ESTIMATES FOR TRANSLATION INVARIANT OPERATORS IN L^p SPACES

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Contents

Preface

The theory of bounded translation invariant operators between L^p spaces in several variables has attracted much interest in the literature during the past decade, partly due to its applications in some fields such as the theory of partial differential equations. Through the work of Calderón, Zygmund and others real variable methods have been introduced which have permitted the extension to several variables of results originally based on complex methods in the case of a single variable. Further, a suitable framework for a general theory is given by the theory of distributions, for translation invariant operators are essentially convolutions with distributions (see section 1.1).

The purpose of this paper is thus the study of the spaces L_p^q of tempered distributions T in R^n such that with a constant C

$$\| T \ast u \|_q \leqslant C \| u \|_p$$

for all infinitely differentiable u with compact support, the norms being L^q and L^p norms. In section 1.2 we discuss those properties of these spaces which follow from M. Riesz' convexity theorem and the theory of the Fourier transformation in L^p spaces. Some of these results are taken over from Schwartz [13], and others have been used implicitly in various papers on convolution transforms. In section 1.3 we study homomorphisms of the Fourier transform M_p^p of L_p^p induced by a mapping in R^n. It turns out that if the mapping is twice continuously differentiable and $p \neq 2$, it must be linear. This improves a result of Schwartz [13], but for $p = 1$ it is weaker than known results concerning the algebra M_1^1 of Fourier-Stieltjes transforms. In section 1.4 we prove that the Wiener-Lévy theorem is valid in a certain subalgebra of M_p^p whose relation to M_p^p is studied. The proof is rather trivial but we have included it because of its similarity with a result in Chapter II which is essential in Chapter III. (Closely related results concerning sequence spaces are due to Devinatz and Hirschman, Amer. J. Math. 80 (1958), 829–842.)

Chapters II and III are devoted to the numerous estimates which originate from Riesz' theorem on conjugate functions (Riesz [10]). In Chapter II we discuss the real variable method introduced by Calderón and Zygmund [2] in the study of conjugate functions in several variables. It has also been used later by Zygmund [18] to prove the Hardy-Littlewood-Sobolev estimates of potentials and also by Stein [15] in studying estimates of the kind which we discuss in Chapter III. The main theorem in section 2.1 describes the general situation in which such arguments apply. In section 2.2 we show first that our theorem contains the results of Calderón and Zygmund [2] and Zygmund [18] mentioned above. We then show that it also gives a short proof and a slight improvement of a theorem of Mihlin [8], [9]. (The proof given by Mihlin depends on a paper of Marcinkiewicz [7] which is based on the Littlewood-Paley theory (see Chapter III) and on the properties of Rademacher functions.) We end the section by proving a theorem of the Wiener-Lévy type for a certain algebra of homogeneous functions of degree 0 contained in M_p^p. Closely related results are due to Calderón and Zygmund [3] but are not sufficient for the applications in Chapter III.

In Chapter III we study estimates of convolution transforms involving parameters. We do not make a systematic theory similar to that in Chapter I for such families of transforms but restrict ourselves to results parallel to those of Chapter II. In sections 3.1.–3.3 estimates involving L^2 norms with respect to the parameters are proved. In section 3.4 they are shown to contain the known results concerning the functions of Littlewood-Paley, Lusin and Marcinkiewicz as well as other estimates which may be of interest in the theory of partial differential equations. The proofs are similar to those in Section 2.1. Real variable methods have previously been used by Stein [15] in studying the Marcinkiewicz function in several variables but our method differs considerably from his. By studying the adjoint transformations which map functions in the product space of R^n and the parameter space on functions in R^n, we obtain estimates also when $2 < p < \infty$. In that case the results known previously are rather incomplete when $n > 1$ and the proofs when $n = 1$ seem difficult. We also obtain simple proofs of general "inverse" estimates. The results concerning M_p^p which follow from the Littlewood-Paley estimates are not studied here so we refer to Littlewood and Paley [5] and Marcinkiewicz [7].

This paper is essentially self-contained, which may be an advantage to the non specialist in view of the extensive literature in the field. Necessary prerequisites are elements of distribution theory, including the Fourier transformation (see [12]); Riesz' convexity theorem (see [11] and [16]), Marcinkiewicz' interpolation theorem (see [18]), and also basic facts concerning bounded operators in Banach spaces. The bibliography is very incomplete so a reader interested in studying the literature closely should consult the references given in the quoted papers also.

CHAPTER I

General theory

1.1. Translation invariant operators as convolutions

We denote by L^p, $1 \leqslant p \leqslant \infty$, the space of measurable functions in R^n with integrable pth power, and write

$$\|u\|_p = \left(\int |u|^p \, dx\right)^{1/p}, \quad u \in L^p. (^1)$$

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(1) It is convenient to set formally $\|u\|_p = \infty$ if $u \notin L^p$.

When $p=\infty$ this shall be understood as the essential supremum of $|f|$. By L_0^∞ we denote the space of functions in L^∞ which tend to 0 at ∞ and by C the space of continuous functions. M will denote the space of bounded measures $d\mu$ normed by $\int |d\mu|$. When $1 \leqslant p \leqslant \infty$ we shall use the notation p' for the conjugate exponent defined by $1/p + 1/p' = 1$.

If $h \in R^n$ we denote by τ_h the operator defined by

$$(\tau_h u)(x) = u(x-h).$$

DEFINITION 1.1. *A bounded linear operator A from L^p to L^q is said to be translation invariant if*

$$\tau_h A = A \tau_h, \quad h \in R^n.$$

Such operators which are non trivial do not exist for all p, q.

THEOREM 1.1. *If A is a bounded translation invariant operator from L^p to L^q and $p>q$ we have $A=0$ if $p<\infty$ and if $p=\infty$ the restriction of A to L_0^∞ is 0.*

Proof. First note that if $p<\infty$

$$\|u+\tau_h u\|_p \to 2^{1/p} \|u\|_p, \quad u \in L^p;\ h \to \infty; \tag{1.1.1}$$

the same is true for $p=\infty$ provided that $u \in L_0^\infty$. In fact, we can write $u=v+w$ where v has compact support and $\|w\|_p < \varepsilon$. For sufficiently large $|h|$ the supports of v and $\tau_h v$ do not meet, hence

$$\|v+\tau_h v\|_p = 2^{1/p} \|v\|_p.$$

Since $|\,\|v\|_p - \|u\|_p| < \varepsilon$ and $|\,\|v+\tau_h v\|_p - \|u+\tau_h u\|_p| < 2\varepsilon$ and ε is arbitrary, we obtain (1.1.1).

Now assume that

$$\|Au\|_q \leqslant C\|u\|_p, \quad u \in L^p, \tag{1.1.2}$$

with $q<p<\infty$. The linearity and translation invariance of A give

$$\|Au+\tau_h Au\|_q = \|A(u+\tau_h u)\|_q \leqslant C\|u+\tau_h u\|_p.$$

When $h\to\infty$ it follows from (1.1.1) that

$$\|Au\|_q \leqslant 2^{1/p-1/q} C\|u\|_p, \tag{1.1.3}$$

which improves (1.1.2) since the exponent is negative. If C denotes the smallest constant such that (1.1.2) holds we thus get a contradiction unless $C=0$, that is, $A=0$. The same arguments apply when $p=\infty>q$ provided that we replace L^∞ by L_0^∞. The proof is complete.

Although somewhat incomplete for $p=\infty$ this result will justify us to assume that $p\leqslant q$ in what follows.

Let S be the space of infinitely differentiable functions u such that

$$\sup |x_\beta D^\alpha u| < \infty$$

for all α and β, and with the topology defined by these seminorms. Here $\alpha=(\alpha_1, \ldots, \alpha_j)$ and $\beta=(\beta_1, \ldots, \beta_k)$ are multi-indices, that is, sequences of indices between 1 and n,

$$D^\alpha = (-i\,\partial/\partial x_{\alpha_1}) \ldots (-i\,\partial/\partial x_{\alpha_j}); \quad x_\beta = x_{\beta_1} \ldots x_{\beta_k}.$$

We use the notation $|\alpha|$ for the length j of the multi-index α. S is dense in L^p if $p<\infty$, and its closure in L^∞ is $C\cap L_0^\infty$. The dual space of S is denoted by S' and its elements are called tempered distributions. (See Schwartz [12].)

THEOREM 1.2. *If A is a bounded translation invariant operator from L^p to L^q, then there is a unique distribution $T\in S'$ such that*

$$Au = T \star u, \quad u\in S.$$

For the proof we need a lemma which is a very special case of Sobolev's lemma.

LEMMA 1.1. *If a function v in R^n and its derivatives of order $\leqslant n$ are in L^p locally, the definition of v may be changed on a set of measure 0 to make it continuous. Then we have with a constant C*

$$|v(x)| \leqslant C \sum_{|\alpha|\leqslant n} \left(\int_{|y-x|\leqslant 1} |D^\alpha v|^p \, dy \right)^{1/p}. \tag{1.1.4}$$

Proof. The assumptions concerning v are also satisfied with $p=1$ and (1.1.4) follows from Hölder's inequality for every p if it is proved for $p=1$. In the proof we may also assume that $x=0$ and that v has compact support in the unit sphere. For let φ be a function in C_0^∞ with support in the unit sphere and which equals 1 in a neighbourhood of 0. Then $w=v\varphi$ has compact support in the unit sphere, and Leibniz' formula shows that

$$\sum_{|\alpha|\leqslant n} \int |D^\alpha w| \, dy \leqslant C \sum_{|\alpha|\leqslant n} \left(\int_{|y|\leqslant 1} |D^\alpha v| \, dy \right).$$

Hence if we prove the statement of the lemma for w, it follows that v is continuos in a neighbourhood of the origin after correction on a null set and that (1.1.4) is valid for $x=0$. The general statement of the lemma then follows by its translation invariance.

Now let $h(x) = H(x_1) \ldots H(x_n)$ where H is the Heaviside function which equals 1 for $x > 0$ and 0 for $x < 0$. We then have

$$\partial^n h / \partial x_1 \ldots \partial x_n = \delta$$

in the distribution sence. Hence, since w has compact support,

$$w = w * \delta = w * (\partial^n h / \partial x_1 \ldots \partial x_n) = (\partial^n w / \partial x_1 \ldots \partial x_n) * h.$$

In the right hand side we have a convolution between an integrable and a bounded function, hence a continuous function. w differs from this continuous function only on a null set and if its definition is changed there we have

$$|w(x)| \leqslant \int |\partial^n w / \partial x_1 \ldots \partial x_n| \, dx,$$

which completes the proof.

Proof of Theorem 1.2. Let A be the operator in the theorem and $u \in S$. We claim that

$$D^\alpha (A u) = A (D^\alpha u) \tag{1.1.5}$$

in the distribution sense. To prove this it is clearly enough to consider a derivative of the first order. Put $v = A u$ and define $u_h(x) = u(x_1 + h, x_2, \ldots, x_n)$ and v_h similarly. Since A is invariant for translation we have $A u_h = v_h$ and hence

$$A\left(\frac{u_h - u}{h}\right) = \frac{v_h - v}{h}.$$

When $h \to 0$ the difference quotient $(u_h - u)/h$ converges to $\partial u / \partial x_1$ in L^p, hence $(v_h - v)/h$ converges to $A(\partial u / \partial x_1)$ in L^q norm. Hence (1.1.5) follows.

Lemma 1.1 now shows that $A u$ is a continuous function after correction on a null set if $u \in S$ and that, this correction being made,

$$|(A u)(0)| \leqslant C \sum_{|\alpha| \leqslant n} \| D^\alpha u \|_p.$$

Hence $(A u)(0)$ is a continuous linear form on S so that it may be written

$$(A u)(0) = T(\check{u}) = (T * u)(0),$$

where $\check{u}(x) = u(-x)$ and $T \in S'$. In view of the invariance for translation of both sides we get

$$(A u)(x) = (T * u)(x)$$

for every x, which proves the theorem since the uniqueness of T follows immediately from the proof.

If $p < \infty$, the space S is dense in L^p and the operator A is obtained as the closure of the operator $u \to T * u$. If $p = \infty$ and $q < \infty$, we have $T = 0$ in virtue of Proposition 1.1; and if $p = q = \infty$ the distribution T is obviously a bounded measure. The case $p = \infty$ therefore does not present great interest. Thus the study of the translation invariant operators is essentially equivalent to the study of the spaces L_p^q of the following definition.

DEFINITION 1.2. *The space of distributions T in S' such that*

$$\|T * u\|_q \leq C \|u\|_p, \quad u \in S, \tag{1.1.6}$$

where C is a constant, is denoted by L_p^q. The smallest constant C which can be used in (1.1.6) *will be denoted by $L_p^q(T)$.*

L_p^q is thus isomorphic to a closed subspace of the Banach space of all bounded linear mappings of L^p into L^q, hence is also a Banach space.

Let $\mathfrak{F}$ denote the Fourier transformation $u \to \hat{u}$,

$$\hat{u}(\xi) = \int e^{-2\pi i \langle x, \xi \rangle} u(x)\, dx, \quad u \in S,$$

extended to all $T \in S'$ by continuity or, equivalently, the formula

$$\hat{T}(u) = T(\hat{u}), \quad u \in S,$$

(see Schwartz [12], Chap. VII). We recall that the Fourier transformation is an isomorphism of S and of S'. Then the mapping $u \to T * u$, $u \in S$, can also be written

$$u \to \mathfrak{F}^{-1}(\hat{T}\, \mathfrak{F} u)$$

and is thus via the Fourier transformation equivalent to multiplication by $\hat{T}$.

DEFINITION 1.3. *The set of Fourier transforms $\hat{T}$ of distributions $T \in L_p^q$ is denoted by M_p^q and we write*

$$M_p^q(\hat{T}) = L_p^q(T).$$

The elements in M_p^q are called multipliers of type (p, q).

Sometimes we shall write $L_p^q{}_{(n)}$ and $M_p^q{}_{(n)}$ in order to emphasize that the number of independent variables is n.

1.2. Basic properties of M_p^q

Our first theorem in this section is very well known but we formulate it for completeness and reference.

THEOREM 1.3. *Let T be a distribution $\neq 0$. Then the set of points $(x, y) \in R^2$ such that $T \in L_{1/x}^{1/y}$ is a convex subset of the triangle*

$$0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1, \quad y \leqslant x, \tag{1.2.1}$$

which is symmetric with respect to the line $x + y = 1$. In this set $\log L_{1/x}^{1/y}(T)$ is a convex function of (x, y) with the corresponding symmetry property.

