = 0, 1, \ldots k;\ g = e^{\frac{\mu}{2} \log q} q^r\right),$$

where the C_i^j are binomial coefficients. Equations (2 b) can be solved in succession uniquely for the $s_j^i (j = \eta, \eta + 1, \ldots;\ i = 1, \ldots k)$.

Suppose now that $\eta = 0$ while for no integer $m\ (\geqq 0)$

(3) $$q^{r + \frac{m}{s}} = 1.$$

We substitute again (2) in (1). The $s_m^k\ (m = 0, 1, \ldots)$ will be defined by the equations

(4) $$\left(q^{r + \frac{m}{s}} - 1\right) s_m^k = v_m^k \qquad (m = 0, 1, \ldots).$$

The s_m^{k-1} $(m=0, 1, \ldots)$ will then be determined uniquely by the equations

$$(4\,a)\qquad \left(q^{r+\frac{m}{s}}-1\right)s_m^{k-1}+q^{r+\frac{m}{s}}C_{k-1}^k s_m^k=v_m^{k-1}\qquad (m=0, 1, \ldots);$$

and so on. Thus, in succession are determined uniquely all the

$$s_m^i\qquad (i=k,\ k-1, \ldots 0;\ m=0, 1, \ldots).$$

It remains to consider the case when $\eta=0$ and for some integer $m(\geqq 0)$ (3) holds. An expression like (2) cannot be made to satisfy (1). We let

$$(5)\qquad y(x)=x^r\left[\left(s_0^0+s_1^0x^{\frac{1}{s}}+\cdots\right)+\cdots+t^k\left(s_0^k+s_1^kx^{\frac{1}{s}}+\cdots\right)+t^{k+1}s_m^{k+1}x^{\frac{m}{s}}\right].$$

Comparison of coefficients gives the equations

$$(5\,a)\qquad q^{\frac{n-m}{s}}\left(C_i^i s_n^i+C_i^{i+1}s_n^{i+1}+\cdots+C_i^k s_n^k\right)-s_n^i=v_n^i$$

$$(i=0, 1, \ldots k;\ n=0, 1, \ldots m-1, m+1, m+2, \ldots),$$

$$(5\,b)\qquad (C_i^i-1)s_m^i+C_i^{i+1}s_m^{i+1}+\cdots+C_i^k s_m^k+C_i^{k+1}s_m^{k+1}=v_m^i.$$

For $i=k$ (5 a), (5 b) define uniquely the s_n^k $(n=0, 1, \ldots m-1, m+1, \ldots)$ and s_m^{k+1} but leave s_m^k undefined. For $i=k-1$ we have

$$\left(q^{\frac{n-m}{s}}-1\right)s_n^{k-1}+q^{\frac{n-m}{s}}C_{k-1}^k s_n^k=v_n^{k-1}\qquad (n=0, 1, \ldots m-1, m+1, \ldots),$$

$$(C_{k-1}^{k-1}-1)s_m^{k-1}+C_{k-1}^k s_m^k+C_{k-1}^{k+1}s_m^{k+1}=v_m^{k-1}.$$

These equations definie the s_n^{k-1} $(n=0, 1, \ldots m-1, m+1, \ldots)$ and s_m^k, leaving s_m^{k-1} undefined. Thus, using (5 a), (5 b) in succession for $i=k,\ k-1, \ldots 1$ it is seen that the s_n^i $(i=k,\ k-1, \ldots 1;\ n=0, 1, \ldots m-1, m+1, \ldots)$ are defined; so are the constants $s_m^{k+1}, s_m^{k-1}, \ldots s_m^2$. However, s_m^1 will be left undefined. Now the last set of equations (5 a), (5 b), formed for $i=0$, is

$$\left(q^{\frac{n-m}{s}}-1\right)s_n^0+\left[C_0^1 s_n^1+\cdots+C_0^k s_n^k\right]=v_n^0$$

$$(n=0, 1, \ldots m-1, m+1, \ldots),$$

$$(C_0^0-1)s_m^0+C_0^1 s_m^1+\cdots+C_0^k s_m^k+C_0^{k+1}s_m^{k+1}=v_m^0.$$

These equations define the s_n^0 $(n=0, 1, \ldots m-1, m+1, \ldots)$ and s_m^1, but s_m^0 will be left undefined. Thus equations (5 a), (5 b) serve to define all the coefficients occurring in (5) except s_m^0. Arbitrariness of s_m^0 corresponds, of course, to the relation

$$g(qx) - g(x) = 0 \qquad \left(g(x) = s_m^0 x^{r+\frac{m}{s}}\right)$$

which takes place whenever $q^{r+\frac{m}{s}} = 1$.

Lemma. *The formal equation*

$$y(qx) - y(x) = V(x),$$

where $V(x)$ is given by (1 a), *always possesses a formal solution*

$$y(x) = e^{\frac{\mu}{2} t^2 \log q} x^r \left[\left(s_0^0 + s_1^0 x^{\frac{1}{s}} + \cdots\right) + t\left(s_0^1 + s_1^1 x^{\frac{1}{s}} + \cdots\right) + \cdots \right.$$
$$\left. + t^k \left(s_0^k + s_1^k x^{\frac{1}{s}} + \cdots\right) + t^{k+1} s_m^{k+1} x^{\frac{m}{s}}\right] \qquad (m \geqq 0),$$

where $s_m^{k+1} \neq 0$ if and only if $\mu = 0$ and $q^{r+\frac{m}{s}} = 1$. If $\mu \neq 0$, the coefficients are uniquely defined by (2 a), (2 b). *If $\mu=0$, but for no integer $m (\geqq 0)$ $q^{r+\frac{m}{s}} = 1$, the coefficients are uniquely determined by* (4), (4 a), ... *When $s_m^{k+1} \neq 0$ the formal solution is specified uniquely, except for the arbitrariness of s_m^0, by the equations* (5 a), (5 b).

§ 4. **Lemmas on Analytic q-Summation.** Consider first the equation

$$(1) \qquad y(qx) - y(x) = H(x) = e^{Q(t)} x^r h(x) \qquad \left(Q(t) = \frac{\mu}{2} t^2 \log q\right)$$

with $H(x) = V(x)$ where $V(x)$ is given by (1 a; § 3), μ being not zero and the involved series being all convergent for $t \left(t = \frac{\log x}{\log q}\right)$ on and to the left of a line D^* whose slope is $\frac{\log |q|}{\bar{q}}$. Whenever $\bar{q} = 0$, it is taken parallel to the axis of imaginaries. An analytic solution of (1),

$$(1\text{ a}) \qquad y(x) = \sum_{\Gamma = x} H(\Gamma)$$

(cf. (1 b; § 3)), will be found under the stated conditions.

Let $L^{(2)}$ be a line either coincident with D^* or parallel to and situated to the left of D^*. Let $L^{(1)}$ be a line parallel and to the left of $L^{(2)}$ at a suitably small distance $\omega (\neq 0)$. Let D_0 be a line parallel to D^*, say, midway $L^{(1)}$ and $L^{(2)}$. A closed region U is then defined as the part of the complex plane bounded on the right by D_0. We have

$$(2) \qquad \sum_{\Gamma = x} H(\Gamma) = \underset{w=t}{\mathbf{S}} H(e^{w \log q}) \qquad \left(t = \frac{\log x}{\log q}\right),$$

where the symbol in the second member is that of ordinary difference summation. That is,

$$\underset{w=t+1}{\mathbf{S}} \Psi(w) - \underset{w=t}{\mathbf{S}} \Psi(w) = \Psi(t).$$

Consider the set of all lines parallel to $L^{(1)}$ and congruent to $L^{(1)}$ from the left. Enclose each of these lines, including $L^{(1)}$, in a strip of width ω, with boundaries parallel to $L^{(1)}$; let each of these lines be in the middle of the strip in which it lies.[1] The totality of these strips will be denoted by $S^{(1)}$. Similarly, denote by $S^{(2)}$ the totality of strips, of width ω, containing the lines parallel and congruent from the left to $L^{(2)}$. With ω sufficiently small,

$$(3) \qquad U = S^{(1)} + S^{(2)} + S^{(3)},$$

where $S^{(3)}$ is a set of strips, each bounded on the right by the left boundary of a strip of $S^{(1)}$ and bounded on the left by the right boundary of a strip of $S^{(2)}$. Let L be a straight line whose position depends on t as follows. When t is in a strip $S^{(1)}$ or on the 'left side' of the right boundary of such a strip, L is $L^{(2)}$. On the other hand, if t is in a strip of $S^{(2)}$ or on the 'right side' of the left boundary of such a strip, L is $L^{(1)}$. Thus, for the left approach to the common boundary of a strip of $S^{(1)}$ and a strip of $S^{(2)}$ we have $L = L^{(2)}$, while for the right approach $L = L^{(1)}$. For t in $S^{(3)}$ we take $L = L^{(1)}$ or $L = L^{(2)}$. With L so defined is follows that no point t in U is at a distance $< \frac{\omega}{2}$ from a position of

[1] The latter condition is assumed only for convenience.

congruency to L. Also, every t in U is to the left of L. Let $t+k_t$ denote the point furthest to the right belonging to the set of points $t, t+1, \ldots$ and lying to the left of L (t in U). It is clear that k_t is uniquely defined in U and does not depend on the choice of L when t is in $S^{(3)}$. The function of t, k_t, depends only on the two positions of L ($L^{(1)}$ and $L^{(2)}$). Let l_t denote a loop enclosing the points $t, t+1, \ldots t+k_t$, not enclosing $t-1$ and passing between $t+k_t$ and L. Such a loop is defined, of course, only for t in U.

Consider the expression

$$\int_{L_t} \frac{e^{\frac{\mu}{2} w^2 \log q}\, e^{w r \log q}\, h(e^{w \log q})\, dw}{1 - e^{2\pi\sqrt{-1}\,(t-w)}} \tag{4}$$

where

$$L_t = L + l_t,$$

with L described upwards and l_t described in the clockwise direction. For t in U and w on L

$$\frac{1}{\left|1 - e^{\pm 2\pi\sqrt{-1}(t-w)}\right|} \leqq d \qquad (d > 0). \tag{4 a}$$

Now the equation of L is of the form

$$u = \frac{\bar{q}\, v}{\log|q|} + c.$$

Accordingly, for w on L the real part of $\frac{\mu}{2} w^2 \log q$ will be

$$\Re\left[\frac{\mu}{2} w^2 \log q\right] = \frac{\mu}{2}\left(c^2 \log|q| - \frac{|\log q|^2}{\log|q|} v^2\right). \tag{4 b}$$

In view of the fact that along L $|h(e^{w \log q})|$ does not increase faster than $|w|^k$ it follows, by (4 a) and (4 b), that the component of the integral (4), along L, is absolutely convergent for t in U. The integrals along l_t and L each represent functions of t analytic in the strips separated from each other by the common boundaries of the strips of $S^{(1)}$ and the strips of $S^{(2)}$. In other words, these are strips whose boundaries D consist of D_0 (the right boundary of U) and of all the lines congruent to D_0 from the left. Each of these integrals represent two different analytic functions on the left and the right side of a line D (excepting D_0), respectively. With t in U there is occasion to consider only the left side of D_0; along this side the two integrals are analytic. We have

$$(5) \qquad \int_{l_t} = -H(e^{t \log q}) - H(e^{(t+1)\log q}) - \cdots - H(e^{(t+k_t)\log q}) = \varphi(t).$$

The sum of the two integrals is, however, analytic in U. In fact, let t be on a line D (different from D_0). Let k_t, for an approach to t from the left, be denoted by k_t^l; for an approach from the right let it be denoted as k_t^r. Then

$$k_t^l = k_t^r + 1.$$

Hence, on applying the corresponding notation to $\varphi(t)$, we have

$$(5\text{ a}) \qquad \varphi^l(t) = \varphi^r(t) - H\left(e^{\left(t+k_t^r+1\right)\log q}\right).$$

Moreover, for a left approach to t

$$\int_L = \int_{L^{(2)}} = \int_L^l$$

and, for a right approach,

$$\int_L = \int_{L^{(1)}} = \int_L^r.$$

To demonstrate that

$$G(t) \equiv \left(\varphi^l(t) + \int_L^l\right) - \left(\varphi^r(t) + \int_L^r\right) \equiv 0$$

we note that

$$G(t) = -H\left(e^{\left(t+k_t^l\right)\log q}\right) + \left(\int_{L^{(2)}} - \int_{L^{(1)}}\right).$$

The common integrand of the two integrals above is analytic in the closed region bounded by $L^{(1)}$ and $L^{(2)}$ except for a pole at $t+k_t^l$ (this point is on D_0). Accordingly, $\int_{L^{(2)}} - \int_{L^{(1)}}$ is representable as an integral described in the counter clockwise direction along a small loop enclosing the point $w = t + k_t^l$. It follows that

$$\int_{L^{(2)}} - \int_{L^{(1)}} = H\left(e^{(t+k_t^{1})\log q}\right).$$

The difference of the analytic function represented by (4) is equal to $H(e^{t\log q})$. We shall define the operation of (2) as follows

(6) $$y(x) = \sum_{\Gamma=x} H(\Gamma) = \int_{L_t} \frac{e^{\frac{\mu}{2}w^2\log q}\, e^{wr\log q}\, h(e^{w\log q})\, dw}{1 - e^{2\pi\sqrt{-1}(t-w)}}.$$

The properties of this analytic solution of the equation (1) will be further investigated. For the component consisting of the integral clong l_t it follows that

(7) $$e^{-Q(t)-rt\log q}\varphi(t) = -h(e^{t\log q}) - e^{Q(t+1)-Q(t)}\, e^{r\log q}\, h(e^{(t+1)\log q}) - \cdots$$
$$- e^{Q(t+k_t)-Q(t)}\, e^{rk_t\log q}\, h(e^{(t+k_t)\log q}).$$

For convenience the following notation will be introduced:

(7 a) $$e^{\frac{t\log q}{s}} = w, \quad g_i = e^{\left(\frac{\eta}{2s}i^2+ri\right)\log q}, \quad p = e^{\frac{\log q}{s}}.$$

Then

(7 b) $$e^{Q(t+i)-Q(t)}\, e^{r_i\log q} = g_i\, w^{i\eta}.$$

Let m be an integer ($m \geqq 0$) of the form $m = (\Gamma+1)\eta - 1$ (Γ, an integer). With m and Γ supposed fixed we assume, for the present and unless stated otherwise, that t is on and to the left of a line $D_{-k'(m)}$, parallel to D_0 and to the left from D_0 at a distance (in the direction of the axis of reals) equal to $k'(m)$. Here $k'(m)$ is such that, for t restricted as above,

$$k_t > \Gamma.$$

We have

(7 c) $$h(e^{t\log q}) = \sum_{\sigma=0}^{k} t^\sigma \left[v_0^\sigma + \cdots + v_m^\sigma w^m + b_m^\sigma(t)\, w^{m+1}\right]$$

$$(|b_m^\sigma(t)| < b_m \text{ for } t \text{ in } U)$$

so that

(7 d) $$e^{Q(t+i)-Q(t)}\, e^{ri\log q}\, h(e^{(t+i)\log q})$$
$$= \sum_{\sigma=0}^{k} (t+i)^\sigma g_i \left[v_0^\sigma w^{i\eta} + \cdots + v_m^\sigma p^{im} w^{i\eta+m}\right]$$
$$+ w^{m+1} \sum_{\sigma=0}^{k} e^{Q(t+i)-Q(t)}\, e^{ri\log q} (t+i)^\sigma b_m^\sigma(t+i)\, p^{i(m+1)}.$$

Now

$$(7\,\mathrm{e})\qquad \sum_{\sigma=0}^{k}(t+i)^{\sigma} g_i[\cdots]$$

$$= \sum_{\sigma=0}^{k}(i^{\sigma} + C_1^{\sigma} i^{\sigma-1} t + \cdots + C_{\sigma}^{\sigma} t^{\sigma}) g_i [v_0^{\sigma} w^{i\eta} + \cdots + v_m^{\sigma} p^{im} w^{i\eta+m}]$$

$$= \lambda_i^0(t) + t\lambda_i^1(t) + \cdots + t^k \lambda_i^k(t),$$

where

$$(7\,\mathrm{f})\qquad \lambda_i^H(t) = l_{i0}^H w^{i\eta} + l_{i1}^H w^{i\eta+1} + \cdots + l_{im}^H w^{i\eta+m},$$

$$l_{iI}^H = g_i p^{iI} \sum_{\sigma=H}^{k} C_H^{\sigma} i^{\sigma-1} v_I^{\sigma}$$

$$(H = 0, 1, \ldots k; \quad I = 0, 1, \ldots m; \quad i = 0, 1, \ldots k_t).$$

Accordingly, we may write

$$(8)\qquad -e^{-Q(t) - rt\log q}\,\varphi(t) = S_m(t) + \Gamma_m(t) + R_m(t).$$

Here we let

$$(8\,\mathrm{a})\qquad S_m(t) = \sum_{i=0}^{\Gamma}\sum_{\sigma=0}^{k}(t+i)^{\sigma} g_i [v_0^{\sigma} w^{i\eta} + \cdots + v_m^{\sigma} p^{im} w^{i\eta+m}]$$

$$= L_m^0(t) + t L_m^1(t) + \cdots + t^k L_m^k(t),$$

$$(8\,\mathrm{b})\qquad L_m^H(t) = \sum_{i=0}^{\Gamma} \lambda_i^H(t) = l_{0,0}^H + l_{0,1}^H w + \cdots + l_{0,\eta-1}^H w^{\eta-1}$$

$$+ (l_{0,\eta}^H + l_{1,0}^H) w^{\eta} + (l_{0,\eta+1}^H + l_{1,1}^H) w^{\eta+1} + \cdots + (l_{0,2\eta-1}^H + l_{1,\eta-1}^H) w^{2\eta-1}$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ (l_{0,\Gamma\eta}^H + l_{1,(\Gamma-1)\eta}^H + \cdots + l_{\Gamma,0}^H) w^{\Gamma\eta} + \cdots$$

$$+ (l_{0,(\Gamma+1)\eta-1}^H + l_{1,\Gamma\eta-1}^H + \cdots + l_{\Gamma,\eta-1}^H) w^{(\Gamma+1)\eta-1} + \Omega^H(t).$$

The expression $\Omega^H(t)$ consist of a sum of terms of the form

$$l\,w^r \qquad (l,\ \text{constant};\ r \geqq m+1).$$

The number of these terms is independent of t. Thus

$$(8\,\mathrm{c})\qquad \Omega^H(t) = w^{m+1}\beta_m^H(t) \qquad (|\beta_m^H(t)| \leqq \beta_m).$$

The function $\Gamma_m(t)$ will be defined as

$$\Gamma_m(t) = \sum_{i=\Gamma+1}^{k_t} \sum_{\sigma=0}^{k} (t+i)^\sigma g_i [v_0^\sigma w^{i\eta} + \cdots + v_m^o p^{im} w^{i\eta+m}].$$