Proof. That the set in question satisfies (1.2.1) follows at once from Theorem 1.1. The symmetry is proved as follows. Let x', y' be defined by $x + x' = y + y' = 1$. Then if

$$\| T \ast u \|_{1/y} \leqslant C \| u \|_{1/x}, \quad u \in S,$$

we get from Hölder's inequality

$$|T \ast u \ast v(0)| \leqslant C \| u \|_{1/x} \| v \|_{1/y'}; \quad u, v \in S.$$

Since convolution products are associative and commutative we get from the converse of Hölder's inequality

$$\| T \ast v \|_{1/x'} \leqslant C \| v \|_{1/y'}.$$

Hence (x, y) and (y', x') belong to the set in the theorem at the same time, and $L_{1/x}^{1/y}(T) = L_{1/y'}^{1/x'}(T)$ since the role of (x, y) and (y', x') may be interchanged in the above argument.

Finally, the convexity follows from Riesz-Thorin's convexity theorem (see Riesz [11], Thorin [16]). The proof is complete.

We next list some cases where L_p^q is easy to describe precisely.

THEOREM 1.4. *We have*

$$L_p^\infty = L_1^{p'} = L^{p'}, \; p < \infty; \; L_\infty^\infty = L_1^1 = M, \tag{1.2.2}$$

with equality also of the norms.

Proof of the theorem. In view of Theorem 1.3 it is enough to prove that $L_p^\infty = L^{p'}$ and that $L_\infty^\infty = M$. The last fact is essentially the definition of bounded measures and was already observed after Theorem 1.2. The first follows from the fact that $L^{p'}$ is the dual space of L^p when $p < \infty$.

COROLLARY 1.1. *$T \in L_{1/x}^{1/y}$ for all (x, y) in the triangle* (1.2.1) *if and only if $T \in L^1 \cap L^\infty$.*

Proof. In view of Theorem 1.4 we have $T \in L_{1/x}^{1/y}$ for all three corners (x, y) of the triangle if and only if $T \in L^1 \cap L^\infty$. But the convexity property in Theorem 1.3 shows that T is then in $L_{1/x}^{1/y}$ for every (x, y) in the triangle.

COROLLARY 1.2. *Let $p \leqslant q$ and set $1/p - 1/q = 1 - 1/a$. Then we have $L^a \subset L_p^q$ and*

$$L_p^q(f) \leqslant \|f\|_a, \quad f \in L^a. \tag{1.2.3}$$

If $a = 1$, one may replace L^1 by M.

Proof. In virtue of Theorem 1.4 the corollary is true for $p = a'$, $q = \infty$ and for $p = 1$, $q = a$. Hence it follows in general from the convexity properties in Theorem 1.3.

Note that (1.2.3) means exactly the well-known inequality

$$\|f * u\|_q \leqslant \|f\|_a \|u\|_p, \quad f \in L^a, \quad u \in S.$$

We next turn to some results which are best expressed in terms of the M_p^q spaces.

THEOREM 1.5. *With equality also of the norms we have*

$$M_2^2 = L^\infty. \tag{1.2.4}$$

Proof. Let $T \in L_2^2$ so that $\hat{T} \in M_2^2$. Then $T * u \in L^2$ for all $u \in S$, hence the Fourier transform $\hat{T}\hat{u} \in L^2$, and

$$\|\hat{T}\hat{u}\|_2 = \|T * u\|_2 \leqslant L_2^2(T)\|u\|_2 = M_2^2(\hat{T})\|\hat{u}\|_2$$

for all $\hat{u} \in S$. This proves first that $\hat{T}$ is a locally square integrable function and then that $|\hat{T}(\xi)| \leqslant M_2^2(\hat{T})$ almost everywhere. On the other hand, if $|\hat{T}(\xi)| \leqslant C$ almost everywhere the same argument proves that $M_2^2(\hat{T}) \leqslant C$. Hence $M_2^2(\hat{T})$ is the essential supremum of $\hat{T}$, which proves the theorem.

COROLLARY 1.3. *For every p we have*

$$M_p^p \subset L^\infty, \tag{1.2.5}$$

$$\|f\|_\infty \leqslant M_p^p(f), \quad f \in M_p^p. \tag{1.2.6}$$

Proof. The convexity and symmetry stated in Theorem 1.3 show that $M_p^p \subset M_2^2$ and that

$$M_2^2(f) \leqslant M_p^p(f).$$

Hence the corollary follows from Theorem 1.5.

For $p \neq q$ we can only prove a weaker regularity result. By L^p_{loc} we shall mean the set of functions which belong to L^p on every compact set.

THEOREM 1.6. *The following inclusions are valid:*

$$M_p^q \subset L^p_{\text{loc}} \text{ if } p \geqslant 2; \quad M_p^q \subset L^{q'}_{\text{loc}} \text{ if } q \leqslant 2. \tag{1.2.7}$$

Proof. Since $M_p^q = M_{q'}^{p'}$ according to Theorem 1.2, it is sufficient to prove the latter half of (1.2.7). Let $T \in L_p^q$, $q \leqslant 2$. For every $u \in S$ we then have $T * u \in L^q$ and in view of the Hausdorff-Young theorem on Fourier transforms of functions in L^q, $q \leqslant 2$, (Zygmund [17]), we obtain $\hat{T}\hat{u} \in L^{q'}$ for every $u \in S$, hence for every $\hat{u} \in S$. This proves the theorem.

If $p \geqslant 2$ or $q \leqslant 2$ we thus have $M_p^q \subset L^2_{\text{loc}}$. Elements in two such M_p^q classes may thus be multiplied together pointwise, giving a locally integrable product. This gives a sense to the statement in the following theorem.

THEOREM 1.7. *Let $2 \leqslant p \leqslant q \leqslant r$ or $p \leqslant q \leqslant r \leqslant 2$. Then if $f \in M_p^q$ and $g \in M_q^r$ we have $fg \in M_p^r$ and*

$$M_p^r(fg) \leqslant M_p^q(f)\, M_q^r(g). \tag{1.2.8}$$

The translation invariant operator corresponding to fg is the product of those corresponding to g and to f.

COROLLARY 1.4. *M_p^p is for every p a normed ring with the operations of pointwise multiplication and addition.*

This result is partly given by Schwartz [13]. Note that Theorem 1.4 shows that $M_1^1 = M_\infty^\infty$ is the algebra of Fourier-Stieltjes transforms.

Proof of the theorem. In view of Theorem 1.3 it is enough to consider the case $p \leqslant q \leqslant r \leqslant 2$. Denote by A_f the closure of the mapping

$$L^p \supset S \ni u \to \mathfrak{F}^{-1}(f\hat{u}) \in L^q.$$

A_f is a bounded operator from L^p to L^q. Similarly we define a bounded operator A_g from L^q to L^r. Then we have

$$\mathfrak{F}(A_f u) = f\hat{u}, \quad u \in L^p; \quad \mathfrak{F}(A_g v) = g\hat{v}, \quad v \in L^q. \tag{1.2.9}$$

In fact, these identities are valid by definition when u and v are in S. To prove the second identity, for example, we note that the Hausdorff-Young inequality shows that the mapping $L^q \ni v \to \mathfrak{F}(A_g v) \in L^{r'}$ is continuous. Since $g \in L^2_{\text{loc}}$ the mapping

$L^q \ni v \to g\hat{v} \in L^1_{\text{loc}}$ is also continuous. S being dense in L^q, the second formula (1.2.9) follows. Taking $v = A_f u$ we thus have

$$\mathcal{F}(A_g A_f u) = f g \hat{u}, \quad u \in L^p,$$

hence in particular this is true when $u \in S$. This proves that fg is the multiplier corresponding to $A_g A_f$, hence (1.2.8) is valid. The proof is complete.

Without the restriction given on the exponents, Theorem 1.7 would not always have a sense (see Theorem 1.9). However, we can always prove a much weaker statement showing that the local smoothness of the elements in M_p^q increases with p and decreases with q. Combining this with Theorem 1.4 we could also get another proof of Theorem 1.6.

THEOREM 1.8. *If $f \in M_p^q$ and $g \in S$ we have*

$$g f \in M_r^q \text{ if } r \leqslant p; \quad g f \in M_p^s \text{ if } s \geqslant q.$$

Proof. Since $M_p^q = M_{q'}^{p'}$ it is sufficient to prove the first statement. Let $f = \hat{T}_f$, $g = \hat{T}_g$. Then $T_g \in S$ and for $u \in S$ we get

$$\| (T_f * T_g) * u \|_q = \| T_f * (T_g * u) \|_q \leqslant M_p^q(f) \| T_g * u \|_p \leqslant M_p^q(f) \, M_r^p(g) \| u \|_r.$$

Hence $T_f * T_g \in L_r^q$ so that the Fourier transform $g f \in M_r^q$ and

$$M_r^q(g f) \leqslant M_r^p(g) \, M_p^q(f).$$

Theorem 1.6 does not give any information when $p < 2 < q$. We shall now study that case starting with the following lemma.

LEMMA 1.2. *If $u \in S$ and $u_t \in S$ is defined by $\hat{u}_t(\xi) = \hat{u}(\xi) e^{it|\xi|^2}$, we have for $p > 2$ with a constant C_p*

$$\| u_t \|_p \leqslant C_p |t|^{1/p - 1/2}, \; t \in R.$$

Proof. First note that Parseval's formula gives

$$\| u_t \|_2 = \| \hat{u}_t \|_2 = \| \hat{u} \|_2 = \| u \|_2.$$

To estimate the maximum of u_t, we introduce polar coordinates,

$$u_t(x) = \int e^{i(2\pi\langle x, \xi\rangle + t|\xi|^2)} \hat{u}(\xi) \, d\xi = \int_0^\infty \int_{|\omega|=1} e^{i(2\pi\langle x, \omega\rangle r + tr^2)} \hat{u}(r\omega) \, r^{n-1} \, dr \, d\omega. \tag{1.2.10}$$

We shall integrate by parts with respect to r. Note that

$$\left|\int_0^R e^{i(2ar+tr^2)}\,dr\right| \leqslant C|t|^{-\frac{1}{2}} \tag{1.2.11}$$

for all real R, a and t. In fact, a change of variables gives that

$$\int_0^R e^{i(2ar+tr^2)}\,dr = t^{-\frac{1}{2}}\,e^{-ia^2/t}\int_b^c e^{ir^2}\,dr$$

if $t>0$, where $b=a/\sqrt{t}$ and $c=R\sqrt{t}+a/\sqrt{t}$. Since $\int_{-\infty}^{+\infty} e^{ir^2}\,dr$ is convergent as a generalized Riemann integral, (1.2.11) follows for $t>0$, hence by complex conjugation for $t<0$. Integrating by parts in (1.2.10) and using (1.2.11), we obtain

$$|u_t(x)| \leqslant C|t|^{-\frac{1}{2}}\int |\partial\hat{u}/\partial r + (n-1)\,r^{-1}\,\hat{u}|\,r^{n-1}\,dr\,d\omega = C_1|t|^{-\frac{1}{2}}.$$

From this estimate and the fact that $\|u_t\|_2$ is constant we obtain

$$\int |u_t|^p\,dx \leqslant (C_1|t|^{-\frac{1}{2}})^{p-2}\int |u_t|^2\,dx = C_p{}^p\,|t|^{1-\frac{1}{2}p}$$

which proves the lemma.

THEOREM 1.9. *If $p<2<q$ there exist elements in M_p^q which are distributions of positive order, that is, which are not measures.*

Proof. Assume that the statement were false, so that every $f\in M_p^q$ is a measure. Mapping f on the restriction to the set $\{\xi;|\xi|\leqslant 1\}$ we get a closed everywhere defined mapping from M_p^q to the space of bounded measures in the unit sphere, with the norm defined as the total variation. In virtue of the theorem on the closed graph the mapping must be continuous. In particular

$$\int_{|\xi|\leqslant 1} |f|\,d\xi \leqslant C\,M_p^q(f), \quad f\in S. \tag{1.2.11}$$

Take a function u in S so that $\hat{u}(0)\neq 0$ and define u_t as in the lemma. With f replaced by $\hat{u}_t$ the left hand side of (1.2.11) is independent of t and $\neq 0$. In virtue of Theorem 1.4 and Lemma 1.2 we obtain $L_{p'}^\infty(u_t)=L_1^p(u_t)=\|u_t\|_p\to 0$ if $p>2$, that is

$$M_{p'}^\infty(\hat{u}_t)=M_1^p(\hat{u}_t)\to 0 \text{ when } t\to\infty \text{ if } p>2.$$

Further, it follows from Theorem 1.5 that

$$M_2^2(\hat{u}_t)=\|\hat{u}\|_\infty$$

which is independent of t. The logarithmic convexity of the M_p^q norm as a function of $1/p$ and $1/q$ which is contained in Theorem 1.3 now immediately shows that

$$M_p^q(\hat{u}_t) \to 0 \text{ if } p < 2 < q,\ t \to \infty.$$

Hence we get a contradiction if f is replaced by $\hat{u}_t$ in (1.2.11), and $t \to \infty$. This proves the theorem.

In particular we get the following familiar result.

COROLLARY 1.5. *If $p > 2$ there exist functions $u \in L^p$ such that $\hat{u}$ is a distribution of positive order.*

Proof. Every element in M_1^p is the Fourier transform of a function in L^p (Theorem 1.4).

When $1 < p \leqslant 2 \leqslant q < \infty$, an important subclass of M_p^q is given by Paley's inequality:

THEOREM 1.10. *Let $\varphi \geqslant 0$ be a measurable function such that*

$$m\{\xi;\ \varphi(\xi) \geqslant s\} \leqslant C/s. \tag{1.2.13}$$

With a constant C_p depending on p and on C we then have when $1 < p \leqslant 2$

$$\left(\int |\hat{u}/\varphi|^p \varphi^2\, d\xi\right)^{1/p} \leqslant C_p \|u\|_p,\ u \in L^p. \tag{1.2.14}$$

Note that the integrand may be written $|\hat{u}|^p \varphi^{2-p}$ so that it is natural that we define it to be 0 when $\varphi = 0$.

Proof. For the sake of completeness we recall the proof, following Zygmund [18]. When $p = 2$ the inequality (1.2.14) follows with $C_2 = 1$ from Parseval's equality. Write $d\mu(\xi) = (\varphi(\xi))^2 d\xi$ and $Tu = \hat{u}/\varphi$. Note that it follows from (1.2.13) that

$$\mu\{\xi;\ \varphi(\xi) \leqslant \sigma\} \leqslant 2C\sigma. \tag{1.2.15}$$

In fact, writing $m(s) = m\{\xi;\ \varphi(\xi) \geqslant s\}$ we have

$$\mu\{\xi;\ \varphi(\xi) \leqslant \sigma\} = \int_0^\sigma s^2\, d(-m(s)) \leqslant 2\int_0^\sigma m(s)\, s\, ds + \lim_{s\to 0} s^2 m(s) \leqslant 2C\sigma,$$

since $s\,m(s) \leqslant C$ in virtue of (1.2.13). We now obtain

$$\mu\{\xi;\ |(Tu)(\xi)| \geqslant s\} \leqslant 2C\|u\|_1/s,\ u \in L^1. \tag{1.2.16}$$

In fact, since $|(Tu)(\xi)| \leqslant \|u\|_1/\varphi(\xi)$, the set in question is contained in the set where $\varphi(\xi) \leqslant \|u\|_1/s$. From the validity of (1.2.14) when $p=2$ it also follows that

$$\mu\{\xi;\ |(Tu)(\xi)|>s\} \leqslant (\|u\|_2/s)^2,\ u \in L^2. \tag{1.2.17}$$

We now only have to invoke Marcinkiewicz' interpolation theorem (Zygmund [18], Theorem 1) in order to conclude from (1.2.16) and (1.2.17) that (1.2.14) is valid.