Thus, by (7 b),

$$\Gamma_m(t) = \sum_{\sigma=0}^{k} \sum_{i=\Gamma+1}^{k_t} e^{Q(t+i)-Q(t)} e^{ri\log q} (t+i)^\sigma [v_0^\sigma + \cdots + v_m^\sigma p^{im} w^m] \tag{8 d}$$

so that, in view of the inequalities

$$| v_0^\sigma + \cdots + v_m^\sigma p^{im} w^m | < N'_m \qquad (i \leqq k_t),$$

$$| \Gamma_m(t) | < N'_m \sum_{\sigma=0}^{k} | \bar{t} |^\sigma \sum_{i=\Gamma+1}^{k_t} | e^{Q(t+i)-Q(t)} e^{ri\log q} | \tag{8 e}$$

$$\leqq N_m | w^{m+1} | \sum_{\sigma=0}^{k} | \bar{t} |^\sigma \sum_{j=0}^{k_t-\Gamma-1} | e^{Q(t_\Gamma+j)-Q(t)} e^{rj\log q} |$$

$$(t_\Gamma = t + \Gamma + 1);$$

here $\bar{t}$ is t or $t+k_t$, according as to whichever of the two numbers $|t|$ and $|t+k_t|$ is the greatest. In deriving (8 e) use is made of the fact that there exists a constant a such that

$$|\bar{t}| \leqq a |t|.$$

Before proceeding further certain inequalities involving $Q(t)$ will be established. Let ϱ be the distance from t to D_0. Then

$$\Re (t \log q) = - (\varrho - h_0) \log |q|$$

where h_0 depends only on the position of D_0 ($h_0=0$ when D_0 goes through the origin). Accordingly,

$$\Re [Q(t+i) - Q(t)] = \frac{\mu}{2} i [i - 2(\varrho - h_0)] \log |q|.$$

Now $\varrho = k_t + \varrho_t (|\varrho_t| \leqq \tfrac{1}{2})$ so that

$$\Re\,[Q(t+i)-Q(t)] \leqq \frac{\mu}{2}\,i\,[k_t - 2\,(k_t + \varrho_t - h_0)]\log|q| \leqq -\frac{\mu}{2}\,i\,(k_t - h')\log|q| \tag{9}$$

$$\left(i = 0, 1, \ldots k_t;\quad |h_0 - \varrho_t| \leqq \frac{h'}{2}\right).$$

Similarly, it is possible to show that

$$\Re\,[Q\,(t_\Gamma + j) - Q\,(t_\Gamma)] \leqq -\frac{\mu}{2}\,j\,[k_t - \Gamma - 1 - h']\log|q| \tag{9 a}$$

$$(j = 0, 1, \ldots k_t - \Gamma - 1).$$

In virtue of (9 a) from (8 e) it follows that

$$|\Gamma_m(t)| \leqq g^{(m)}\,|w^{m+1}|\sum_{\sigma=0}^{k}|t|^\sigma \tag{10}$$

whenever t is sufficiently far to the left, that is whenever t is on and to the left of a line $D_{-k''(m)}$ ($k''(m)$ sufficiently great). In fact, with the inequality

$$k_t - \Gamma - 1 - h' \geqq \lambda \qquad (> 0)$$

secured for λ suitably great it would follow that

$$\sum_{j=0}^{k_t-\Gamma-1}\left|e^{Q(t_\Gamma+j)-Q(t_\Gamma)}\,e^{rj\log q}\right| \tag{10 a}$$

$$\leqq \sum_{j=0}^{k_t-\Gamma-1} e^{-j\left(\frac{\mu}{2}\lambda\log|q| - \Re(r\log q)\right)} < \frac{1}{1 - e^{-\frac{\mu}{2}\lambda\log|q| + \Re(r\log q)}}$$

provided that $\frac{\mu}{2}\lambda\log|q| - \Re\,(r\log q) > 0$.

It remains to consider the last term in (8)

$$|R_m(t)\,w^{-m-1}| \leqq \sum_{\sigma=0}^{k}\sum_{i=0}^{k_t}\left|e^{Q(t+i)-Q(t)}\,e^{ri\log q}\,(t+i)\;b_m^\sigma\,(t+i)\,p^{i\,(m+1)}\right| \tag{10 b}$$

$$< b''_m \sum_{\sigma=0}^{k}|t|^\sigma \sum_{i=0}^{k_t}\left|e^{Q(t+i)-Q(t)}\,e^{ri\log q}\,e^{i(m+1)\frac{\log|q|}{s}}\right|.$$

Applying (6) and using the reasoning employed in deriving (10 a) we conclude that

(10 c) $$|R_m(t)| \leqq g_1^{(m)} |w^{m+1}| \sum_{\sigma=0}^{k} |t|^{\sigma}$$

for t on and to the left of a line $D_{-k_1''(m)}$.

By virtue of (8), (8 a), (8 b), (8 c), (10) and (10 c) the function $\varphi(t)$ is seen to be expressible in the form

(11) $$\varphi(t) = e^{Q(t)} e^{rt \log q} \sum_{H=0}^{k} t^H \varphi^H(t).$$

There exists a set of lines $D_{-k(m)}$, parallel to D_0, such that

$$k(m_2) > k(m_1) > 0 \qquad (\text{for } m_2 > m_1)$$

and such that for t on and to the left of $D_{-k(m)}$

(11 a) $$\varphi^H(t) \sim \varphi_0^H + \varphi_1^H e^{\frac{t \log q}{s}} + \cdots \qquad (H = 0, 1, \ldots k)$$

to m terms in the sense that

(11 b) $$\varphi^H(t) = \varphi_0^H + \varphi_1^H e^{\frac{t \log q}{s}} + \cdots + \varphi_m^H e^{\frac{m t \log q}{s}} + \varphi_m^H(t)\, e^{\frac{(m+1) t \log q}{s}}$$

$$(|\varphi_m^H(t)| \leqq \varphi_m^{\mathbf{H}} \text{ for } t \text{ on and to the left of } D_{-k(m)}).$$

For t in U the functions $\varphi^H(t)$ $(H = 0, 1, \ldots k)$ are bounded.

Let U^{ε} denote a subregion of U bounded on the right by curves C^u, C^l with limiting directions at infinity, extending upward sand downwards from the point P of intersection of D_0 and of the axis of reals and such that if t is on C^u (or C^l) and t' (with $\Im t' = \Im t$) is on D_0, we have

(11 c) $$\Re(t' - t) \rightarrow \infty$$

$$(\text{as } \Im t \rightarrow \infty \ (\text{or } \Im t \rightarrow -\infty)).$$

In U^{ε} the asymptotic relation (11 a) holds to infinitely many terms (that is, in the usual sense).

Consider the function represented by the integral along L; denote it by $g(t)$. Let $\Im t' = \Im t$, while t' is on L. Designate the parts of L above and below t' by L_1 and L_2, respectively. Noting that

$$h(e^{t\log q}) = \sum_{\sigma=0}^{k} t^\sigma h^\sigma(t) \qquad (|h^\sigma(t)| < h \text{ in } U)$$

and using (4 a) we have

$$\left|\int_{L_1}\right| = \left|\int_{L_1} \frac{e^{Q(\Gamma)+\Gamma r \log q - 2\pi\sqrt{-1}(t-\Gamma)}\, h(e^{\Gamma \log q})\, d\Gamma}{-\left(1 - e^{-2\pi\sqrt{-1}(t-\Gamma)}\right)}\right|$$

$$< hd\left|e^{-2\pi\sqrt{-1}\,t}\right| \sum_{\sigma=0}^{k} \int_{L_1} \left|e^{Q(\Gamma)+(r\log q + 2\pi\sqrt{-1})\Gamma + \sigma \log \Gamma}\right| |d\Gamma|.$$

Suppose, for the present, that $\Im t \geqq 0$. On taking note of (4 b) it is seen that the integrand in any of the $k+1$ integrals above diminishes, as $\Im \Gamma$ approaches infinity, rapidly enough to insure the inequality

(12)
$$\left|\int_{L_1}\right| < h' \left|e^{-2\pi\sqrt{-1}\,t}\right| \sum_{\sigma=0}^{k} \left|e^{Q(t')+(r\log q + 2\pi\sqrt{-1})t' + \sigma \log t'}\right|$$

$$= h' \left|e^{Q(t')+rt'\log q}\right| \sum_{\sigma=0}^{k} |t'|^\sigma. \qquad (\Im t \geq 0)$$

On the other hand,

$$\left|\int_{L_2}\right| \leqq dh \sum_{\sigma=0}^{k} \int_{L_2} \left|e^{Q(\Gamma)+r\Gamma\log q + \sigma\log\Gamma}\right| |d\Gamma|,$$

the maximum of an integrand above being attained near the axis of reals. Such an integrand diminishes sufficiently rapidly, away from the axis of reals, upwards and downwards, so that

(12 a)
$$\left|\int_{L_2}\right| < h_2 \qquad (\Im t \geqq 0).$$

For $\Im t \leqq 0$ the rôles of the integrals along L_1 and along L_2 are interchanged. Thus, for t in U,

(12 b)
$$|g(t)| \leqq g.$$

By a special example it can be shown that, in general, $|g(t)|$ cannot satisfy an inequality like (12).