If we combine Theorem 1.10 with the Hausdorff-Young inequality

$$\|\hat{u}\|_{p'} \leqslant \|u\|_p,\ 1 \leqslant p \leqslant 2, \tag{1.2.18}$$

and use Hölder's inequality, we obtain the following

COROLLARY 1.6. *If φ satisfies* (1.2.13) *and $1<p\leqslant r\leqslant p'<\infty$, we have*

$$\left(\int |\hat{u}\,\varphi^{(1/r-1/p')}|^r\,d\xi\right)^{1/r} \leqslant C_p\|u\|_p,\ u\in L^p. \tag{1.2.19}$$

This reduces to (1.2.18) when $r=p'$ and to (1.2.14) when $r=p$.

THEOREM 1.11. *Let f be a measurable function such that, with $1<b<\infty$, we have for some constant C*

$$m\{\xi;\ |f(\xi)|\geqslant s\} \leqslant C/s^b. \tag{1.2.20}$$

Then $f\in M_p^q$ if

$$1<p\leqslant 2\leqslant q<\infty,\ 1/p-1/q=1/b. \tag{1.2.21}$$

Proof. Since $M_p^q = M_{q'}^{p'}$ we may assume that $p\leqslant q'$, for otherwise we have $q'\leqslant(p')'=p$. With $\varphi=|f|^b$ and $r=q'$, the assumptions of Corollary 1.6 are then satisfied and since $1/q'-1/p'=1/p-1/q=1/b$ we obtain

$$\|f\hat{u}\|_{q'} \leqslant C_p\|u\|_p,\ u\in L^p.$$

Let T be the distribution with $\hat{T}=f$. When $u\in S$ the Hausdorff-Young inequality gives since $q'\leqslant 2$

$$\|T*u\|_q \leqslant \|f\hat{u}\|_{q'} \leqslant C_p\|u\|_p,$$

which proves that $T\in L_p^q$ and hence that $f\in M_p^q$. The proof is complete.

When $p\leqslant 2\leqslant q$ we can thus give bounds on the absolute value of a function f which ensure that the function is in M_p^q. That this is not possible for other values of p and q is shown by the following result.

THEOREM 1.12. *Suppose that there exists a measurable function $F\geqslant 0$ which is not 0 almost everywhere, such that every measurable function f satisfying the condition $|f|\leqslant F$ belongs to M_p^q. Then we have $p\leqslant 2\leqslant q$.*

Proof. We may assume that F is bounded. The assumption means that $Fg \in M_p^q$ if $g \in L^\infty$. Thus the mapping

$$L^\infty \ni g \to Fg \in M_p^q$$

is defined everywhere in L^∞ and it is obviously closed since it is continuous for the topology of L^∞ on the right hand side. Hence the closed graph theorem shows that the mapping is continuous, that is,

$$M_p^q(Fg) \leqslant C \|g\|_\infty.$$

In view of the definition of M_p^q this means that for all u and $v \in S$

$$\left| \int Fg\hat{u}\hat{v}\,d\xi \right| \leqslant M_p^q(Fg)\|u\|_p \|v\|_{q'} \leqslant C\|g\|_\infty \|u\|_p \|v\|_{q'}.$$

Hence
$$\int F|\hat{u}\hat{v}|\,d\xi \leqslant C\|u\|_p \|v\|_{q'}. \tag{1.2.22}$$

More generally, we get for any η

$$\int F(\xi-\eta)|\hat{u}(\xi)\hat{v}(\xi)|\,d\xi \leqslant C\|u\|_p \|v\|_{q'}, \qquad (1.2.22)'$$

if in (1.2.22) we replace $u(x)$ by $u(x)e^{-2\pi i\langle x,\eta\rangle}$ and make a similar substitution for v. If g is a continuous positive function with $\int g\,d\eta = 1$ and $G = F * g$, we get by multiplying (1.2.22)′ with $g(\eta)$ and integrating

$$\int G|\hat{u}\hat{v}|\,d\xi \leqslant C\|u\|_p \|v\|_{q'},\ u,\ v \in S. \tag{1.2.23}$$

This inequality has the advantage over (1.2.22) that G is continuous und positive everywhere. Now take v fixed with $\hat{v} \neq 0$ when $|\xi| \leqslant 1$. It then follows from (1.2.23) that

$$\int_{|\xi|\leqslant 1} |\hat{u}|\,d\xi \leqslant C'\|u\|_p,\ u \in S.$$

Thus, if we replace u by the function u_t defined in Lemma 1.2 a contradiction results unless $p \leqslant 2$. Similarly, taking u fixed we obtain $q' \leqslant 2$, that is, $2 \leqslant q$, which completes the proof.

1.3. Homomorphisms of M_p^p

We shall only study homomorphisms of M_p^p which are induced by a mapping $\xi \to a(\xi)$ of R^n into R^m. If f is a function in R^m a function $a^* f$ in R^n is defined by

$$(a^* f)(\xi) = f(a(\xi)), \quad \xi \in R^n.$$

We first consider the case where a is an affine mapping

$$(a(\xi))_j = a_{j0} + \sum_{k=1}^{n} a_{jk} \xi_k, \quad j = 1, \ldots, m.$$

THEOREM 1.13. *If a is an affine mapping of R^n onto R^m, the mapping a^* is an isometric mapping of $M_p^p{}_{(m)}$ into $M_p^p{}_{(n)}$, for every p. If $m = n$, the mapping is onto.*

Proof. The definition of the L^p spaces and hence of L_p^q was independent of the system of coordinates except that it used a particular Lebesgue measure. However, the norm of an element in L_p^p is obviously independent of the Lebesgue measure chosen. Further, if T is a distribution whose Fourier transform has a density $\hat{T}(\xi)$, this density is independent of the choice of Lebesgue measure. (This is most easily seen when T is a measure and $\hat{T}(\xi)$ the Fourier-Stieltjes transform.) Hence M_p^p is invariant for every change of coordinates.

Changing coordinates in R^n and in R^m (considered as different spaces even if $n = m$) we may assume that a is given by

$$(a(\xi))_j = \xi_j + a_{j0}, \quad j = 1, \ldots, m.$$

Let $T \in L_p^p{}_{(m)}$ and form
$$T_1 = e^{-2\pi i l} T \otimes \delta,$$

where $l(x) = x_1 a_{10} + \cdots + x_m a_{m0}$ and δ is the Dirac measure in the variables $x_{m+1}, \ldots, x_n$. The Fourier transform of T_1 as a distribution in R^n is

$$\hat{T}_1 = \hat{T}(\xi_1 + a_{10}, \cdots, \xi_m + a_{m0}),$$

so that what we have to prove is that $T_1 \in L_p^p{}_{(n)}$ and has the same norm there as T has in $L_p^p{}_{(m)}$. Now, if $u \in S_{(n)}$,

$$T_1 \ast u = e^{-2\pi i l} (T \ast (e^{2\pi i l} u)),$$

where the convolution in the right hand side is taken with respect to $x_1, \cdots, x_m$, the other variables being fixed. If $C = L_p^p(T)$ we have

$$\int |T_1 \ast u|^p \, dx_1 \cdots dx_m \leqslant C^p \int |u|^p \, dx_1 \cdots dx_m$$

for fixed $x_{m+1}, \cdots, x_n$. Integrating with respect to these variables, we get

$$\| T_1 \ast u \|_p \leqslant C \| u \|_p,$$

which proves that $T_1 \in L_p^p{}_{(n)}$ and has a norm which is at most C. That the norm cannot be smaller than C is immediately seen be considering functions u which are products of functions of $x_1, \cdots, x_m$ and a fixed function $\not\equiv 0$ of $x_{m+1}, \cdots, x_n$. Since the mapping a has an affine inverse if $n = m$, the proof is complete.

We omit the similar but less simple and useful result concerning M_p^q when $p \neq q$. In that case one can only take $m = n$.

Under certain regularity assumptions it will now be proved that the assumption in Theorem 1.13 that the mapping a is affine is necessary. The essential step in the proof is the following lemma.

LEMMA 1.3. *$e^{iA|\xi|^2}$ is not in M_p^p for any $p \neq 2$ if A is a real constant $\neq 0$.*

Proof. Suppose that $e^{iA|\xi|^2} \in M_p^p$ and that $A \neq 0$, $p \neq 2$. Since $M_p^p = M_{p'}^{p'}$ we may assume that $p > 2$, and since M_p^p is invariant for conjugation we may also assume that $A < 0$. In virtue of Theorem 1.13 applied to the mapping $\xi \to (-t/A)^{\frac{1}{2}} \xi$, the function $e^{-it|\xi|^2}$ is in M_p^p for every $t > 0$ and $M_p^p(e^{-it|\xi|^2})$ is independent of t. Hence we get when $u \in S$

$$\| u \|_p \leqslant C \| u_t \|_p \tag{1.3.1}$$

if u_t is defined as in Lemma 1.2 by the equation $e^{-it|\xi|^2} \hat{u}_t(\xi) = \hat{u}(\xi)$. When $t \to \infty$ (1.3.1) contradicts Lemma 1.2 which completes the proof.

LEMMA 1.4. *Let $A(\xi)$ be a real quadratic form. If $e^{iA} \in M_p^p$ vhere $p \neq 2$, it follows that $A = 0$.*

Proof. Assume that A does not vanish identically. With suitable coordinates we may write

$$A(\xi) = a_1 \xi_1^2 + \cdots + a_n \xi_n^2$$

where $a_1 \neq 0$. In view of Theorem 1.13 we may even assume that

$$|a_1| > |a_2| + \cdots + |a_n|. \tag{1.3.2}$$

If k is a permutation $(k_1, \cdots, k_n)$ of the integers $1, \cdots, n$ we write

$$A_k(\xi) = a_1 \xi_{k_1}{}^2 + \cdots + a_n \xi_{k_n}{}^2.$$

Since $e^{iA} \in M_p^p$ we have also $e^{iA_k} \in M_p^p$ in virtue of Theorem 1.13. Hence Corollary 1.4 shows that

$$\prod_k e^{iA_k} \in M_p^p.$$

Now $\sum A_k = (n-1)!\ (a_1 + \cdots + a_n)\,|\xi|^2 = a\,|\xi|^2$ where $a \neq 0$ in view of (1.3.2). This contradicts Lemma 1.3 and hence proves the lemma.

We are now able to prove the main theorems in this section.

THEOREM 1.14. *Let f be a real valued function $\in C^2$. Suppose that there exists a sequence t_k of real numbers such that $t_k \to +\infty$ and $e^{it_k f} \in M_p^p$,*

$$M_p^p\,(e^{it_k f}) < C, \quad k = 1, \cdots \tag{1.3.3}$$

where C is a constant and $p \neq 2$. Then f is a linear function.

On the other hand, if $f(\xi) = a + 2\pi\langle h, \xi\rangle$ is a real linear function then e^{itf} is the Fourier transform of the mass e^{ita} at $-th$, hence $M_p^p\,(e^{itf}) = 1$ for every p. (This follows for $p = 1,\ 2$ and ∞ from Theorems 1.4 and 1.5, and then in general from the convexity in Theorem 1.3.)

Proof of Theorem 1.14. We shall prove that the second derivatives of f vanish. It is sufficient to do so for $\xi = 0$, for every translation of f also satisfies (1.3.3) in view of Theorem 1.13. Since $f \in C^2$ we have

$$f(\xi) = a + \langle h, \xi\rangle + A(\xi) + o(|\xi|^2), \quad \xi \to 0,$$

where a is a real number, h a real vector, A a real quadratic form. Write

$$g(\xi) = f(\xi) - a - \langle h, \xi\rangle.$$

It follows from Corollary 1.4 and the remark above after Theorem 1.14 that g satisfies the same assumptions in the theorem as f does. But now we have $g(\xi) = A(\xi) + o(|\xi|^2)$, and writing $g_k(\xi) = t_k\, g(\xi/t_k^{\frac{1}{2}})$ it thus follows that

$$g_k(\xi) \to A(\xi),$$

uniformly on every compact set. It follows from (1.3.3) and Theorem 1.13 that

$$M_p^p\,(e^{ig_k}) \leqslant C.$$

From the following lemma it follows that $e^{iA} \in M_p^p$. Hence $A = 0$ in view of Lemma 1.4, which completes the proof.

LEMMA 1.5. *The unit spheres in M_p^q and in L_p^q are closed in S'.*

Proof. Only the statement concerning L_p^q needs to be proved, for the Fourier transformation is an isomorphism of S' mapping the unit sphere in L_p^q onto that in M_p^q. Now the unit sphere in L_p^q is by definition the set $\{T;\ T \in S'$ and $|T * u * v\,(0)| \leqslant \|u\|_p \|v\|_{q'};\ u, v \in S\}$, and since the left hand side of the inequality is the absolute value of a continous linear form on S', the assertion is obvious.

THEOREM 1.15. *Let a be a C^2 mapping of R^n into R^m. Assume that a^* maps $M_p^p{}_{(m)}$ into $M_p^p{}_{(n)}$ and that $p \neq 2$. Then a is affine and onto.*

Proof. Since the mapping a^* is obviously closed, it follows from the theorem on the closed graph that a^* maps $M_p^p{}_{(m)}$ continuously into $M_p^p{}_{(n)}$. If l is a linear function in R^m, the norm of e^{itl} in $M_p^p{}_{(m)}$ is 1 for every t. In view of the continuity of a^*, it follows that the norm of e^{ita^*l} in $M_p^p{}_{(n)}$ is bounded for all t. Hence Theorem 1.14 shows that a^*l is a linear function. Applying this with l equal to the jth coordinate in R^m, it follows that $(a(\xi))_j$ is a linear function of $\xi \in R^n$ for $j = 1, \cdots, m$. This proves the theorem. For if a were not onto, its range were a null set and every function would be in $M_p^p{}_{(n)}$.

For $p = 1$, that is, for the algebra of Fourier-Stieltjes transforms, a much more precise result has been given by Beurling and Helson [1] (see also Helson [4]). In particular, it is not necessary in that case to assume that $a \in C^2$. (These authors also treat more general homomorphisms. However, the proof of Theorem 1.14 immediately extends to that case if a smoothness assumption replacing the assumption $a \in C^2$ is made.) For $p \neq 1$ and ∞, however, some smoothness assumption is needed in Theorem 1.14. It may be sufficient to assume that $a \in C^1$ but not merely that a is Lipschitz continuous. In fact, using Riesz' theorem on conjugate functions (see Chapter II), Corollary 1.4 and Theorem 1.13 it is easily seen that if a is pieceweise linear (and has only a finite number of pieces) then a^* maps M_p^p into itself, if $1 < p < \infty$. For further details see Schwartz [13].

1.4. Analytic operations in M_p^p

Our purpose here is to prove an analogue of the Wiener-Lévy theorem concerning M_1^1, or rather the subspace of M_1^1 consisting of the Fourier transforms of functions in L^1. This subset of M_1^1 can also be regarded as the closure of S in M_1^1 and we are thus led to introduce the following definition.

DEFINITION 1.4. *The closure of S in M_p^p will be denoted by m_p^p.*

It is clear that m_p^p is also a normed ring. Since $M_p^p(f) \geq \|f\|_\infty$ according to (1.2.6), it follows that $m_p^p \subset C \cap L_0^\infty$. On the other hand, we can prove an opposite result which is only slightly weaker.

THEOREM 1.16. *If $|1/q - 1/2| < |1/p - 1/2|$ we have $M_p^p \cap C \cap L_0^\infty \subset m_q^q$.*

Remark. If $p = 1$ the result is not valid with $q = 1$ since there are singular measures with Fourier transforms converging to 0 at infinity. We do not know if it is possible to take $q = p$ for some other value of $p \neq 2$.

Proof of the theorem. Let $f \in M_p^p \cap C$ and assume first that f has compact support. Take a non negative function $\varphi \in C_0^\infty$ such that $\int \varphi \, d\xi = 1$ and form

$$f_\varepsilon(\xi) = \int f(\xi - \varepsilon\eta)\,\varphi(\eta)\,d\eta.$$

As is well known, $f_\varepsilon \in C_0^\infty$ and converges to f uniformly when $\varepsilon \to 0$, hence $M_2^2(f - f_\varepsilon) \to 0$ as $\varepsilon \to 0$ (Theorem 1.5). The convexity of the norm in M_p^p and Theorem 1.13 also give that $M_p^p(f_\varepsilon) \leqslant M_p^p(f)$, hence $M_p^p(f - f_\varepsilon) \leqslant 2\,M_p^p(f)$. Replacing if necessary q by q' we may assume that $1/q = \alpha/p + (1-\alpha)/2$ where $0 \leqslant \alpha < 1$. Hence the logarithmic convexity of the M_q^q norm as function of $1/q$ (Theorem 1.3) shows that

$$M_q^q(f - f_\varepsilon) \leqslant (M_p^p(f - f_\varepsilon))^\alpha (M_2^2(f - f_\varepsilon))^{1-\alpha} \to 0 \text{ as } \varepsilon \to 0,$$

which proves that $f \in m_q^q$. Next let f be an arbitrary function in $M_p^p \cap C \cap L_0^\infty$. Let $\psi \in C_0^\infty$ be equal to 1 when $|\xi| \leqslant 1$ and set $f_\varepsilon(\xi) = f(\xi)\,\psi(\varepsilon\xi)$. Since M_p^p is an algebra containing S we get $f_\varepsilon \in M_p^p$, and from what we have already proved it thus follows that $f_\varepsilon \in m_q^q$. Since $M_p^p(f_\varepsilon - f) \leqslant M_p^p(f)\,(1 + M_p^p(\psi))$ and

$$M_2^2(f - f_\varepsilon) \leqslant (1 + \sup|\psi|) \sup_{\varepsilon|\xi|>1} |f| \to 0 \text{ as } \varepsilon \to 0,$$

it follows again from the logarithmic convexity of the M_q^q norm as a function of $1/q$ that $M_q^q(f - f_\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence $f \in m_q^q$.

THEOREM 1.17. *The maximal ideal space of the algebra m_p^p can be identified with R^n; the characters are the mappings $f \to f(\xi)$, $\xi \in R^n$.*

Proof. The restriction of a continuous character in m_p^p to m_1^1 is a continuous character in m_1^1, hence of the form $f \to f(\xi)$ since m_1^1 is the Fourier transform of L^1. In view of the definition of m_p^p, the set S and a fortiori m_1^1 is dense in m_p^p. Hence all continuous characters on m_p^p are of the form $f \to f(\xi)$. Since $S \subset m_p^p \subset C$ it is obvious that the topology of the space of maximal ideals is the usual topology in R^n.

Remark. It is not known to the author whether R^n is the maximal ideal space of $M_p^p \cap C$ for some $p \neq 2$. That this is not true for $p = 1$ is well known. (Cf. Šreider [14].)

From Theorem 1.17 and the basic results on commutative Banach algebras (see Loomis [6], pp. 78 and 79), we obtain the following theorem.

THEOREM 1.18. *If $f \in m_p^p$ and Φ is analytic in a neighbourhood of the closed range of f and $\Phi(0) = 0$, then $\Phi(f) \in m_p^p$.*

Combination of Theorems 1.16 and 1.18 also gives

THEOREM 1.19. *If $f \in M_p^p \cap C \cap L_0^\infty$ and Φ is analytic in a neighbourhood of the closed range of f and $\Phi(0)=0$, then $\Phi(f) \in M_q^q$ if $|1/q-1/2|<|1/p-1/2|$.*

For another subalgebra of M_p^p we shall in Chapter II discuss similar results, which are closely related to some theorems of Calderón and Zygmund [3].

CHAPTER II

Estimates for some special operators

2.1. Main theorem

Corollary 1.2 shows that $T \in L^a$ (or $T \in M$ if $a=1$) implies $T \in L_p^q$ if $1 \leqslant p \leqslant q \leqslant \infty$ and

$$1/p-1/q=1-1/a. \tag{2.1.1}$$

Theorem 1.4 shows that these conditions on T are also necessary in order that $T \in L_p^q$ for all p, q satisfying (2.1.1) and $1 \leqslant p \leqslant q \leqslant \infty$. The purpose here is to show that if the condition $T \in L^a$ (or M) is slightly weakened we still have $T \in L_p^p$ if (2.1.1) is fulfilled and $1<p \leqslant q<\infty$.

Let k be a locally integrable function. If $k \in L_p^q$ and we set with $t>0$

$$k_t^{(a)}(x)=t^{-n/a}\, k(x/t), \tag{2.1.2}$$

we have also $k_t^{(a)} \in L_p^q$ and, if (2.1.1) holds,

$$L_p^q(k_t^{(a)})=L_p^q(k). \tag{2.1.3}$$

This follows from Theorem 1.10 when $p=q$ and in fact by a trivial computation for all p and q. It is therefore natural that we now introduce a condition involving the family of functions $k_t^{(a)}$.

DEFINITION 2.1. *We shall say that the locally integrable function k is almost in L^a and write $k \in K^a$ if there is a compact set M, a neighbourhood N of 0 and a constant C such that*

$$\left(\int_{\complement M} |k_t^{(a)}(x-y)-k_t^{(a)}(x)|^a\, dx\right)^{1/a} \leqslant C, \quad y \in N,\ 0<t. \tag{2.1.4}$$

Remark. When $a=1$ it would have been enough to assume that k is a measure and that the analogue of (2.1.4) is valid. However, we do not consider this simple generalization in order not to complicate the notations.

Examples of functions in K^a will be given in the next section. The main result we shall prove is

THEOREM 2.1. *Let $k \in K^a$. Then $k \in L_p^q$ either for all p and q satisfying* (2.1.1) *with $1 < p \leqslant q < \infty$ or else for no such p and q.*

In the applications we shall use Theorem 1.5 or 1.11 to prove that $k \in L_p^q$ for some p and q.

We shall prepare the proof of Theorem 2.1 by rewriting the property (2.1.4) in a more useful form. Let $u \in L^1$ vanish outside N and form the convolution

$$(k_t^{(a)} * u)(x) = \int k_t^{(a)}(x-y)\, u(y)\, dy \tag{2.1.5}$$

which exists almost everywhere (and is the density of the convolution in the distribution sense.) If

$$\int u\, dx = 0 \tag{2.1.6}$$

we can also write

$$(k_t^{(a)} * u)(x) = \int (k_t^{(a)}(x-y) - k_t^{(a)}(x))\, u(y)\, dy. \tag*{(2.1.5)$'$}$$

Using Minkowski's inequality for integrals and (2.1.4) we thus obtain

$$\left(\int_{\complement M} |k_t^{(a)} * u|^a\, dx\right)^{1/a} \leqslant C \int |u|\, dy. \tag{2.1.7}$$

That (2.1.7) is practically equivalent to (2.1.4) is seen by letting u in (2.1.7) converge to the difference between the Dirac measures at $y \in N$ and 0. (2.1.4) then follows with C replaced by $2C$.

Let I_0 be a cube $\subset N$ with centre at 0 and let I_0^* be another cube with centre at 0 containing M. If I is an arbitrary cube we denote by I^* the cube with the same centre such that $m(I^*)/m(I) = m(I_0^*)/m(I_0) = \gamma$. (By a cube we always mean a cube with edges parallel to the coordinate axes.) When $I = I_0$ it then follows from (2.1.7) that

$$\left(\int_{\complement I^*} |k_t^{(a)} * u|^a\, dx\right)^{1/a} \leqslant C \int |u|\, dy \quad \text{if} \int u\, dx = 0 \text{ and } u = 0 \text{ outside } I. \tag{2.1.8}$$

This inequality is in fact valid for every cube I. For since (2.1.8) is invariant for translation we may assume that I has its centre at 0 so that $I = s^{-1} I_0$ for some s.

If u vanishes outside I it follows that $v(x)=s^n u(sx)$ vanishes outside I_0 and an easy computation gives that

$$(k_t^{(a)}*v)(x)=s^{n/a}(k_{st}^{(a)}*u)(sx).$$

Applying (2.1.8) with u replaced by v and I by I_0, and substituting x for sx we thus obtain (2.1.8) with t replaced by st. Since t is arbitrary, this proves (2.1.8) for all t. In particular, for $t=1$ we obtain the following lemma.

LEMMA 2.1. *Let $k \in K^a$. Then with the same constant C as in Definition 2.1, we have for every cube I with I^* defined as above*

$$\left(\int_{\complement I^*} |k*u|^a\,dx\right)^{1/a} \leqslant C\int_I |u|\,dx \quad \textit{if} \int u\,dx=0 \textit{ and } u=0 \textit{ outside } I. \tag{2.1.9}$$

For the proof of Theorem 2.1 we also need a fundamental "covering lemma" due to Calderón and Zygmund [2] (see also Zygmund [18] and Stein [15]). We give it a slightly different form.

LEMMA 2.2. *Let $u \in L^1$ and let s be a number >0. Then we can write*

$$u=v+\sum_1^\infty w_k, \tag{2.1.10}$$

where v and all $w_k \in L^1$,

$$\|v\|_1+\sum_1^\infty \|w_k\|_1 \leqslant 3\|u\|_1, \tag{2.1.11}$$

$$|v(x)| \leqslant 2^n s \textit{ almost everywhere}, \tag{2.1.12}$$

and for certain disjoint cubes I_k

$$\int w_k\,dx=0, \textit{ and } w_k(x)=0 \textit{ if } x \notin I_k, \tag{2.1.13}$$

$$\sum_1^\infty m(I_k) \leqslant s^{-1}\int |u|\,dx. \tag{2.1.14}$$

If u has compact support, the supports of v and all w_k are contained in a fixed compact set.

Proof. Divide the whole space R^n into a mesh of cubes of volume $>s^{-1}\int |u|\,dx$. The mean value of $|u|$ over every cube is thus $<s$. Divide each cube into 2^n equal cubes and let $I_{11}, I_{12}, I_{13}, \ldots$ be those (open) cubes so obtained over which the mean value of $|u|$ is $\geqslant s$. We have

$$s\,m\,(I_{1k}) \leqslant \int_{I_{1k}} |u|\,dx < 2^n s\,m\,(I_{1k}). \tag{2.1.15}$$

For if I_{1k} was obtained by subdivision of the cube I', the construction gives

$$s\,m\,(I_{1k}) \leqslant \int_{I_{1k}} |u|\,dx \leqslant \int_{I'} |u|\,dx < s\,m\,(I') = 2^n s\,m\,(I_{1k}).$$

We set $$v\,(x) = \frac{1}{m\,(I_{1k})} \int_{I_{1k}} u\,dy, \quad x \in I_{1k}; \quad w_{1k}\,(x) = \begin{cases} u\,(x) - v\,(x), & x \in I_{1k} \\ 0 & , x \notin I_{1k} \end{cases}. \tag{2.1.16}$$

Next we make a new subdivision of the cubes which are not among the cubes I_{1k}, select those new cubes $I_{21}, I_{22}, \ldots$ over which the mean value of $|u|$ is $\geqslant s$, and extend the definitions (2.1.16) to these cubes. Continuing in this way be obtain disjoint cubes I_{jk} and functions w_{jk}; for convenience in notations we rearrange them as a sequence. If the definition of v is completed by setting $v\,(x) = u\,(x)$ when $x \notin O = \bigcup I_k$, it is clear that (2.1.10) holds. To prove (2.1.11) we first note that

$$\int_{I_k} (|v| + |w_k|)\,dx \leqslant 3 \int_{I_k} |u|\,dx.$$

Since the cubes are disjoint, w_k vanishes outside I_k and $\int_{\complement O} |v|\,dx = \int_{\complement O} |u|\,dx$, we immediately get (2.1.11). Further (2.1.12) follows from (2.1.15) if $x \in O$. On the other hand, if $x \notin O$, there are arbitrarily small cubes containing x over which to mean value of $|u|$ is $< s$. Hence $|u\,(x)| \leqslant s$ at every Lebesgue point in $\complement O$, that is, almost everywhere. (2.1.13) follows from the construction. To prove (2.1.14) we only note that since the cubes I_k are disjoint we get by adding the inequalities (2.1.15)

$$s \sum m\,(I_k) \leqslant \int_O |u|\,dx.$$

The proof is complete.

We now prove an estimate for the case $p = 1$, $q = a$, which is then a substitute for Theorem 2.1. Using this result it will be easy to prove Theorem 2.1.

THEOREM 2.2. *Let $k \in K^a$ and assume that $k \in L_p^q$ for some p and q satisfying (2.1.1) with $1 < p \leqslant q < \infty$. Then we have, when u has compact support and $u \in L^1$,*

$$m\,\{x;\ |k \ast u\,(x)| > \sigma\} \leqslant C_1\,(\|u\|_1/\sigma)^a, \quad \sigma > 0, \tag{2.1.17}$$

where m denotes Lebesgue measure and C_1 a constant.