The function defined by (6), $y(x)=\varphi(t)+g(t)$, will be now considered. It is observed that along the lines

$$v = a_1 u, \tag{13}$$

$$v = a_2 u \qquad (t = u + \sqrt{-1}\, v)$$

(cf. § 2) the real part of $Q(t)$ vanishes. The asymptotes of the hyperbola

$$\Re\,[Q(t) + rt\log q] = 0 \tag{13 a}$$

are parallel to the lines (13), respectively. The hyperbola degenerates into the pair of lines (13) when $r=0$. The portions of these asymptotes, which lie in U, will be denoted by B_u (above the negative axis) and by B_l (below the negative axis). The subregion of U enclosed by B_u and B_l (and, in some cases, by a part of D_0) will be denoted by W in accordance with § 4. Moreover, letting B_0 denote the complete boundary of W, we take B_{-k} to designate B_0 shifted to the left (in the direction of the negative axis) through the distance k.

Taking into consideration the established properties of $\varphi(t)$, (12 b) and the way in which $\Re\,[Q(t)+rt\log q]$ increases, as $|t|$ increases in W, and of the fact that in $U-W$ the function $\Re\,[Q(t)+rt\log q]$ approaches minus infinity (as $|t|\to\infty$) we establish the lemma.

Lemma 1. *The function*

$$y(x) = e^{Q(t)+rt\log q}\,\eta(t), \tag{14}$$

defined by (6), *is a solution of* (1) *for t in U. It is analytic in U and it has the following properties.*

There exists a set of curves $B_{-k(m)}$ (congruent from the left to the boundary of a region W),

$$k(m_2) > k(m_1) > 0 \qquad (\textit{when } m_2 > m_1),$$

such that, on writing

$$\eta(t) = \sum_{H=0}^{k} t^H \eta^H(t), \tag{14 a}$$

the asymptotic relations

$$\eta^H(t) \sim \varphi_0^H + \varphi_1^H e^{\frac{t\log q}{s}} + \varphi_2^H e^{\frac{2t\log q}{s}} + \cdots \qquad (H=0, 1, \ldots k) \tag{14 b}$$

hold to m terms, provided that t is on or to the left of $B_{-k(m)}$; in a region W^ε the *asymptotic relations* (14 b) *hold to infinitely many terms. In* W *the functions* $|\eta^H(t)|$ *are bounded. In the region* $U-W$ *the solution* (14) *is bounded.*

The series in the second members of (14 b) are in general divergent. The formal expression

$$e^{Q(t)+rt\log q}\sum_{H=0}^{k} t^H\left(\varphi_0^H+\varphi_1^H e^{\frac{t\log q}{s}}+\cdots\right)$$

is clearly a formal solution of the equation (1); it is identical with the formal series solution of Lemma 1 (§ 3). The lemma continues to hold true, when $V(x)$ ($V(x)$ given by (1 a; § 3)) is not convergent for t in U, provided that $H(x)$ is defined, for every integer $m(\geqq 0)$, by an expression

$$H(x)=e^{Q(t)+rt\log q}\sum_{\sigma=0}^{k} t^\sigma\left[v_0^\sigma+\cdots+v_m^\sigma w^m+b_m^\sigma(t)\,w^{m+1}\right]$$

$$\left(\left|b_m^\sigma(t)\right|<b_m \text{ for } t \text{ in } U\right).$$

It will be now assumed that the function $H(x)\,[=e^{Q(t)}x^r h(x)]$, *involved in* (1), *has the following pvoperties. If we write*

$$h\left(e^{\frac{t\log q}{s}}\right)=\sum_{H=0}^{k} t^H h^H(t) \tag{15}$$

then the functions $|h^H(t)|$ *are bounded in a region* $W(u)$ (*cf.* § 2). *In* $W^\varepsilon(u)$

$$h^H(t)\sim v_0^H+v_1^H e^{\frac{t\log q}{s}}+v_2^H e^{\frac{2t\log q}{s}}+\cdots.^1 \tag{15 a}$$

As before it is assumed that $\mu>0$; *moreover,* $H(x)$ *is analytic in* t *for* t *in* $W(u)$ (*and* $t\neq\infty$).

Denote the lower boundary of $W(u)$ by h. Let T be a point on h. A solution of (1) analytic in t for t in a region $W_1(u)$ slightly interior to $W(u)$, will be defined by the integral

$$y(t)=\int_{L_t}\frac{e^{Q(\Gamma)}e^{\Gamma r\log q}e^{2\pi\sqrt{-1}\,\lambda(t-\Gamma)}h\left(e^{\frac{\Gamma\log q}{s}}\right)}{1-e^{2\pi\sqrt{-1}\,(t-\Gamma)}}\,d\Gamma \tag{16}$$

(λ, a suitable integer).

[1] Unless stated to the contrary, asymptotic relations are taken in the ordinary sense, that is, to infinitely many terms.

We need only to take $L_t = L + l_t$, where L is formed exactly as before except that it is in $W(u)$ and that the two positions of L are terminated below in the fixed point T. The contour l_t (and L_t) is defined for t in $W(u)$ on and to the left of a line L_0 extending from T upwards between the two positions of L; moreover, t is to be on and above a line h' parallel to the line h and slightly above it. It may be assumed that the portions above h' of L_0 and of the two positions of L can be obtained by translation of the right boundary of $W(u)$ in the direction of the negative axis.[1] The part of $W(u)$ bounded on the right by L_0 and bounded on the left by h' will de denoted by $W_1(u)$ (which is a region of type $W(u)$). The definitions of l_t and of L, for t restricted as above, are entirely analogous to those given before. Moreover, L and the integer λ can and will be supposed to be so chosen that, along each of the two positions of L,

$$(16\,\mathrm{a}) \qquad \Re\left[Q(\Gamma) - 2\pi\sqrt{-1}\,(\lambda - 1)\,\Gamma\right] \to -\infty,$$

$$\Re\left[Q(\Gamma) - 2\pi\sqrt{-1}\,\lambda\,\Gamma\right] \to +\infty$$

as the imaginary part of Γ increases. It is clear then that the integral (16) converges absolutely for the specified values of t. It is noteworthy that a translation (in the direction of the axis of reals) of L may necessitate a change in λ. This is unlike any corresponding situation of the paper (BT). The fact is thus peculiar to the theory of q-difference equations; it constitutes one of the reasons why some of the summation methods of (BT) do not apply in the present paper.

A reasoning closely similar to that used before in deriving the asymptotic form of $\varphi(t)\left[=\int_{L_t}\right]$ will show that, on writing (11), the asymptotic relations (11 a) will be valid in the ordinary sense in a region $W^{\varepsilon}(u; -c')$ (cf. § 2). Here $-c'$ is the distance of h' from the axis of reals. On the other hand, on taking account of (4 a) (valid for w on L and for t on and to the left of L_0 and on and above h'), of (16 a) and of the fact that $\Re\, Q(t)$ increases sufficiently rapidly as t moves to the left in the direction parallel to the axis of reals, it follows that each of the integrals in the second member of the relation

[1] Compare with certain analogous situations in (BT).

$$\int_L - \int_T^{t'} + \int_{t'}^{\infty} \qquad (\Re t' = \Re t),$$

if added to $\varphi(t)$, will not produce a change in the asymptotic form when t is in $W^\varepsilon(u; -c')$. Thus the function $y(t)$, defined by (16), is asymptotic in $W^\varepsilon(u; -c')$ to a formal expression which necessarily will be the one whose existence was established (for the corresponding formal problem) in Lemma 1 (§ 3).

If (11) is written for $\varphi(t)$ it can be concluded that the $|\varphi^H(t)|$ are bounded in $W_1(u)$. Consideration of the integral along L, when t lies between L_0 and the right boundary of $W^\varepsilon(u; -c')$ leads us, in view of the properties of $\Re\, Q(t)$ and in conjunction with the above property of $\varphi(t)$, to the conclusion that $y(t)$ is of the form (14), (14 a) where the $|\eta^H(t)|$ are bounded in $W_1(u)$.

Lemma 2. *Let $H(x)$ of equation (1) satisfy the hypotheses of the statement in italics in connection with (15) and (15 a). The function*

$$(17) \qquad y(x) = e^{Q(t)+rt\log q}\,\eta(t)$$

defined by (16), will then be a solution of (1) for t in a region $W_1(u)$, slightly interior to $W(u)$. This solution is analytic in $W_1(u)$. Moreover,

$$(17\text{ a}) \qquad \eta(t) = \sum_{H=0}^{k} t^H \eta^H(t)$$

where

$$(17\text{ b}) \qquad \eta^H(t) \sim \varphi_0^H + \varphi_1^H e^{\frac{t\log q}{s}} + \cdots \qquad (H=0, \ldots k)$$

in a region $W^\varepsilon(u)$. On the other hand, the functions $|\eta^H(t)|$ are bounded in $W_1(u)$.