Proof. We may assume in the proof that $\|u\|_1 = 1$. To simplify the notations we write $\tilde{u}(x) = k * u(x)$, which exists almost everywhere as an absolutely convergent integral. Form the decomposition of u given by Lemma 2.2. Then we have

$$|\tilde{u}(x)| \leq |\tilde{v}(x)| + \sum_1^\infty |\tilde{w}_k(x)|, \tag{2.1.18}$$

for every x such that $\int |k(x-y)| \, (|v(y)| + \sum |w_k(y)|) \, dy < \infty$, hence almost everywhere. In virtue of Lemma 2.1 we have

$$\left(\int_{\complement I_k^*} |\tilde{w}_k|^a \, dx \right)^{1/a} \leq C \, \|w_k\|_1. \tag{2.1.19}$$

and if $O = \bigcup I_k^*$ it follows from (2.1.14) that

$$m(O) \leq \gamma s^{-1} \|u\|_1 = \gamma s^{-1}.$$

If we restrict the integration in the left hand side of (2.1.19) to $\complement O$ and use Minkowski's inequality, we get writing $\tilde{w} = \sum |\tilde{w}_k|$

$$\left(\int_{\complement O} \tilde{w}^a \, dx \right)^{1/a} \leq C \sum \|w_k\|_1 \leq 3 C \|u\|_1 = 3 C.$$

Hence the measure of the set of points in $\complement O$ where $\tilde{w}(x) \geq \frac{1}{2}\sigma$ is at most $(6 C/\sigma)^a$. Choosing $s = \sigma^a$ we thus have $\tilde{w}(x) < \frac{1}{2}\sigma$ except in a set of measure at most $(\gamma + (6 C)^a)/\sigma^a$.

Now the assumption that $k \in L_p^q$ for some p and q means that

$$\|k * u\|_q \leq C' \|u\|_p, \quad u \in S. \tag{2.1.20}$$

S is dense in L^p since $p < \infty$. Hence (2.1.20) follows for every $u \in L^p$ with compact support (with the convolution defined in the distribution sense, which however is well known to be equivalent to the classical sense since k is locally integrable). In particular, (2.1.20) may be applied to v which gives

$$\|\tilde{v}\|_q \leq C' \|v\|_p \leq (2^n s)^{1-1/p} \|v\|_1^{1/p} \leq C'' s^{(1/a - 1/q)} = C'' \sigma^{1 - a/q} \tag{2.1.21}$$

in view of (2.1.11), (2.1.12) and (2.1.1). Hence the measure of the set where $|\tilde{v}| > \frac{1}{2}\sigma$ is at most $(2 C'')^q \sigma^{-a}$. Since (2.1.18) shows that the set where $|\tilde{u}| > \sigma$ is contained in the union of the set where $|\tilde{v}| > \frac{1}{2}\sigma$ and that where $\tilde{w} > \frac{1}{2}\sigma$, the inequality (2.1.17) follows.

Proof of Theorem 2.1. Let $k \in K^a$ and $k \in L_{p_0}^{q_0}$ where p_0 and q_0 satisfy (2.1.1). Then it follows from Theorem 2.2 that (2.1.17) holds. But this means that Marcinkiewicz' interpolation theorem (Zygmund [18], Theorem 1) can be applied, and we obtain if $1 < p < p_0$ and q is defined by (2.1.1)

$$\|k * u\|_q \leqslant C \|u\|_p \text{ if } u \in L^p \text{ and } u \text{ has compact support.} \tag{2.1.22}$$

In particular, this holds when $u \in C_0^\infty$ and since C_0^∞ is dense in S for the L^p norm we obtain (2.1.22) for $u \in S$. Hence $k \in L_p^q$. To remove the restriction $p < p_0$ we only have to use Theorem 1.3. The proof is complete.

Remark. It is important in the applications that the proof gives an estimate of $L_p^q(k)$ which only depends on $p, q, p_0, q_0, L_{p_0}^{q_0}(k)$, the constants C and γ connected with (2.1.4) and the dimension n. This fact is often useful in estimating $L_p^q(k)$ even when $k \in L^a$. (See for example the proof of Theorem 2.5 below.)

2.2. Applications

Our first example is that of Calderón and Zygmund [2], where the methods of section 2.1 were originally introduced. Thus k is a locally integrable function satisfying

$$k(x) = 0 \text{ if } |x| < 1, \quad k(tx) = t^{-n} k(x) \text{ if } t \geqslant 1, \ |x| \geqslant 1. \tag{2.2.1}$$

Assume further that $k \in K^1$. (In virtue of the footnote on p. 95 in Calderón and Zygmund [2], this follows if k satisfies a Dini condition when $|x| = 1$.) We have to examine when the Fourier transform $\hat{k}$ is in L^∞ so that $k \in L_2^2$. First note that if $y \in N$ we have $k(x-y) - k(x) \in L^1$ as a function of x, since $k \in K^1$. Hence the Fourier transform $(e^{-2\pi i \langle y, \xi \rangle} - 1)\hat{k}$ is continuos. Since this is true for all y in the neigbourhood N of 0 it follows that $\hat{k}$ is a continuous function for $\xi \neq 0$ and bounded when $\xi \to \infty$. It remains to study the behaviour of $\hat{k}(\xi)$ as $\xi \to 0$. Noting that the Fourier transform of $\hat{k}_t^{(1)}$ is $\hat{k}(t\xi)$, we obtain

$$\hat{k}(t\xi) - \hat{k}(\xi) = \int_{|\omega|=1} \int_t^1 e^{-2\pi i \langle r\omega, \xi \rangle} k(\omega)\, d\omega\, dr/r.$$

Hence
$$\lim_{\xi \to 0} (\hat{k}(t\xi) - \hat{k}(\xi)) = -\log t \int k(\omega)\, d\omega.$$

Letting $t \to 0$ we find that if $\hat{k}$ is a bounded function we must have

$$\int_{|\omega|=1} k(\omega)\, d\omega = 0. \tag{2.2.2}$$

Conversely, if this condition is fulfilled, one can write

$$\hat{k}(t\xi) - \hat{k}(\xi) = \int\limits_{|\omega|=1} \int\limits_{t}^{1} (e^{-2\pi i r \langle \omega, \xi \rangle} - 1)\, k(\omega)\, d\omega\, dr/r.$$

If we take ξ as a unit vector and $0 < t \leqslant 1$, the right hand side is bounded by $2\pi \int |k(\omega)|\, d\omega$, which proves the boundedness of $\hat{k}$ for $\xi \neq 0$. Thus $\hat{k}$ is the sum of a bounded function and—possibly—a distribution with support at 0, that is, a linear combination of the Dirac measure at 0 and its derivatives. A component of that form is impossible, however. To see this it is sufficient to show that if $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$, where $\varphi \in C_0^\infty$, it follows that $\hat{k}(\varphi_\varepsilon) \to 0$ as $\varepsilon \to 0$. Now we have

$$\hat{k}(\varphi_\varepsilon) = k(\hat{\varphi}_\varepsilon) = \int k(x)\, \hat{\varphi}(\varepsilon x)\, \varepsilon^n\, dx.$$

Using the facts that $|\hat{\varphi}(y)| \leqslant C/(|y|+1)$ since $\hat{\varphi} \in S$ and that $\int |k(r\omega)|\, d\omega \leqslant C/r^n$, we get by introducing polar coordinates and computing the integral

$$|\hat{k}(\varphi_\varepsilon)| \leqslant C\, \varepsilon^n \log(1 + 1/\varepsilon),$$

and this tends to 0 as $\varepsilon \to 0$. Hence the following theorem follows from Theorem 2.1.

THEOREM 2.3. *If k is in K^1 and satisfies* (2.2.1), *it follows that $k \in L_p^p$ for $1 < p < \infty$ if* (2.2.2) *is fulfilled whereas k is not in L_p^p for any p if* (2.2.2) *is not valid.*

We briefly recall the consequences of this result for the singular integrals corresponding to k. First note that it follows from (2.2.2) that $k_0^{(1)} = \lim k_t^{(1)}$ exists in the topology of S'. In fact, if $u \in S$ we have

$$k_t^{(1)}(u) = \int\limits_{t}^{\infty} \int\limits_{|\omega|=1} u(r\omega)\, k(\omega)\, d\omega\, dr/r,$$

and in view of (2.2.2)

$$\int u(r\omega)\, k(\omega)\, d\omega = \int (u(r\omega) - u(0))\, k(\omega)\, d\omega = O(r)$$

as $r \to 0$. Hence the integral

$$k_0^{(1)}(u) = \int\limits_{0}^{\infty} dr/r \int u(r\omega)\, k(\omega)\, d\omega$$

exists and is the limit of $k_t^{(1)}$ as $t \to 0$. It is clear that $k_0^{(1)}$ is a distribution in S'. Since $\| k_t^{(1)} * u \|_p \leqslant C_p \| u \|_p$ if $u \in S$, $t > 0$ and $1 < p < \infty$, the same estimate for $t = 0$ follows in the limit. (Cf. Lemma 1.5.) Hence we obtain

COROLLARY 2.1. *Let $k \in K^1$ and let* (2.2.1), (2.2.2) *be valid. Then* $\lim_{t \to 0} k_t^{(1)}$ *exists in S' and is an element of L_p^p, $1 < p < \infty$.*

Next consider the potential kernel

$$k(x) = |x|^{-n/a}, \quad 1 < a < \infty.$$

k is locally integrable and $\in K^a$. In fact, $k_t^{(a)} = k$ so all we have to estimate is

$$\left(\int_{\complement M} |k(x-y) - k(x)|^a \, dx \right)^{1/a}, \quad y \in N,$$

where N is the sphere $|y| < 1$ and M the set $|x| \leqslant 2$. The mean value theorem gives the estimate

$$|k(x-y) - k(x)| \leqslant (|y| \, n/a)/(|x| - 1)^{1+n/a}.$$

Since the function $1/(|x|-1)^{n+a}$ is integrable over $\complement M$, it follows that $k \in K^a$. Further, it is well known that the Fourier transform of $|x|^{-n/a}$ is $C_a |\xi|^{-n/a'}$ where C_a is a constant. (This may be proved as follows: k is in L^2 in a neighbourhood of infinity if $a < 2$, hence $\hat{k}$ is a function. Considerations of orthogonal invariance and homogeneity immediately show that $\hat{k}$ is proportional to $|\xi|^{-n/a'}$. For $a > 2$ the same result now follows from Fourier's inversion formula and for $a = 2$ it is obtained as a limiting case.) Using Theorem 1.11 and Theorem 2.1 we obtain that $k \in L_p^q$ if (2.1.1) is valid. Hence

THEOREM 2.4. *If k is a locally integrable function such that*

$$|k(x)| \leqslant C |x|^{-n/a}, \quad a > 1, \tag{2.2.3}$$

then $k \in L_p^q$ if $1 < p < q < \infty$ and (2.1.1) *is valid.*

The proof we have given is essentially the same as that in Zygmund [18]. The theorem itself is due to Hardy and Littlewood when $n = 1$ and to Sobolev when $n > 1$. (See the references in Zygmund [18].)

The next application is an improvement of a result proved by Mihlin [8], [9].

THEOREM 2.5. *Let $f \in L^\infty$ and assume that*

$$\int_{\frac{1}{2}R \leqslant |\xi| \leqslant 2R} |R^{|\alpha|} D_\alpha f|^2 \, d\xi / R^n \leqslant B^2, \; 0 < R < \infty, \; |\alpha| \leqslant \varkappa, \tag{2.2.4}$$

where B is a constant and $\varkappa$ is the least integer $> \frac{1}{2} n$. Then it follows that $f \in M_p^p$, $1 < p < \infty$.

Remark. Mihlin's hypotheses are of the form (2.2.4) with maximum norms instead of L^2 norms and involve derivatives up to the order n.

In the proof of Theorem 2.5 we need a simple lemma.

LEMMA 2.3. *There is a function $\varphi \in C_0^\infty$ with support in the set $\frac{1}{2} < |\xi| < 2$ such that*

$$\sum_{-\infty}^{+\infty} \varphi(2^{-j}\xi) = 1, \quad \xi \neq 0. \tag{2.2.5}$$

Proof. Let $\Phi \geqslant 0$ be a function in C_0^∞ with support in the set $\frac{1}{2} < |\xi| < 2$ and let $\Phi(\xi) > 0$ when $1/\sqrt{2} \leqslant |\xi| \leqslant \sqrt{2}$. Set

$$\varphi(\xi) = \Phi(\xi) / \sum_{-\infty}^{+\infty} \Phi(2^{-j}\xi).$$

Since the denominator is never 0 for $\xi \neq 0$ it follows that $\varphi \in C_0^\infty$, and (2.2.5) follows immediately.

Proof of Theorem 2.5. We shall apply Theorem 2.2 to the inverse Fourier transform of f. To do so we first decompose f into a sum $\sum f_j$ by setting

$$f_j(\xi) = f(\xi)\, \varphi(2^{-j}\xi). \tag{2.2.6}$$

The support of f_j belongs to the spherical shell $2^{j-1} < |\xi| < 2^{j+1}$. Leibniz' formula gives

$$D_\alpha f_j(\xi) = \sum_{\beta+\gamma=\alpha} 2^{-j|\beta|} (D_\gamma f(\xi))\, ((D_\beta \varphi)(2^{-j}\xi)).$$

Using (2.2.4) with $R = 2^j$ and the fact that the derivatives of φ are bounded, we obtain

$$\int \sum_{|\alpha| \leqslant \varkappa} |2^{j|\alpha|} D_\alpha f_j|^2 \, d\xi / 2^{nj} \leqslant C B^2. \tag{2.2.7}$$

(In the whole proof C will denote constants depending only on n but C may have different values in different formulas.)

Let g_j be the inverse Fourier transform of f_j,

$$g_j(x) = \int e^{2\pi i \langle x, \xi \rangle} f_j(\xi)\, d\xi.$$

Parseval's formula gives in view of (2.2.7)

$$\int (1 + 2^{2j}|x|^2)^\varkappa |g_j|^2 \, dx \leqslant C B^2\, 2^{nj}, \tag{2.2.8}$$

and hence it follows from Cauchy-Schwarz' inequality that

$$\int |g_j|\,dx \leqslant CB\left(2^{nj}\int dx/(1+2^{2j}|x|^2)^{\varkappa}\right)^{\frac{1}{2}} = C'B. \tag{2.2.9}$$

(The integral is convergent since $2\varkappa > n$.) Note that this also shows that $|f_j| = |\hat{g}_j|$ is $\leqslant C'B$ almost everywhere, hence that $|f| = |\sum f_j| \leqslant 2\,C'B$ since at most two f_j can be $\neq 0$ at any point. Application of Cauchy-Schwarz' inequality also gives, if we drop the term 1 in the brackets on the left hand side in (2.2.8),

$$\int_{|x|\geqslant t} |g_j|\,dx \leqslant CB\,(2^j t)^{(\frac{1}{2}n-\varkappa)}. \tag{2.2.10}$$

Write
$$F_N = \sum_{-N}^{N} f_j, \quad G_N = \sum_{-N}^{N} g_j. \tag{2.2.11}$$

We then have $|F_N| \leqslant 2\,C'B$, hence

$$L_2^2(G_N) = M_2^2(F_N) \leqslant 2\,C'B. \tag{2.2.12}$$

Further, we shall estimate

$$\int_{|x|\geqslant 2t} |G_N(x-y) - G_N(x)|\,dx, \quad |y| \leqslant t.$$

To do so we first note that (2.2.10) gives

$$\int_{|x|\geqslant 2t} |g_j(x-y) - g_j(x)|\,dx \leqslant CB\,(2^j t)^{(\frac{1}{2}n-\varkappa)},$$

which is a good estimate when $2^j t \geqslant 1$, the exponent being negative. Further, (2.2.9) and Bernstein's inequality give

$$\int |g_j(x-y) - g_j(x)|\,dx \leqslant CB\,2^{j+1}t, \quad |y| \leqslant t, \tag{2.2.13}$$

since the spectrum of g_j is contained in the sphere with radius 2^{j+1}. (It is also easy to obtain (2.2.13) by a direct estimation without using Bernstein's inequality.) Hence, when $|y| \leqslant t$,

$$\int_{|x|\geqslant 2t} |G_N(x-y) - G_N(x)|\,dx \leqslant CB \sum_{-\infty}^{+\infty} \min\,(2^j t,\ (2^j t)^{(\frac{1}{2}n-\varkappa)}) \tag{2.2.14}$$

and since the sum is obviously a bounded function of t we get

$$\int_{|x| \geq 2t} |G_N(x-y) - G_N(x)| \, dx \leq CB, \quad |y| \leq t, \tag{2.2.15}$$

which is equivalent to

$$\int_{|x| \geq 2} |G_{N_t}^{(1)}(x-y) - G_{N_t}^{(1)}(x)| \, dx \leq CB, \quad |y| \leq 1. \tag{2.2.15)'$$

From (2.2.12) and (2.2.15)′ it now follows in view of the remark at the end of section 2.1 that

$$M_p^p(F_N) = L_p^p(G_N) \leq C_p B, \quad 1 < p < \infty, \tag{2.2.15}$$

where C_p only depends on p and the dimension n. Since $F_N(\xi) \to F(\xi)$ for $\xi \neq 0$ and is uniformly bounded, we have $F_N \to F$ in S', $N \to \infty$. Hence it follows from Lemma 1.5 that $F \in M_p^p$ and that

$$M_p^p(F) \leq C_p B. \tag{2.2.16}$$

The proof is complete.

Using Theorem 2.5 we shall now prove some results similar to those in section 1.4. They are modifications of theorems proved by Calderón and Zygmund [3].

Definition 2.2. *We shall denote by* H_p, $1 < p < \infty$, *the closure in* M_p^p *of the set* h^∞ *of functions* f *which are in* C^∞ *for* $\xi \neq 0$ *and homogeneous of degree* 0, *that is,*

$$f(t\xi) = f(\xi), \quad t > 0.$$

Note that Theorem 2.5 shows that $h^\infty \subset M_p^p$, and that H_p is obviously a normed ring.

Theorem 2.6. *Let* $f \in M_p^p$ *be continuous for* $\xi \neq 0$ *and homogeneous of degree* 0. *Then we have* $f \in H_q$ *if* $|1/q - 1/2| < |1/p - 1/2|$.

Proof. The theorem is quite parallel to Theorem 1.16 and so is the proof. Let O_n be the orthogonal group, with elements denoted by $A, B, \ldots$ and let dA be the Haar measure in O_n. For $j = 1, 2, \ldots$ take a function $\varphi_j \geq 0$ in O_n which is infinitely differentiable so that any neighbourhood of the identity in O_n contains the supports of all φ_j except a finite number. Let $\int \varphi_j(A)\, dA = 1$ and set

$$f_j(x) = \int f(Ax)\, \varphi_j(A)\, dA.$$

It is obvious that f_j is homogeneous of degree 0 and in view of Theorem 1.13 and the convexity of the norm M_p^p we have $M_p^p(f_j) \leqslant M_p^p(f)$, hence $M_p^p(f_j - f) \leqslant 2 M_p^p(f)$. Further $f_j \to f$ uniformly when $j \to \infty$. It is sufficient to verify this when ξ is on the unit sphere and then it follows at once from the uniform continuity of f. Thus $M_2^2(f_j - f) \to 0$, and as in the proof of Theorem 1.16 it follows that $M_q^q(f_j - f) \to 0$ as $j \to \infty$. It remains to prove that $f_j \in h^\infty$, that is that the restriction of f_j to the unit sphere is infinitely differentiable. But this is obvious since

$$f_j(Bx) = \int f(ABx)\,\varphi_j(A)\,dA = \int f(Ax)\,\varphi_j(AB^{-1})\,dA, \quad B \in O_n$$

and the right hand side is an infinitely differentiable function of B.

THEOREM 2.7. *The only continuous multiplicative linear forms on H_p are the mappings $f \to f(\xi)$, $|\xi| = 1$.*

Proof. Let $f \to T(f)$ be such a form. If h^∞ is identified with $C^\infty(\Sigma)$, where $\Sigma = \{\xi;\ |\xi| = 1\}$, the restriction to h^∞ of T can be considered as a distribution T in Σ. In fact, Theorem 2.5 or more precisely inequality (2.2.16) shows that $M_p^p(f_n) \to 0$, hence $T(f_n) \to 0$, if $f_n \in h^\infty$ and the derivatives of f_n of order $\leqslant \frac{1}{2}(n+1)$ converge to 0 uniformly in Σ. Thus $\hat{T}(x) = T(e^{-2\pi i \langle x, \cdot \rangle})$ is continuous and satisfies the equation $\hat{T}(x+y) = \hat{T}(x)\,\hat{T}(y)$. Hence $\hat{T}(x) = e^{-2\pi i \langle x, \xi \rangle}$ for some complex ξ, but since T has support in $\sum$ it follows that $\xi \in \Sigma$. Hence

$$T(f) = f(\xi), \quad f \in h^\infty. \tag{2.2.17}$$

Since h^∞ is dense in H_p, the equality (2.2.17) is valid for every $f \in H_p$, the two sides being continuous functions of $f \in H_p$. The proof is complete.

The result we need in Chapter III is the following

COROLLARY 2.1. *Let f be continuous for $\xi \neq 0$ and homogeneous of degree 0. Further assume that $f \in M_p^p$ for all p with $1 < p < \infty$ and that $f(\xi) \neq 0$ when $\xi \neq 0$. Then $1/f \in M_p^p$ when $1 < p < \infty$.*

Proof. Theorem 2.6 shows that $f \in H_q$ for all q with $1 < q < \infty$. Hence the corollary follows from Theorem 2.7 and basic facts concerning commutative Banach algebras (Loomis [6], pp. 78 and 79).

Chapter III

Estimates for some families of operators

3.1. Preliminaries

Our aim in this chapter is to supplement the investigation of the convolution transforms $k_t^{(a)} * u$ made in Chapter II by studying these as functions of t also. For simplicity we only consider the case $a=1$, but in view of some important applications we shall admit several parameters. Thus let T be a set in R^m such that $0 \notin T$ and $T \cup \{0\}$ is a closed(1) cone. Thus

$$\alpha t \in T \text{ if } \alpha > 0 \text{ and } t \in T. \tag{3.1.1}$$

Write $R^n = X$. We shall consider a function $K(x, t)$ defined and measurable in $X \times T$, such that

$$K(\alpha x, \alpha t) = \alpha^{-n} K(x, t), \quad (x, t) \in X \times T, \quad \alpha > 0. \tag{3.1.2}$$

With suitable assumptions on K and on u we shall study the convolution

$$\int K(x-y, t)\, u(y)\, dy, \quad (x, t) \in X \times T.$$

Writing $\tilde{K}(x, t) = \overline{K(-x, t)}$ we shall also for functions U in $X \times T$ study the adjoint transformation which maps U on the function

$$\int_X \int_T \tilde{K}(x-y, t)\, U(y, t)\, dy\, dt/|t|^m.$$

In the estimates we shall use norms on measurable functions U in $X \times T$ of the form (3.1.3) with $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$,

$$X^p T^q(U) = \left(\int_X \left(\int_T |U(x, t)|^q\, dt/|t|^m\right)^{p/q} dx\right)^{1/p}. \tag{3.1.3}$$

(3.1.3) has a sense, finite or infinite, in view of the Fubini theorem. (It is clear how (3.1.3) should be interpreted if p or q is infinite.) We write $X^p T^q$ for the set of measurable U such that the norm (3.1.3) is finite. This is a Banach space. In fact, the convexity of $X^p T^q(U)$ follows by repeated use of Minkowski's inequality in L^p spaces. Similarly one proves the convergence in the $X^p T^q$ norm of a series of elements in $X^p T^q$ whose $X^p T^q$ norms form a convergent series. Hence $X^p T^q$ is complete.

(1) This assumption is merely made to get shorter statements.

We shall also use the notation

$$T^q(U)=\left(\int |U|^q\, dt/|t|^m\right)^{1/q}.$$

If $U \in X^p T^q$ this function of x is finite almost everywhere.

Hölder's inequality takes the following form.

THEOREM 3.1. *Let U and V be measurable in $X\times T$. Then*

$$\int\limits_{X\times T}\int |U|\,|V|\,dx\,dt/|t|^m \leqslant X^p T^q(U)\, X^{p'} T^{q'}(V). \tag{3.1.4}$$

Proof. We may assume in the proof that the factors in the right hand side are finite. For almost all x we can then use Hölder's inequality for the integral with respect to t and another use of Hölder's inequality completes the proof of (3.1.4).

It would be easy to show that the dual space of the set of all measurable U with the norm $X^p T^q(U)<\infty$ is the corresponding space with p and q replaced by p' and q', provided that p and q are finite. However, we content ourselves with the following converse of Hölder's inequality.

THEOREM 3.2. *Let U be a measurable function in $X\times T$ and assume that for all measurable V with compact support in $X\times T$ we have with constant C*

$$\left|\int\limits_{X\times T}\int U V\,dx\,dt/|t|^m\right| \leqslant C\, X^{p'} T^{q'}(V), \tag{3.1.5}$$

which shall mean in particular that the integral in the left hand side exists if the right hand side is finite. Then it follows that

$$X^p T^q(U)\leqslant C. \tag{3.1.6}$$

Proof. We may assume that $U\geqslant 0$, that $X^p T^q(U)<\infty$ and that U has compact support. In fact, if the theorem is known when this hypothesis is made a priori we only need to form U_N defined as $\min(|U|, N)$ when $|x|\leqslant N$, $1/N\leqslant |t|\leqslant N$ and as 0 otherwise. U_N satisfies (3.1.5) and our requirements, so we will get $X^p T^q(U_N)\leqslant C$. Letting $N\to\infty$, we obtain (3.1.6). Now if the assumptions made above are fulfilled and $p<\infty$, $q<\infty$, we take

$$V=U^q/U\left(\int\limits_T U^q\,dt/|t|^m\right)^{1-p/q}$$

which we define as 0 when a factor in the denominator vanishes. A simple computation then gives (3.1.6). The case when p or $q=\infty$ is easily handled in a similar way.

The following variant of Lemma 2.2 is important in this chapter.

LEMMA 3.1. *Let $U \in X^1 T^q$. For every $s > 0$ we can then write*

$$U = V + \sum_1^\infty W_k, \tag{3.1.7}$$

where V and all $W_k \in X^1 T^q$ and for certain disjoint cubes I_k in X

$$X^1 T^q (V) + \sum_1^\infty X^1 T^q (W_k) \leqslant 3\, X^1 T^q (U). \tag{3.1.8}$$

$$(T^q (V))(x) \leqslant 2^n s \text{ for almost all } x, \tag{3.1.9}$$

$$\int W_k (x, t)\, dx = 0 \text{ for almost all } t \text{ and } W_k (x, t) = 0 \text{ if } x \notin I_k, \tag{3.1.10}$$

$$\sum_1^\infty m (I_k) \leqslant s^{-1} X^1 T^q (U). \tag{3.1.11}$$

If $U = 0$ outside a compact set, then there is another compact set such that V and all W_k vanish outside that set.

Proof. Let $u = T^q (U)$. We have $u \in L^1$ so we can form the decomposition of u given by Lemma 2.2. We take the cubes I_k associated with u and s in that lemma and set

$$V (x, t) = (m (I_k))^{-1} \int_{I_k} U (y, t)\, dy, \quad x \in I_k \tag{3.1.12}$$

$$W_k (x, t) = U (x, t) - V (x, t), \quad x \in I_k \tag{3.1.13}$$

which has as sense for almost all t. When $x \notin I_k$ we set $W_k = 0$ and when $x \notin \bigcup I_k$ we set $V = U$. Minkowski's inequality and (2.1.12) give

$$\left(\int |V (x, t)|^q\, dt / |t|^m\right)^{1/q} \leqslant (m (I_k))^{-1} \int_{I_k} u\, dy = (m (I_k))^{-1} \int_{I_k} v\, dy < 2^n s, \quad x \in I_k. \tag{3.1.14}$$

When $x \notin \bigcup I_k$ we have

$$\left(\int |V (x, t)|^q\, dt / |t|^m\right)^{1/q} = u (x) \leqslant s$$

almost everywhere which proves (3.1.9). (3.1.11) follows trivially from Lemma 2.2 for the right hand side is $s^{-1} \int u\, dx$. Since (3.1.14) gives

$$\int_{I_k} \left(\int |V (x, t)|^q\, dt / |t|^m\right)^{1/q} dx \leqslant \int_{I_k} u\, dy$$

we obtain $$X^1 T^q(W_k) \leqslant 2 \int_{I_k} u\,dy, \quad X^1 T^q(V) \leqslant \int u\,dy.$$

This proves (3.1.8).

3.2. L^2 estimates

Let K be a locally integrable function in $X \times T$ and form

$$(K * u)(x, t) = \int K(x-y, t)\, u(y)\, dy, \tag{3.2.1}$$

where u is integrable and has compact support. The integral exists for almost all (x, t) and is a locally integrable function in $X \times T$ in view of the Fubini theorem. We shall study conditions in order that the inequality

$$X^2 T^2 (K * u) \leqslant \| u \|_2 \tag{3.2.2}$$

shall hold.

THEOREM 3.3. *Let $K(x,t)$ be in S' for almost all fixed $t \in T$, and assume that there is a locally integrable function $\hat{K}(\xi, t)$ which defines the Fourier transform of $K(x,t)$ as a function of x for almost all t. If*

$$T^2(\hat{K}) \leqslant C \text{ almost everywhere,} \tag{3.2.3}$$

the inequality (3.2.2) *holds for all $u \in L^2$ with compact support.*

Proof. For almost all t we get by Parseval's formula

$$\int |K * u|^2\, dx = \int |\hat{K}(\xi, t)\, \hat{u}(\xi)|^2\, d\xi.$$

Hence

$$\int\int |K * u|^2\, dx\, dt/|t|^m = \int |\hat{u}(\xi)|^2\, d\xi \int |\hat{K}(\xi, t)|^2\, dt/|t|^m \leqslant C^2 \int |\hat{u}(\xi)|^2\, d\xi, \tag{3.2.4}$$

which proves (3.2.2).