§ 5. **Factorization.** As remarked in § 2, under the transformation (4; § 2) the q-difference equation (1 a; § 1) assumes the form of a difference equation (5 a; § 2),

$$(1) \qquad M_n(y) \equiv y(t+n) + b_1(t)\,y(t+n-1) + \cdots + b_n(t)\,y(t) = 0,$$

which has a set of n formal series solutions (7; § 2). Let $1 \leqq \Gamma < n$. The system (10, 10 a; § 2; with $k=\Gamma$), of order C_Γ^n and associated with the system (8; § 2) (which, in turn, corresponds to (1)), possesses a formal matrix solution (10 b; § 2),

$$S_\Gamma(t) = \left(e^{Q_{j_1}(t)+\cdots+Q_{j_\Gamma}(t)}\, e^{t(r_{j_1}+\cdots+r_{j_\Gamma})\log q}\, \sigma^\Gamma_{i_1\ldots i_\Gamma;\, j_1\ldots j_\Gamma}(t)\right).$$

In the first column $(j_1\ldots j_\Gamma)=(1\ldots\Gamma)$ and the value of μ $(=\mu_1+\cdots+\mu_\Gamma)$ corresponding to this column is equal to or is less than any of the other values μ $(=\mu_{j_1}+\mu_{j_2}+\cdots+\mu_{j_\Gamma};\ j_1<j_2<\cdots<j_\Gamma)$. The elements of this column,

$$(2)\qquad u_{i_1\ldots i_\Gamma}(t) = e^{Q_1(t)+\cdots+Q_\Gamma(t)}\, e^{t(r_1+\cdots+r_\Gamma)\log q}\, \sigma^\Gamma_{i_1\ldots i_\Gamma;\,1\ldots\Gamma}(t),$$

$$\sigma^\Gamma_{i_1\ldots i_\Gamma;\,1\ldots\Gamma}(t) = \left|\left(e^{t(i_r-1)\eta_c\frac{\log q}{s_c}}\,\sigma_{i_r,c}(t)\right)\right| \qquad (r, c=1,\ldots\Gamma),$$

converge in a closed region U bounded on the right by a line D (cf. § 2).[1] These elements constitute therefore an actual solution of the system (10; § 2). Accordingly, there exist Γ solutions of the system (8; § 2),

$$(2\text{ a})\qquad y_{ij}(t) = y_{1j}(t+i-1)$$
$$(j=1,\ldots\Gamma;\ i=1,\ldots n),$$

such that the elements (2) are correspondingly equal to the determinants

$$(2\text{ b})\qquad |(y_{i_r,c}(t))| = |(y_{1,c}(t+i_r-1))|$$
$$(r, c=1,\ldots\Gamma;\ i_1<i_2<\cdots<i_\Gamma=1, 2,\ldots n).$$

It is to be noted that an element given by (2) is also representable by the determinant

$$(2\text{ c})\qquad |(e^{Q_c(t+i_r-1)}\, e^{(t+i_r-1)r_c\log q}\,\sigma_{1,c}(t+i_r-1)|$$
$$(r, c=1,\ldots\Gamma)$$

the Γ series

$$(2\text{ d})\qquad e^{Q_c(t)}\, e^{tr_c\log q}\,\sigma_{1,c}(t) \qquad (c=1,\ldots\Gamma)$$

being formal solutions of the equation (1).

Consider the operator

$$(3)\qquad M_\Gamma(y) = y(t+\Gamma)+c_1(t)\,y(t+\Gamma-1)+\cdots+c_\Gamma(t)\,y(t),$$

$$c_i(t) = \frac{(-1)^i u_{1,2,\ldots\Gamma-i,\Gamma-i+2,\ldots\Gamma+1}(t)}{u_{1\ldots\Gamma}(t)} \qquad (i=1, 2,\ldots\Gamma).$$

[1] Some of the $\sigma_{i_r,c}$ may diverge. However, formal computation of (2) will always give a convergent result inasmuch as $\mu_1+\cdots+\mu_\Gamma \leqq \mu_{j_1}+\mu_{j_2}+\cdots+\mu_{j_\Gamma}$ $(j_1<j_2<\cdots<j_\Gamma)$.

The coefficients $c_i(t)$ will be of the same type as the $b_i(t)$ $(i = 1, \ldots n)$, in (1), at least when

$$(3\text{ a}) \qquad \mu_\Gamma < \mu_{\Gamma+1}.$$

As the more general case, when $\mu_\Gamma = \mu_{\Gamma+1}$, will not be needed for the purposes at hand (3 a) will be assumed. The truth of the statement concerning the nature of the $c_i(t)$ is a consequence of the following considerations. The elements of the determinant $u_{1\ldots\Gamma}(t)$ may contain positive integral powers of t (cf. 2 c); these enter in the consecutive columns in such a way that by suitably combining the columns it is possible to show that in the formal series expression for $u_{1\ldots\Gamma}(t)$ no such powers actually enter (compare with analogous situations in (BT)). In this connection it is also essential that any μ is a positive rational fraction. In view of the linear independence of the formal series we have $u_{1\ldots\Gamma}(t) \not\equiv 0$. For same reason $c_\Gamma(t)$ will not be identically zero. Hence the difference operator $M_\Gamma(y)$ is actually of order Γ.

Since the $u_{i_1\ldots i_\Gamma}(t)$ are representable by the determinants (2 c), respectively, it follows that, except for a factor, $M_\Gamma(y)$ is of the form

$$(3\text{ b}) \qquad \begin{vmatrix} y(t) & e^{Q_1(t)} e^{t r_1 \log q} \sigma_{1,1}(t) & \ldots & e^{Q_\Gamma(t)} e^{t r_\Gamma \log q} \sigma_{1,\Gamma}(t) \\ y(t+1) & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ y(t+\Gamma), & e^{Q_1(t+\Gamma)} e^{(t+\Gamma) r_1 \log q} \sigma_{1,1}(t+\Gamma), & \ldots & e^{Q_\Gamma(t+\Gamma)} e^{(t+\Gamma) r_\Gamma \log q} \sigma_{1,\Gamma}(t+\Gamma) \end{vmatrix}.$$

Consequently the series (2 d) constitute a set of linearly independent formal solutions of the equation

$$(4) \qquad M_\Gamma(y) = 0.$$

As the $u_{i_1\ldots i_\Gamma}(t)$ are also representable by the determinants (2 b) it is concluded that there exist solutions $y_{1c}(t)$ $(c = 1, \ldots \Gamma)$ of the equation $M_n(y) = 0$ which also satisfy the equation (4).

If we write

$$(4\text{ a}) \qquad M_n(y) = M_{n-\Gamma} M_\Gamma(y),$$

$M_{n-\Gamma}(z)$ will be a difference operator of order $n - \Gamma$

$$(4\text{ b}) \qquad M_{n-\Gamma}(z) = z(t+n-\Gamma) + d_1(t)\, z(t+n-\Gamma-1) + \cdots + d_{n-\Gamma}(t)\, z(t).$$

The coefficients $d_i(t)$ $(i=1, \dots n-\Gamma)$ can be obtained by comparison and they are of the same nature as the $b_i(t)$. The equation

$$M_{n-\Gamma}(z) = 0 \tag{4 c}$$

possesses a linearly independent set of formal solutions

$$M_\Gamma(e^{Q_{\Gamma+i}(t)} e^{(t+i) r_c \log q} \sigma_{1, \Gamma+i}(t)) \qquad (i=1, \dots n-\Gamma). \tag{5}$$

Lemma. *Suppose that*

$$\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_\Gamma < \mu_{\Gamma+1} \leqq \cdots \leqq \mu_n$$
$$(1 \leqq \Gamma < n).$$

The equation (1) *will be then necessarily factorable,*

$$M_n(y) \equiv M_{n-\Gamma} M_\Gamma(y),$$

in such a way that the difference operators $M_\Gamma(y)$ (*of order* Γ) *and* $M_{n-\Gamma}(z)$ (*of order* $n-\Gamma$) *have analytic coefficients of the same nature as those of* $M_n(y)$. *Moreover, the first* Γ *formal series solutions of the equation* $M_n(y)=0$ *will also satisfy*

$$M_\Gamma(y) = 0,$$

while the series (5) *will constitute a set of formal solutions of the equation*

$$M_{n-\Gamma}(z) = 0.$$

§ 6. **The Preliminary Lemma.** We now proceed to the actual construction of solutions.

The equation $M_n(y) = 0$ (1; § 5) can be factored in the sense of the Lemma of § 5 if, as will be supposed, there are more than one μ-groups; that is, if we have (1 c; § 2) with $\sigma > 1$. The contrary case has been already treated by Carmichael and Birkhoff. Several successive applications of the factorization Lemma will result in the following factorization (valid in U; cf. § 5)

$$M_n(y) \equiv M_{\Gamma_\sigma - \Gamma_{\sigma-1}} M_{\Gamma_{\sigma-1} - \Gamma_{\sigma-2}} \dots M_{\Gamma_2 - \Gamma_1} M_{\Gamma_1}(y) \qquad (\Gamma_\sigma = n). \tag{1}$$

Here the σ factors correspond to the σ μ-groups. Now the equation $M_n(y)=0$ possesses n linearly independent formal series solutions $e^{Q_j(t)} e^{r_j t \log q} \sigma_j(t)$ $(j=1, \dots n)$, where the $\sigma_j(t)$ are σ-series (cf. § 2). An equation

$$M_{\Gamma_\lambda-\Gamma_{\lambda-1}}(y^{\lambda,\lambda-1}) = 0 \qquad (1 \leqq \lambda \leqq \sigma;\ \Gamma_0 = 0) \tag{1 a}$$

will be of order $\Gamma_\lambda - \Gamma_{\lambda-1}$. It will be satisfied by $\Gamma_\lambda - \Gamma_{\lambda-1}$ linearly independent formal solutions

$$e^{Q_j(t)}\, e^{r_j t \log q}\, \sigma_j^{\lambda,\lambda-1}(t) \qquad (j = \Gamma_{\lambda-1}+1, \ldots \Gamma_\lambda), \tag{1 b}$$

where the $\sigma_j^{\lambda,\lambda-1}(t)$ are σ-series. The values of μ in (1 b) are all equal and so these series converge in U. The right boundary of U will be taken suitably far to the left so as to enable taking the same region U for all equation (1 a).

First the equation

$$M_{\Gamma_2}(y^2) \equiv M_{\Gamma_2-\Gamma_1}\, M_{\Gamma_1}(y^2) = 0 \tag{2}$$

will be solved. The equation $M_{\Gamma_1}(y^1)=0$ has solutions

$$y_j^1(t) = e^{Q_j(t)}\, e^{r_j t \log q}\, \sigma_j(t) \qquad (j = 1, \ldots \Gamma_1). \tag{2 a}$$

Elements $\bar{y}_{ij}^1(t)$ $(i, j = 1, \ldots \Gamma_1)$ will be defined by the relation

$$(\bar{y}_{ij}^1(t) = (y_j^1(t+i-1))^{-1} \qquad (i, j = 1, \ldots \Gamma_1). \tag{2 b}$$

On the other hand, in view of (1 b), the equation $M_{\Gamma_2-\Gamma_1}(y^{2,1})=0$ is satisfied by the solutions

$$y_j^{2,1}(t) = e^{Q_j(t)}\, e^{r_j t \log q}\, \sigma_j^{2,1}(t) \qquad (j = \Gamma_1+1, \ldots \Gamma_2).$$

The equation (2) will possess the Γ_1 solutions (2 a),

$$y_j^2(t) = y_j^1(t) \qquad (j = 1, \ldots \Gamma_1).$$

To find the remaining solutions the equations

$$M_{\Gamma_1}(y^2) = y_j^{2,1}(t) \qquad (\Gamma_1 + 1 \leqq j \leqq \Gamma_2) \tag{3}$$

are to be solved. An equation (3) can be solved by formula (7) in (BT; § 7), which was used in (BT) for an analogous purpose. Taking account of the difference in notation we have

$$y_j^2(t) = \sum_{\lambda=1}^{\Gamma_1} y_\lambda^1(t) \mathop{\mathfrak{S}}_{\Gamma=t} \bar{y}_{\lambda\Gamma_1}^1(\Gamma+1)\, y_j^{2,1}(\Gamma) \tag{3 a}$$

$$(\Gamma_1 + 1 \leqq j \leqq \Gamma_2).$$

Here the summation methods of § 4 are to be applied.

The summand in (3 a) can be expressed in the form

$$\bar{y}^1_{\lambda \Gamma_1}(t+1)\, y^{2,1}_j(t) = e^{Q_{j,\lambda}(t)+r^2_{j,\lambda} t \log q} \qquad [\sigma\text{-series}], \tag{4}$$

$$\left[Q_{j,\lambda}(t) = Q_j(t) - Q_\lambda(t);\ r^2_{j,\lambda} = r_{j,\lambda} + \frac{N}{s}(N, \text{integer});\ r_{j,\lambda} = r_j - r_\lambda\right].[1]$$

The σ-series involved in (4) converges in U. Application of Lemma 1 (§ 4) will give

$$\mathop{\mathfrak{S}}_{\Gamma=t} \bar{y}^1_{\lambda \Gamma_1}(\Gamma+1)\, y^{2,1}_j(\Gamma) = e^{Q_{j,\lambda}(t)+r^2_{j,\lambda} t \log q} \sum_{H=0} t^H \eta^H_{\lambda,j}(t), \tag{4 a}$$

where the $\eta^H_{\lambda,j}(t)$ $(H=0, 1, \ldots)$ are analytic and bounded in a region W, while

$$\eta^H_{\lambda,j}(t) \sim \eta^H_{\lambda,j;\,0} + \eta^H_{\lambda,j;\,1}\, e^{-\frac{t \log q}{s}} + \cdots \tag{4 b}$$

for t in a region W^ε. Therefore, by (3 a) and in virtue of the known form of the $y^1_\lambda(t)$ $(\lambda = 1, \ldots \Gamma_1)$,

$$y^2_j(t) = e^{Q_j(t)}\, e^{r^2_j t \log q} \sum_{H=0} t^H \eta^{2;\,H}_j(t) \tag{4 c}$$

$$\left(\Gamma_1 + 1 \leqq j \leqq \Gamma_2;\ r^2_j = r_j + \frac{N}{s}\right).$$

The functions $\eta^{2;\,H}_j(t)$ $(H=0, 1, \ldots)$ are analytic and bounded in W; moreover, in W^ε,

$$\eta^{2;\,H}_j(t) \sim \sigma^{2;\,H}_{j;\,0} + \sigma^{2;\,H}_{j;\,1}\, e^{-\frac{t \log q}{s}} + \cdots. \tag{4 d}$$

In this sense, for t in W^ε,

$$y^2_j(t) \sim e^{Q_j(t)}\, e^{r^2_j t \log q}\, \sigma^2_j(t) \qquad (\Gamma_1 + 1 \leqq j \leqq \Gamma_2), \tag{4 e}$$

the symbol $\sigma^2_j(t)$ denoting a σ-series. We get a result of this kind for $j = \Gamma_1 + 1$, $\Gamma_1 + 2, \ldots \Gamma_2$.

[1] In the derivation of this formula use is made of the fact that $\left|(y^1_j(t+i-1))\right|$ $(i, j = 1, \ldots \Gamma_1)$ is expressible as a product of an exponential factor by a σ-series factor. The latter factor involves no powers of t; this can be shown by suitably combining columns of the determinant.

Consider the equation

$$M_{\Gamma_3}(y^3) \equiv M_{\Gamma_3-\Gamma_2} M_{\Gamma_2}(y^3) = 0. \tag{5}$$

The nature of a set of solutions of $M_{\Gamma_2}(y^2) = 0$ in a region W has been determined above. We define the elements $\bar{y}^2_{ij}(t)$ by the relation

$$(\bar{y}^2_{ij}(t)) = (y^2_j(t+i-1))^{-1} \qquad (i, j = 1, \ldots \Gamma_2). \tag{5 a}$$

On the other hand, the functions (1 b), formed for $\lambda = 3$, constitute a set of solutions of the equation $M_{\Gamma_3-\Gamma_2}(y^{3,2}) = 0$,

$$y^{3,2}_j(t) = e^{Q_j(t)} e^{r_j t \log q} \sigma^{3,2}_j(t) \qquad (j = \Gamma_2 + 1, \ldots \Gamma_3).$$

The equation (5) is satisfied by the solutions of $M_{\Gamma_2}(y^2) = 0$ so that it will be consistent to write

$$y^3_j(t) = y^2_j(t) \qquad (j = 1, 2, \ldots \Gamma_2).$$

The remaining $\Gamma_3 - \Gamma_2$ solutions are obtained by solving each one of the equations

$$M_{\Gamma_2}(y^3) = y^{3,2}_j(t) \qquad (\Gamma_2 + 1 \leqq j \leqq \Gamma_3). \tag{6}$$

Now

$$y^3_j(t) = \sum_{\lambda=1}^{\Gamma_2} y^2_\lambda(t) \mathop{S}_{\Gamma=t} \bar{y}^2_{\lambda \Gamma_2}(\Gamma + 1)\, y^{3,2}_j(t). \tag{6 a}$$

Application of the summation method of Lemma 2 (§ 4) gives, after a simple computation, the following result:

$$y^3_j(t) = e^{Q_j(t)} e^{r^3_j t \log q} \sum_{H=0} t^H \eta^{3;H}_j(t) \tag{6 b}$$

$$\left(\Gamma_2 + 1 \leqq j \leqq \Gamma_3;\ r^3_j = r_j + \frac{N_j}{s};\ N_j, \text{ integers}\right).$$

The functions $\eta^{3;H}_j(t)$ are analytic and bounded in a region of type $W(u)$ while the asymptotic relations

$$\eta^{3;H}_j(t) \sim \sigma^{3;H}_{j;0} + \sigma^{3;H}_{j;1} e^{\frac{t \log q}{s}} + \cdots \qquad (H = 0, 1, \ldots) \tag{6 c}$$

will hold in a region $W^\varepsilon(u)$. In this sense, for t in $W^\varepsilon(u)$, (6 d) $$y_j^3(t) \sim e^{Q_j(t)} e^{r_j^3 t \log q} \sigma_j^3(t) \qquad (\Gamma_2 + 1 \leqq j \leqq \Gamma_3)$$

where $\sigma_j^3(t)$ is a σ-series. There are $\Gamma_3 - \Gamma_2$ such solutions.

In the indicated manner the equations

$$M_{\Gamma_1}(y^1) = 0, \ M_{\Gamma_2}(y^2) = 0, \ldots, \ M_n(y^\sigma) = 0$$

are solved in succession.

The Preliminary Lemma. *The equation $M_n(y) = 0$ possesses a linearly independent set of solutions*

(7) $$y_j(t) = e^{Q_j(t)} e^{r_j^{(n)} t \log q} \sum_{H=0}^{k_j} t^H \eta_j^H(t) \qquad (j = 1, \ldots n)$$

such that the $\eta_j^H(t)$ $(H = 0, 1, \ldots k_j;\ j = 1, \ldots n)$ are analytic (for $t \neq \infty$) and are bounded in a region $W(u)$; moreover, they are such that in a region of type $W^\varepsilon(u)$

(7 a) $$\eta_j^H(t) \sim \sigma_j^H(t) = \sigma_{j;\,0}^H + \sigma_{j;\,1}^H e^{\frac{t \log q}{s}} + \cdots$$
$$(H = 0, 1, \ldots k_j;\ j = 1, \ldots n).$$

The formal expressions

(7 b) $$e^{Q_j(t)} e^{r_j^{(n)} t \log q} \sum_{H=0}^{k_j} t^H \sigma_j^H(t) \qquad (j = 1, \ldots n)$$

will constitute a linearly independent set of formal series solutions of the equation. Similar results hold in regions $W(l)$, $W^\varepsilon(l)$. There is an analogous situation, with respect to a set of formal series solutions corresponding to $x = \infty$, in a right t-half plane.[1]

Let us consider a set of solutions $y_j(t)$ $(j = 1, \ldots n)$ of the equation $M_n(y) = 0$, whose existence in the above lemma has been asserted in connection with a region $W(u)$, for instance. The nature of the $y_j(t)$, in the upper t-half plane, outside of $W(u)$ can be determined by consideration of the corresponding system (8; § 2),

(8) $$Y(t + 1) = D(t)\, Y(t).$$

In $W^\varepsilon(u)$

(8 a) $$Y(t) = (y_j(t + i - 1)) \sim (e^{Q_j(t)} e^{t \delta_{ij}} \sigma_{ij}(t))$$

[1] Provided that the coefficients of the equation are of the nature, in the vicinity of $x = \infty$, specified in § 1.