When (3.2.3) holds, the mapping $u \to K * u$ can thus be extended to a continuous mapping A of L^2 into $X^2 T^2$. To compute the adjoint A^* let $U \in C_0^\infty(X \times T)$ (1) and form with $u \in C_0^\infty(X)$, which is a dense set in L^2,

$$\int\int (K * u)\, \bar{U}\, dx\, dt/|t|^m = \int\int \bar{U}\, dx\, dt/|t|^m \int K(x-y, t)\, u(y)\, dy.$$

---

(1) By this we mean that U is the restriction to $X \times T$ of a function in $C_0^\infty(R^n \times (R^m - \{0\}))$.

The integrations may be interchanged since the integrand vanishes except when (x, y, t) is in a compact set, and K is locally integrable. Hence

$$\int (K*u)\,\bar{U}\,d\,x\,d\,t/|t|^m = \int u\,\overline{\tilde{K}*U}\,d\,x \tag{3.2.5}$$

where we have used the notation $\tilde{K}(x, t) = \overline{K(-x, t)}$ and

$$(\tilde{K}*U)(x) = \int \tilde{K}(x-y, t)\,U(y, t)\,d\,y\,d\,t/|t|^m.$$

Since an operator between Hilbert spaces and its adjoint have the same bounds, we have

THEOREM 3.4. *If the hypotheses of Theorem* 3.3 *are fulfilled it follows that*

$$\|\tilde{K}*U\|_2 \leqslant C\,X^2\,T^2(U), \quad U \in C_0^\infty(X \times T), \tag{3.2.6}$$

where C is the same constant as in (3.2.3) *and* (3.2.2).

A^* is thus the closure of the mapping $X^2\,T^2 \supset C_0^\infty(X \times T) \ni U \to \tilde{K}*U \in L^2(X)$. We shall now compute A^*A. When $u \in C_0^\infty(X)$ (3.2.4) may be written

$$(X^2\,T^2(A\,u))^2 = \int |\hat{u}|^2\,(T^2(\hat{K}))^2\,d\,\xi,$$

that is,

$$(A^*A\,u, u) = \int |\hat{u}|^2\,(T^2(\hat{K}))^2\,d\,\xi, \quad u \in C_0^\infty(X). \tag{3.2.7}$$

Polarization of this identity gives since A^*A is a self-adjoint operator

$$(A^*A\,u, v) = \int \hat{u}\,\bar{\hat{v}}\,(T^2(\hat{K}))^2\,d\,\xi, \quad u, v \in C_0^\infty(X).$$

Thus, if $\mathfrak{F}$ denotes the Fourier transformation, we have

$$A^*A\,u = \mathfrak{F}^{-1}((T^2(\hat{K}))^2\,\mathfrak{F}\,u), \quad u \in C_0^\infty, \tag{3.2.8}$$

so that A^*A is the convolution operator corresponding to the multiplier $(T^2(\hat{K}))^2$. This will be essential later on. For future reference we also remark here that it follows immediately from (3.1.2) that $T^2(\hat{K})$ is homogeneous of degree 0, that is,

$$(T^2(\hat{K}))(\alpha\,\xi) = (T^2(\hat{K}))(\xi), \quad \alpha > 0. \tag{3.2.9}$$

3.3. Main theorem on mixed L^2 estimates

In this section we assume throughout that K is locally in $X^1 T^2$. By this we mean that a function K_1 which is equal to K in a compact subset of $X \times T$ and equal to 0 elsewhere in $X^1 T^2$. Then

$$(\tilde{K} * U)(x) = \int \int \tilde{K}(x-y, t)\, U(y, t)\, dy\, dt / |t|^m \tag{3.3.1}$$

exists as an absolutely convergent integral almost everywhere and defines a locally integrable function, if U is in $X^1 T^2$ and has compact support. To prove this it is sufficient to study (3.3.1) with K replaced by K_1. But then we can apply the Fubini theorem to the measurable function $\tilde{K}_1(x-y, t)\; U(y, t)/|t|^m$ of (x, y, t), for

$$\int\int\int |\tilde{K}_1(x-y, t)\, U(y, t)|\, dx\, dy\, dt/|t|^m \leqslant \int\int T^2(\tilde{K}_1(x-y, \cdot))\, T^2(U(y, \cdot))\, dx\, dy =$$
$$= X^1 T^2(\tilde{K}_1)\, X^1 T^2(U) < \infty.$$

Also note that the same argument shows that (3.2.5) is valid for all $u \in C_0^\infty$ if $U \in X^1 T^2$ and has compact support. Thus the operator A^* discussed in section 3.2 is defined by $A^* U = \tilde{K} * U$ and (3.2.6) is valid for all $U \in X^2 T^2$ with compact support.

The principal aim of this section is to prove the following theorem.

THEOREM 3.5. *Let K be locally in $X^1 T^2$ and satisfy the hypotheses of Theorem 3.3. Further assume that there is a neighbourhood N of 0 in X and a compact set M in X such that*

$$\int_{\complement M} dx \left(\int_T |K(x-y, t) - K(x, t)|^2\, dt/|t|^m \right)^{\frac{1}{2}} \leqslant C, \quad y \in N, \tag{3.3.2}$$

where C is a constant. For $1 < p < \infty$ there then exists a constant C_p such that if $u \in L^p$ has compact support in X

$$X^p T^2(K * u) \leqslant C_p \|u\|_p, \tag{3.3.3}$$

and if $U \in X^p T^2$ and has compact support in $X \times T$.

$$\|\tilde{K} * U\|_p \leqslant C_{p'}\, X^p T^2(U). \tag{3.3.4}$$

Proof. It is sufficient to prove (3.3.3) and (3.3.4) when $1 < p \leqslant 2$, for (3.3.3) is equivalent to (3.3.4) with p replaced by p' in view of the duality used in the proof of Theorem 3.4. In fact, if (3.3.4) is known with p replaced by p' we get using (3.2.5) and Hölder's inequality

$$\left|\int (K*u)\,\bar{U}\,dx\,dt/|t|^m\right| \leqslant \|u\|_p\, C_p\, X^{p'} T^2(U)$$

hence (3.1.5) gives $X^p T^2 (K*u) \leqslant C_p \|u\|_p$. The argument may of course be reversed. Also note that Theorems 3.3 and 3.4 show that the theorem is valid for $p=2$.

We now prove a theorem concerning the mapping $u \to T^2 (K*u)$ which combined with Marcinkiewicz' interpolation theorem gives (3.3.3) for $1 < p \leqslant 2$.

THEOREM 3.6. *Let the assumptions of Theorem 3.5 be fulfilled. Then there is a constant C_1 such that if $u \in L^1$ and has compact support we have*

$$m\{x;\ (T^2 (K*u))(x) > \sigma\} \leqslant C_1 \|u\|_1/\sigma, \quad \sigma > 0. \tag{3.3.5}$$

Proof. To prove this theorem we have essentially only to repeat the proof of Theorem 2.2. There is no restriction in assuming that the sets N and M in (3.3.2) are cubes with centre at 0; we may write $N = I_0$ and $M = I_0^*$. We then define a cube I^* for every cube I as in section 2.1 so that $m(I^*)/m(I) = m(I_0^*)/m(I_0) = \gamma$. In view of (3.1.2) it then follows by a trivial computation that the inequality

$$\int_{x \notin I^*} dx \left(\int_T |K(x-y, t) - K(x, t)|^2\, dt/|t|^m \right)^{\frac{1}{2}} \leqslant C, \quad y \in I \tag{3.3.2$'$}$$

is valid for every cube with centre at 0. Now let $u \in L^1$ have support in I and assume that

$$\int u\, dx = 0. \tag{3.3.6}$$

Then we have $$(K*u)(x, t) = \int (K(x-y, t) - K(x, t))\, u(y)\, dy.$$

Hence Minkowski's inequality for the $X^1 T^2$-norm and (3.3.2)$'$ give

$$\int_{x \notin I^*} T^2 (K*u)\, dx \leqslant C \|u\|_1. \tag{3.3.7}$$

In view of its invariance for translation, (3.3.7) must hold also for cubes with arbitrary centre.

In proving (3.3.5) we may assume that $\|u\|_1 = 1$. To simplify the notations we write $u^*(x, t) = (K*u)(x, t)$, which exists almost everywhere as an absolutely convergent integral. Form the decomposition of u given by Lemma 2.2 with $s = \sigma$. Then we have

$$|u^*(x,t)| \leqslant |v^*(x,t)| + \sum_1^\infty |w_k^*(x,t)| \tag{3.3.8}$$

for all (x, t) such that $(|K| * (|v| + \sum |w_k|))(x, t) < \infty$, hence almost everywhere in $X \times T$. In virtue of (3.3.7) we have

$$\int_{x \notin I_k^*} T^2(w_k^*)\, dx \leqslant C \| w_k \|_1. \tag{3.3.9}$$

(2.1.14) gives an estimate of the measure of the set $O = \bigcup I_k^*$,

$$m(O) \leqslant \gamma \sigma^{-1} \| u \|_1 = \gamma / \sigma.$$

Restricting the integration in the left hand side of (3.3.9) to $\complement O$ and adding, we get if $w^* = \sum |w_k^*|$

$$\int_{x \notin O} T^2(w^*)\, dx \leqslant C \sum_1^\infty \| w_k \|_1 \leqslant 3\, C \| u \|_1 = 3\, C,$$

where the last inequality follows from (2.1.11). Hence the measure of the set of points in $\complement O$ where $T^2(w^*) \geqslant \frac{1}{2}\sigma$ is at most $6\, C/\sigma$. Thus it follows that $T^2(w^*) \leqslant \frac{1}{2}\sigma$ except in a set of measure at most $(\gamma + 6\, C)/\sigma$.

Further, since $v \in L^2$ and has compact support, we may apply (3.2.2) to v and obtain

$$X^2\, T^2(v^*) \leqslant C \| v \|_2 \leqslant C (2^n \sigma)^{\frac{1}{2}} \| v \|_1^{\frac{1}{2}} \leqslant \sqrt{3}\, C (2^n \sigma)^{\frac{1}{2}}.$$

Hence

$$m\{x;\ T^2(v^*) > \tfrac{1}{2}\sigma\} \leqslant 3\, C^2\, 2^{n+2}/\sigma.$$

Since (3.3.8) shows that the set where $T^2(u^*) > \sigma$ is contained in the union of the set where $T^2(v^*) > \frac{1}{2}\sigma$ and that where $T^2(w^*) > \frac{1}{2}\sigma$, the inequality (3.3.5) follows. This proves Theorem 3.6 and thus (3.3.3) follows from Marcinkiewicz' interpolation theorem when $1 < p \leqslant 2$.

We next prove a theorem from which (3.3.4) follows for $1 < p \leqslant 2$. Since Marcinkiewicz' interpolation theorem has to be somewhat modified in order to be applicable here, we have to supply an extra argument after Theorem 3.7 in order to obtain (3.3.4).

THEOREM 3.7. *Let the assumptions of Theorem* 3.5 *be valid. Then there is a constant* C_1 *such that if* $U \in X^1 T^2$ *and has compact support in* $X \times T$, *we have*

$$m\{x;\ |(\tilde{K} * U)(x)| > \sigma\} \leqslant C_1\, X^1\, T^2(U)/\sigma, \quad \sigma > 0. \tag{3.3.10}$$

Proof. The proof is parallel to that of Theorem 3.6 but uses Lemma 3.1 instead of Lemma 2.2. First we shall prove that if $U(x, t)=0$ when $x \notin I$, where I is a cube, and if

$$\int U(x, t)\,dx=0 \tag{3.3.11}$$

for almost all t, then

$$\int_{x \notin I^*} |\tilde{K} * U|\,dx \leqslant C X^1 T^2(U). \tag{3.3.12}$$

To prove this it is sufficient to assume that I is a cube with centre at 0, for (3.3.11) is invariant for translation. In view of (3.3.11) we can write

$$(\tilde{K} * U)(x)=\int U(y, t)(\tilde{K}(x-y, t)-\tilde{K}(x, t))\,dy\,dt/|t|^m.$$

Hence it follows from Cauchy-Schwarz' inequality that

$$|(\tilde{K} * U)(x)| \leqslant \int (T^2(U))(y)\left(\int_T |\tilde{K}(x-y, t)-\tilde{K}(x, t)|^2\,dt/|t|^m\right)^{\frac{1}{2}} dy.$$

Integrating this inequality over $\complement I^*$ and using (3.3.2)$'$, we obtain (3.3.12).

In proving (3.3.10) we assume that $X^1 T^2(U)=1$. To simplify the notations we write $U^*(x)=(\tilde{K} * U)(x)$. Form the decomposition of U given by Lemma 3.1 with $s=\sigma$ and $q=2$. Then we have

$$|U^*(x)| \leqslant |V^*(x)|+\sum_1^\infty |W_k^*(x)| \tag{3.3.13}$$

for all x where $(|\tilde{K}| *(|V|+\sum |W_k|))(x)<\infty$, hence almost everywhere in X. In virtue of (3.3.12) we have

$$\int_{x \notin I_k^*} |W_k^*|\,dx \leqslant C X^1 T^2(W_k). \tag{3.3.14}$$

(3.1.11) gives an estimate of the measure of the set $O=\bigcup I_k^*$,

$$m(O) \leqslant \gamma \sigma^{-1} X^1 T^2(U)=\gamma / \sigma.$$

Restricting the integration in the left hand side of (3.3.14) to $\complement O$ and adding, we get if $W^*=\sum |W_k^*|$

$$\int_{x \notin O} W^*\,dx \leqslant C \sum_1^\infty X^1 T^2(W_k) \leqslant 3\, C X^1 T^2(U)=3\, C,$$

where the last inequality follows from (3.1.8). Hence the measure of the set of points in $\complement O$ where $W^* \geqslant \frac{1}{2}\sigma$ is at most $6\,C/\sigma$. Thus it follows that $W^* = \sum_1^\infty |W_k^*| < \frac{1}{2}\sigma$ except in a set of measure at most $(\gamma + 6\,C)/\sigma$.

Further, since $V \in X^2 T^2$ and has compact support in $X \times T$, we may apply (3.2.6) to V and obtain

$$\| V^* \|_2 \leqslant C\, X^2\, T^2 (V) \leqslant C\, (2^n \sigma\, X^1\, T^2 (V))^{\frac{1}{2}} \leqslant \sqrt{3}\, C\, (2^n \sigma)^{\frac{1}{2}}.$$

Hence
$$m\{x;\ |V^*| > \tfrac{1}{2}\sigma\} \leqslant 3\, C^2\, 2^{n+2}/\sigma.$$

Since (3.3.13) shows that the set where $|U^*| > \sigma$ is contained in the union of the set where $|V^*| > \frac{1}{2}\sigma$ and that where $\sum_1^\infty |W_k^*| > \frac{1}{2}\sigma$, the inequality (3.3.10) follows.