(cf. (8 a; § 2)). Here the asymptotic relations hold with respect to the σ-series $\sigma_{ij}(t)$ $(i, j = 1, \ldots n)$. By (8) it follows that

$$Y(t) = D(t-1) \ldots D(t-m_t)\, Y(t-m). \tag{9}$$

Let $P(u)$ be the half plane bounded below by the line, parallel to the axis of reals, which is obtainable by extending the lower boundary of $W(u)$. For t in $P(u)$ the relation (9) will express the values of the $y_j(t)$ in terms of their values in $W(u)$ (or $W^{\varepsilon}(u)$), provided that (depending on t) m is taken sufficiently great and provided that t is not congruent from the right to a singularity of an element of $D(t)$. If the assumptions of § 1 hold both at $x=0$ and at $x=\infty$ all such singularities will be in a strip bounded by lines with the slope $\frac{\log |q|}{\bar{q}}$. In general, it cannot be expected that asymptotic relations of any kind will be valid for the solutions when t is in U outside of $W(u)$ and above the axis of reals (or outside of $W(l)$ in U and below the axis of reals, in the case when solutions corresponding to $W(l)$ are considered).

Analogous facts can be stated for a right t-half plane (that is, a neighborhood of $x=\infty$) by means of the relation

$$Z(t) = D^{-1}(t)\, D^{-1}(t+1) \ldots D^{-1}(t+r-1)\, Z(t+r). \tag{9 a}$$

Here r is taken sufficiently great, depending on t. Moreover, t should be not congruent from the left to a singularity of an element of $D^{-1}(t-1)$.

§ 7. **The Fundamental Existence Theorem.** The regions $W(u)$, $W(l)$ overlap along a strip S^{-} (bounded above and below by portions of lines parallel to the axis of reals). This strip contains a part of the axis of reals.

Let

$$y_j^u(t) \qquad (j = 1, 2, \ldots n) \tag{1}$$

be a set of solutions, corresponding $W(u)$, and let

$$y_j^l(t) \qquad (j = 1, 2, \ldots) \tag{1 a}$$

be a set of solutions, corresponding to $W(l)$. It is assumed that in $W^{\varepsilon}(u)$ and $W^{\varepsilon}(l)$, respectively, the sets (1) and (1 a) are asymptotic to certain sets of formal series solutions (as asserted in the Preliminary Lemma). These sets of formal series solutions are essentially the same. Thus it may be assumed that

$$Y^u(t) \equiv (y_j^u(t+i-1)) \sim (e^{Q_j(t)}\, e^{t\delta_{ij}}\, \sigma_{ij}(t)) \equiv S(t) \tag{2}$$

$$(\delta_{ij} = [(i-1)\,\mu_j + r_j] \log q;\ \sigma_{ij}(t),\ \sigma\text{-series};\ i, j = 1, \ldots n)$$

for t in $W^\epsilon(u)$, while

$$Y^l(t) \equiv (y_j^l(t+i-1)) \sim (e^{Q_j(t)}\, e^{t\delta_{ij}}\, \sigma_{ij}(t)) \tag{2 a}$$

for t in $W^\epsilon(l)$. The matrix $P'(t)$ $(=p'_{ij}(t))$ of functions of period unity, defined by the relation

$$Y^u(t) = Y^l(t)\, P'(t), \tag{3}$$

consists of elements analytic (for $t \neq \infty$) in S^- and, necessarily, analytic in a strip H containing the axis of reals. In virtue of (2) and (2 a) it follows that

$$(p'_{ij}(t)) = \left(e^{Q_{ji}(t)}\, e^{r_{ji} t \log q}\left[\xi_{ij} + e^{-\frac{tk\log q}{s}}\, b_{ij}^k(t)\right]\right) \tag{3 a}$$

$$(Q_{ji}(t) = Q_j(t) - Q_i(t);\ r_{ji} = r_j - r_i)$$

where $|b_{ij}^k(t)| < b^k$ $(i, j = 1, \ldots n;\ t$ in $S^-)$ and k can be taken arbitrarily great. Here $(\xi_{ij}) = I$ (the identity matrix). In view of the ordering of the μ_j it follows that the $\Re\, Q_{ji}(t)$ $(j < i)$ approach minus infinity, as $|t| \to \infty$ in the strip S^-, rapidly enough to insure the relations

$$p'_{ij}(t) = \lim_{|t'| \to \infty} p'_{ij}(t') = 0$$

$$(\Im x = \Im x' = 0;\quad \Re x' < \Re x;\quad x - x',\ \text{integer})$$

for $i > j$. Similarly,

$$p'_{ii}(t) = \lim_{|t'|} p'_{ii}(t') = 1 \qquad (i = 1, \ldots n).$$

For $j > i$ we still have

$$p'_{ij}(t) = \lim_{|t'|} p'_{ij}(t') = 0$$

provided that i and j belong to the same μ-group, that is provided that for some r $(1 \leqq r \leqq \sigma)$

$$\Gamma_{r-1} + 1 \leqq i < j \leqq \Gamma_r \qquad (\Gamma_0 = 0) \tag{4}$$

(cf. (1 c; § 2)). Thus

$$(p'_{ij}(t)) = \begin{pmatrix} 1, & p'_{12}(t) & \ldots & p'_{1n}(t) \\ 0, & 1, & \ldots & p'_{2n}(t) \\ \cdot & \cdot & \cdot & \cdot \\ 0, & \ldots & & 1 \end{pmatrix}. \tag{5}$$

A matrix of the general type of the second member of (5) will be called, in accordance with Definition 11 of (BT), a half matrix. It is noted that in (5) the only elements, to the right of the principal diagonal, which could at all be not identically zero are the elements $p'_{ij}(t)$ for which no relation (4) holds.

Apply the substitution

$$z = e^{2\pi\sqrt{-1}\,t}.$$

On writing

$$P'(t) \equiv G(z) = (g_{ij}(z)),$$

it is observed that in a Laurent ring (c_1, c_2), formed by circles $c_1\,(|z| = e^{-2\pi\sigma})$ and $c_2\,(|z| = e^{2\pi\sigma};\ \sigma > 0)$, the $g_{ij}(z)$ are analytic. As in (BT; pp. 74—75), it is noted that, if $g(z)$ is analytic in a ring (c_1, c_2), it follows that

$$g(z) = a(z) + b(z),$$

$$a(z) = \frac{1}{2\pi\sqrt{-1}} \int\limits_{c_2} \frac{g(\zeta)\,d\zeta}{\zeta - z}, \quad b(z) = \frac{1}{2\pi\sqrt{-1}} \int\limits_{c_1} \frac{g(\zeta)\,d\zeta}{\zeta - z}. \tag{6}$$

Here the direction of integration is as in (BT; p. 75). The function $a(z)$ will be analytic interior c_2 while $b(z)$ is analytic exterior c_1. We now proceed to the determination of the half matrices $A(z)$ $(\equiv (a_{ij}(z)))$, with elements analytic interior to c_2^*, and $B(z)$ $(\equiv (b_{ij}(z)))$, with elements analytic exterior to c_1^*, such that

$$G(z) = B(z)\,A(z). \tag{7}$$

The ring (c_1^*, c_2^*) is to be slightly interior to (c_1, c_2). If $A(z)$ and $B(z)$ are half matrices it is clear that (7) is satisfied for $i \geqq j$. It remains to consider (7) with $i < j$. We have

$$g_{ij}(z) = \sum_{\lambda = i}^{j} b_{i\lambda}(z)\,a_{\lambda j}(z) \qquad (i < j). \tag{7 a}$$

Choose the $a_{ij}(z)$ and the $b_{ij}(z)$ so that, whenever (4) holds,

$$a_{ij}(z) = b_{ij}(z) = 0. \tag{7 b}$$

Then the equations (7 a) will be satisfied whenever i and j are of the same μ-group. Thus it is necessary to consider (7 a) only for values of i and j such that

$$\Gamma_{\varrho-1} + 1 \leqq i \leqq \Gamma_\varrho < \Gamma_{r-1} + 1 \leqq j \leqq \Gamma_r \qquad (\Gamma_0 = 1).$$

We group equations (7 a), for $\sigma = 1, 2, \ldots n-1$, in sets

$$g_{i,i+\sigma}(z) = \sum_{\lambda=i}^{i+\sigma} b_{i\lambda}(z)\, a_{\lambda,i+\sigma}(z) \qquad (i = 1, \ldots n-\sigma). \tag{8}$$

These sets are solved in succession for $\sigma = 1, 2, \ldots$, as in (BT; pp. 75—78) and on making use of (6). The sets of functions

$$\begin{aligned} g^{m}_{i,i+m}(z) \equiv g_{i,i+m}(z) + [b_{i,i+1}(z)\, a_{i+1,i+m}(z) + b_{i,i+2}(z)\, a_{i+2,i+m}(z) + \cdots \\ + b_{i,i+m-1}(z)\, a_{i+m-1,i+m}(z)] \qquad (i = 1, \ldots n-m) \end{aligned} \tag{8 a}$$

are determined for $m = 1, 2, \ldots n$, while we also obtain in succession, for $m = 1, 2, \ldots n$, the functions

$$a_{i,i+m}(z) = \frac{1}{2\pi\sqrt{-1}} \int_{c_2^m} \frac{g^m_{i,i+m}(\zeta)\, d\zeta}{\zeta - z}, \tag{8 b}$$

$$b_{i,i+m}(z) = \frac{1}{2\pi\sqrt{-1}} \int_{c_1^m} \frac{g^m_{i,i+m}(\zeta)\, d\zeta}{\zeta - z} \qquad (i = 1, 2, \ldots n-m).$$[1]

Let

$$(\alpha_{ij}(t)) \equiv (a_{ij}(z)), \qquad (\beta_{ij}(t)) \equiv (b_{ij}(z)).$$

On making use of (8 b) the following facts are proved. The $\alpha_{ij}(t)$ are of period unity, are analytic for $\Im t \geqq -c$ $(\varrho > c > 0)$ and the $\beta_{ij}(t)$ are of period unity, analytic for $\Im t \leqq -c$. Moreover,

$$\alpha_{ij}(t) = e^{2\pi\sqrt{-1}\,\alpha_{ij}t}\,\alpha^*_{ij} + \cdots \tag{8 c}$$

$$(\alpha_{ij}, \text{ integer}, \geqq 0),$$

where the second member is a series in positive powers of $e^{2\pi\sqrt{-1}t}$ convergent for $\Im t \leqq c$. On the other hand,

$$\beta_{ij}(t) = e^{-2\pi\sqrt{-1}\,\beta_{ij}t}\,\beta^*_{ij} + \cdots \tag{8 d}$$

$$(\beta_{ij}, \text{ integer}, \geqq 1),$$

where the second member is a series in positive powers of $e^{-2\pi\sqrt{-1}t}$ convergent

[1] Here the Laurent ring (c_1^m, c_2^m) is slightly interior to (c_1^{m-1}, c_2^{m-1}), all such rings being interior to (c_1, c_2).

for $\Im t \leqq c$. Furthermore, $(\alpha_{ij}(t))^{-1} \equiv \bar{\alpha}_{ij}(t))$ is a half matrix, with elements analytic for $\Im t \geqq -c$ and such that

$$\bar{\alpha}_{ij}(t) = e^{2\pi\sqrt{-1}\,\bar{\alpha}_{ij}t}\,\bar{\alpha}^*_{ij} + \cdots \tag{8 e}$$

$(\bar{\alpha}_{ij}$, integer; $i < j)$.

Whenever i and j satisfy (4)

$$\alpha_{ij}(t) = \beta_{ij}(t) + \bar{\alpha}_{ij}(t) = 0.\text{[1]} \tag{9}$$

Now

$$Y^u(t)\,(\bar{\alpha}_{ij}(t)) = Y^l(t)\,(\beta_{ij}(t)) \qquad (\equiv Y^*(t)). \tag{10}$$

It is clear that $Y^*(t)$ is a matrix solution of the system (8; § 6), corresponding to the equation $M_n(y) = 0$. Its elements, $y^*_{ij}(t)$, are analytic (for $t \neq \infty$) in the combined region $W = W(u) + W(l)$.

By (2)

$$(y^u_{ij}(t)) = \left(e^{Q_j(t)}\,e^{t\delta_{ij}}\left[\sigma^k_{ij}(t) + e^{\frac{kt\log q}{s}}\,b^k_{ij}(t)\right]\right). \tag{11}$$

Here $\sigma^k_{ij}(t)$ is $\sigma_{ij}(t)$ with the power series factors terminated after m_k ($m_k \to \infty$, as $k \to \infty$) terms and k can be taken arbitrarily great; moreover,

$$|\,b^k_{ij}(t)\,| \leqq b^k \qquad (\text{in } W^\varepsilon(u)).$$

On using (8 c) and (10) it follows that, for t in $W^\varepsilon(u; -c)$,

$$\begin{aligned} y^*_{ij}(t) &= y^u_{ji}(t) + \sum_{\lambda=1}^{j-1} y^u_{i\lambda}\,\bar{\alpha}_{\lambda j}(t) \\ &= e^{Q_j(t)}\,e^{t\delta_{ij}}\left[\sigma^k_{ij}(t) + e^{\frac{kt\log q}{s}}\,b^k_{ij}(t) + r^u_{ij}(t)\right], \end{aligned} \tag{11 a}$$

where

$$r^u_{ij}(t) = \sum_{\lambda=1}^{j-1} e^{Q_\lambda(t)-Q_j(t)}\,e^{t(\delta_{i\lambda}-\delta_{ij})}\left(\sigma^k_{i\lambda}(t) + e^{\frac{kt\log q}{s}}\,b^k_{i\lambda}(t)\right)\left(e^{2\pi\sqrt{-1}\,\bar{\alpha}_{\lambda j}t}\,\bar{\alpha}^*_{\lambda j} + \cdots\right). \tag{11 b}$$

In the second member of (11 b) only those terms may be present for which λ and j are in different μ-groups. Hence, if we write

$$Q_\lambda(t) - Q_j(t) = \mu t^2 \log q,$$

[1] This follows by the reasoning employed in (BT) for an analogous purpose.

the constant μ will be negative for each term in (11 b). As $|t| \to \infty$ in $W^{\varepsilon}(u)$ (and in $W^{\varepsilon} = W^{\varepsilon}(u) + W^{\varepsilon}(l)$) $\Re\,[Q_{\lambda}(t) - Q_{j}(t)]$ will approach minus infinity rapidly enough to insure the asymptotic relationship

$$r_{ij}^{u}(t) \sim 0, \tag{11 c}$$

which will be valid in $W^{\varepsilon}(u; -c)$. In the latter region, then,

$$Y^{*}(t) \sim S(t). \tag{12}$$

Similarly, using (10), (2 a) and (8 d), it can be shown that (12) holds in a region $W^{\varepsilon}(l; c)$. Hence (12) holds in a region W^{ε}.

Let $\Gamma(t)$ denote a formal matrix solution corresponding to $x = \infty$. Then there exists a matrix solution

$$Z^{*}(t) = (z_{ij}^{*}(t)) \sim \Gamma(t) \qquad (t \text{ in } V^{\varepsilon}) \tag{12 a}$$

with elements analytic in V.

Results of the above type will also hold for any system (5; § 2)

$$Y(t+1) = B(t)\,Y(t), \tag{13}$$

corresponding to a q-difference system (1; § 1).

The Fundamental Existence Theorem. *The difference system* (13), *corresponding to a q-difference system* (5; § 2), *will always possess a matrix solution* $Y^{*}(t)$, *of elements analytic in* W, *such that*

$$Y^{*}(t) \sim S(t) \tag{13 a}$$

for t *in* W^{ε}. *This system will also possess a matrix solution* $Z^{*}(t)$, *of elements analytic in* V, *such that*

$$Z^{*}(t) \sim \Gamma(t) \tag{13 b}$$

for t *in* V^{ε}. *Here* $S(t)$ *and* $\Gamma(t)$ *are formal matrix solutions corresponding to* $x = 0$ *and* $x = \infty$, *respectively.*

The implications of this theorem for a q-difference system are obvious.

§ 8. **Connection between the 'left' and the 'right' Solutions.** The limiting directions at infinity of the upper and lower boundaries of W are at right angles. The same is true of V. These regions, as well as W^{ϵ} and V^{ϵ}, each extend indefinitely upwards and downwards from the axis of reals. Let $Y^*(t)$ and $Z^*(t)$ be matrix solutions of the Fundamental Existence Theorem. The elements $p_{ij}(t)$ of the matrix $P(t)$, defined by the relation

$$Y^*(t) = Z^*(t)\,P(t), \tag{1}$$

are of period unity. On making use of the matrix equation it follows that

$$P(t) = Z^{*-1}(t+r_t)\,B(t+r_t-1)\,B(t+r_t-2)\ldots B(t)\,B(t-1)\ldots B(t-l_t)\,Y^*(t-l_t). \tag{2}$$

The important case when the coefficients of the q-difference equation (or system) are rational in x $\left(\text{or } x^{\frac{1}{s}}\right)$ is reducible to the case when they are polynomials in x $\left(\text{or } x^{\frac{1}{s}}\right)$. In the latter case the coefficients of the corresponding difference equation (or system) will be entire in t $\left(\text{being polynomials in } e^{\frac{t \log q}{s}}\right)$. The $p_{ij}(t)$ will be then entire in t. This can be seen by taking the integers r_t, l_t in (2), depending on t, sufficiently great so that $t+r_t$ is in V^{ϵ}, say, and $t-l_t$ is in W^{ϵ}. Moreover, by taking r_t also so that $|t+r_t| \geqq \varrho$ (ϱ independent of t and sufficiently great), the non vanishing of the determinant $|Z^*(t+r_t)|$ will be secured in virtue of (13 b; § 7).

As shown by Birkhoff, in the case when $\mu_1 = \mu_2 = \cdots = \mu_n$ (and when the coefficients are polynomials in x) the $p_{ij}(t)$ are expressible in terms of the Weierstrass sigma-functions (B; pp. 561—569). In the general case at hand, however, even if the formal series solutions, corresponding to $x = 0$ and $x = \infty$, converge nothing of corresponding simplicity can be obtained.