End of proof of Theorem 3.5. As we have already mentioned, the inequality (3.3.4) for $1 < p \leqslant 2$ follows from Theorem 3.7 by means of the usual proof of Marcinkiewicz' interpolation theorem although the theorem itself does not seem applicable. Thus let $U \in X^p T^2$ and assume that U has compact support in $X \times T$. Take a number $s > 0$ and set $U_1(x, t) = U(x, t)$ if $(T^2(U))(x) \leqslant s$ and $U_1(x, t) = 0$ otherwise. Define $U_2(x, t) = U(x, t) - U_1(x, t)$. Then we have $|U^*| \leqslant |U_1^*| + |U_2^*|$ and hence

$$m(s) = m\{x;\ |U^*(x)| > s\} \leqslant m\{x;\ |U_1^*| > \tfrac{1}{2}s\} + m\{x;\ |U_2^*| > \tfrac{1}{2}s\}.$$

To estimate the terms in the right hand side we use (3.2.6) and (3.3.10) respectively. This gives

$$m(s) \leqslant (2/s)^2\, C^2\, (X^2\, T^2 (U_1))^2 + (2/s)\, C_1\, X^1\, T^2 (U_2) =$$
$$= (2/s)^2\, C^2 \int_{T^2(U) \leqslant s} (T^2 (U))^2\, dx + (2/s)\, C_1 \int_{T^2(U) \geqslant s} T^2 (U)\, dx.$$

With a change of the order of integrations we now obtain

$$\int |U^*|^p\, dx = \int_0^\infty m(s)\, d(s^p) \leqslant (4\, C^2 p/(2-p) + 2\, C_1 p/(p-1)) \int |T^2(U)|^p\, dx,$$

which proves (3.3.3) for $1 < p < 2$. This completes the proof of Theorem 3.5.

Our results easily give "inverse estimates" also:

THEOREM 3.8. *Let K_j, $j = 1, \ldots, J$, be kernels satisfying the hypotheses of Theorem 3.5 and assume that*

$$\varkappa = \sum_1^J (T^2 (\hat{K}_j))^2$$

is continuous and $\neq 0$ *when* $\xi \neq 0$. *Then for* $1 < p < \infty$ *there is a constant* C_p *such that if* $u \in L^p$ *and has compact support*

$$C_p^{-1} \| u \|_p \leqslant \sum_1^J X^p T^2 (K_j * u) \leqslant C_p \| u \|_p. \tag{3.3.15}$$

Proof. The last inequality follows from Theorem 3.5. Let A_{jp} be the bounded mapping of L^p into $X^p T^2$ obtained by closing the mapping $u \to K_j * u$ defined for all $u \in L^p$ with compact support. In view of the remarks at the beginning of this section, the adjoint A_{jp}^* is the closure of the mapping $U \to \tilde{K}_j * U$ defined for all $U \in X^{p'} T^2$ with compact support. If $u \in L^p \cap L^2$ we have $A_{jp} u = A_{j2} u$ and if $U \in X^{p'} T^2 \cap X^2 T^2$ we have $A_{jp}^* U = A_{j2}^* U$. Hence if $u \in C_0^\infty$ it follows from (3.2.8) that

$$\sum_1^J A_{jp}^* A_{jp} u = \sum_1^J A_{j2}^* A_{j2} u = \mathfrak{F}^{-1} (\varkappa \mathfrak{F} u). \tag{3.3.16}$$

Let T be the distribution in S' such that $\hat{T} = \varkappa^{-1}$. Since it follows from (3.3.16) that $\varkappa \in M_p{}^p$, Corollary 2.2 implies that $\varkappa^{-1} \in M_p{}^p$, hence $T \in L_p{}^p$. Let B be the closure in L^p of the mapping $S \ni v \to T * v$. The operator B is then bounded and Theorem 1.7 gives

$$B \sum_1^J A_{jp}^* A_{jp} u = u.$$

Hence $$\| u \|_p \leqslant \| B \| \sum_1^J \| A_{jp}^* \| X^p T^2 (A_{jp} u) \leqslant C_p \sum_1^J X^p T^2 (K_j * u).$$

This proves (3.3.15) if $u \in C_0^\infty$ and by approximation (3.3.15) follows for arbitrary u in L^p with compact support.

3.4. Examples of mixed L^2 estimates

As a first example we shall study the Marcinkiewicz' function (see Stein [15] and the references given there). Thus let $\Omega(x)$ be a (positively) homogeneous function of degree 0, that is,

$$\Omega(t x) = \Omega(x), \quad t > 0,$$

which for $x \neq 0$ satisfies a Dini condition. By this we mean that there is an increasing function $\delta(t)$, $t > 0$, such that

$$| \Omega(x) - \Omega(y) | \leqslant \delta(|x - y|) \text{ if } |x| \geqslant 1, |y| \geqslant 1,$$

and
$$\int_0^1 \delta(t)\,dt/t < \infty. \tag{3.4.1}$$

The homogeneity of Ω gives that when x and y are $\neq 0$

$$|\Omega(x) - \Omega(y)| \leqslant \delta(|x-y|/\min(|x|, |y|)). \tag{3.4.2}$$

Let α be a positive constant and set $T = (0, \infty)$. For $t \in T$ we define

$$\begin{aligned} K(x, t) &= \Omega(x)\,|x|^{\alpha-n}\,t^{-\alpha} && \text{if } |x| \leqslant t \\ &= 0 && \text{if } |x| > t. \end{aligned} \tag{3.4.3}$$

($\alpha = 1$ in Stein [15]). Then K satisfies the homogeneity condition (3.1.2). We have to examine if (3.2.3) holds. First note that $\hat{K}(\xi, t) = \hat{K}(\xi t, 1)$. Since $\hat{K}(\xi, 1)$ is an analytic function of ξ, it is clear that if $T^2(\hat{K})$ is finite we must have $\hat{K}(0, 1) = 0$, that is,

$$\int_{|\omega|=1} \Omega(\omega)\,d\omega = 0. \tag{3.4.4}$$

Conversely, if this inequality is fulfilled it follows that $\hat{K}(\xi, 1)/|\xi|$ is bounded, hence $|\hat{K}(\xi, t)| \leqslant C|\xi|t$. Thus the integral

$$\int_0^1 |\hat{K}(\xi, t)|^2\,dt/t \tag{3.4.5}$$

is uniformly bounded when $|\xi| = 1$. To estimate the integral from 1 to ∞ we have to use the Dini condition. Noting that the Fourier transform of $K(x+h, 1) - K(x, 1)$ is $(e^{2\pi i\langle h, \xi\rangle} - 1)\hat{K}(\xi, 1)$ we obtain

$$2|\sin \pi\langle h, \xi\rangle|\,|\hat{K}(\xi, 1)| \leqslant \int |K(x+h, 1) - K(x, 1)|\,dx.$$

Taking $h = \xi/2|\xi|^2$ and estimating the integral on the right hand side separately for $|x| < |h|$, for $|h| < |x| < 1 - |h|$ and for $|x| > 1 - |h|$ one easily obtains

$$|\hat{K}(\xi, 1)| \leqslant C'(|\xi|^{-\frac{1}{2}} + |\xi|^{-\alpha/2} + \delta(4|\xi|^{-\frac{1}{2}}), \; |\xi| \geqslant 1.$$

The detailed verification may be left to the reader. In view of (3.4.1) it follows that the integral

$$\int_1^\infty |\hat{K}(\xi, t)|^2\,dt/t = \int_1^\infty |\hat{K}(t\xi, 1)|^2\,dt/t$$

is uniformly convergent when $|\xi|=1$. Hence $T^2(\hat{K})$ is a continuous function of ξ for $\xi \neq 0$.

It remains to prove that (3.3.2) is valid. We shall do so taking for N the unit sphere and for M the concentric sphere with radius 2. Let $z=x-y$. Since $|x|\geqslant 2$ and $|y|\leqslant 1$ we have $|z|\geqslant|x|-1\geqslant\frac{1}{2}|x|$, hence $|\Omega(z)-\Omega(x)|\leqslant\delta(2/|x|)$. Assuming for example that $|z|\leqslant|x|$ we have

$$\int_0^\infty |K(z,t)-K(x,t)|^2\,dt/t=|\Omega(z)|^2\,|z|^{2(\alpha-n)}\int_{|z|}^{|x|} t^{-1-2\alpha}\,dt+$$
$$+|\Omega(z)\,|z|^{\alpha-n}-\Omega(x)\,|x|^{\alpha-n}|^2\int_{|x|}^\infty t^{-1-2\alpha}\,dt.$$

Since $|x|\leqslant|z|+1$ the first integral in the right hand side is at most $|z|^{-1-2\alpha}$, the second integral is $|x|^{-2\alpha}/2\alpha$. The mean value theorem gives

$$\big||z|^{\alpha-n}-|x|^{\alpha-n}\big|\leqslant|\alpha-n|\,(|x|-1)^{\alpha-n-1}$$

and combining this with the Dini condition for Ω we obtain

$$\left(\int_0^\infty |K(z,t)-K(x,t)|^2\,dt/t\right)^{\frac{1}{2}}\leqslant C\,(|x|^{-n-\frac{1}{2}}+\delta(2/|x|)\,|x|^{-n}). \tag{3.4.6}$$

In the same way we also obtain (3.4.6) if $|x|\leqslant|z|$. From (3.4.1) it follows again that the right hand side of (3.4.6) is integrable over the set $|x|\geqslant 2$. Hence Theorems 3.5 and 3.8 give

THEOREM 3.9. *If K is defined by* (3.4.3) *where Ω satisfies the Dini condition* (3.4.1), (3.4.2) *and also* (3.4.4), *then for all u in L^p, $1<p<\infty$, with compact support*

$$X^p\,T^2(K*u)\leqslant C_p\,\|u\|_p.$$

If K_j are a finite number of kernels of this type and if there does not exist any real $\xi\neq 0$ such that the integral of all the kernels $K_j(x,t)$ over the plane $\langle x,\xi\rangle=1$ vanishes for all t, then for $1<p<\infty$ and the same u as above

$$C_p^{-1}\,\|u\|_p\leqslant\sum_j X^p\,T^2(K_j*u)\leqslant C_p\,\|u\|_p.$$

We next give an application containing estimates for the Littlewood-Paley and Lusin functions (see Stein [15]) as well as for similar functions connected with elliptic partial differential equations other than the Laplacean.

Thus let K be a homogeneous function of degree $-n$ in $X\times T$ which is Hölder continuous of order α, $0<\alpha<1$, in $X\times\overline{T}$ except at (0, 0). We also require that

$$K(x, 0) = 0, \quad x \neq 0. \tag{3.4.7}$$

From the Hölder continuity it then follows that $|K(x, t)| \leqslant C|t|^\alpha$ if $|x| + |t| = 1$, hence the homogeneity gives

$$|K(x, t)| \leqslant C|t|^\alpha/(|t| + |x|)^{n+\alpha} \tag{3.4.8}$$

in $X \times T$. This means in particular that K is integrable for fixed t and that the Fourier transform $\hat{K}(\xi, t)$ is continuous in $X \times T$. The homogeneity of K gives

$$\hat{K}(\varepsilon \xi, t) = \hat{K}(\xi, \varepsilon t) \quad \varepsilon > 0. \tag{3.4.9}$$

If $T^2(\hat{K}) < \infty$ we must thus have $\hat{K}(0, t) = 0$, that is,

$$\int K(x, t)\, dx = 0, \quad t \in T. \tag{3.4.10}$$

We shall now show that the conditions listed are sufficient to make the results in section 3.3 applicable. First note that if $|t| = 1$ the estimate

$$|\hat{K}(\xi, t)| = |\hat{K}(\xi, t) - \hat{K}(0, t)| \leqslant C \int |e^{-2\pi i \langle x, \xi \rangle} - 1|/(1 + |x|)^{n+\alpha}\, dx \leqslant C_1 |\xi|^\alpha$$

follows if the integrals when $|x| \leqslant 1/|\xi|$ and when $|x| > 1/|\xi|$ are estimated separately. Hence (3.4.9) gives for all $(\xi, t) \in X \times T$

$$|\hat{K}(\xi, t)| \leqslant C_1 (|\xi|\,|t|)^\alpha. \tag{3.4.11}$$

Further, since the Fourier transform of $K(x + h, t) - K(x, t)$ with respect to x is $(e^{2\pi i \langle h, \xi \rangle} - 1)\hat{K}(\xi, t)$, we obtain

$$2|\sin \pi \langle h, \xi \rangle|\, |\hat{K}(\xi, t)| \leqslant \int |K(x + h, t) - K(x, t)|\, dx.$$

Now the Hölder continuity gives that

$$|K(x + h, t) - K(x, t)| \leqslant C|h|^\alpha/(|x| + |t|)^{\alpha+n}, \quad |h| \leqslant \tfrac{1}{2}|t|. \tag{3.4.12}$$

In fact, in view of the homogeneity it is sufficient to prove this when $|x| + |t| = 1$ and then it follows at once from the Hölder continuity. Hence we obtain if $|t| = 1$

$$|\sin \pi \langle h, \xi \rangle|\, |\hat{K}(\xi, t)| \leqslant C_1 |h|^\alpha, \quad |h| < \tfrac{1}{2}.$$

Taking $h = \xi/2|\xi|^2$ we get $|\hat{K}(\xi, t)| \leqslant C_2 |\xi|^{-\alpha}$ if $|\xi| \geqslant 1$, hence

$$|\hat{K}(\xi, t)| \leqslant C_2 |t|^{-\alpha} |\xi|^{-\alpha} \text{ if } |t|\,|\xi| \geqslant 1. \tag{3.4.13}$$

The estimates (3.4.11) and (3.4.13) together show that $\int |\hat{K}(\xi, t)|^2 dt/|t|^m$ is uniformly convergent when $|\xi| = 1$. Hence $T^2(\hat{K})$ is bounded and continuous, and since (3.3.2) follows from (3.4.8) we can thus apply Theorems 3.5 and 3.8 and obtain

THEOREM 3.10. *If K is Hölder continuous in $X \times \overline{T}$ except at* (0, 0) *and satisfies* (3.4.7), (3.4.10) *then for all u in L^p, $1 < p < \infty$, with compact support*

$$X^p T^2 (K * u) \leqslant C_p \| u \|_p.$$

If K_j are a finite number of kernels of this type and if there does not exist any real $\xi \neq 0$ such that the integral of all the kernel $K_j(x, t)$ over the plane $\langle x, \xi \rangle = 1$ vanishes for all t, we have for the same u

$$C_p^{-1} \| u \|_p \leqslant \sum_j X^p T^2 (K_j * u) \leqslant C_p \| u \|_p.$$

A particular case in which these conditions are fulfilled is obtained in the following way. Let $K(x, t)$, $t \in T = (0, \infty)$, be a homogeneous function of degree $-n$ with Hölder continuous first derivatives in $X \times \overline{T}$ except at (0, 0), which is integrable and has a non vanishing integral for some t. Then the kernels $K_j(x, t) = t \partial K(x, t)/\partial x_j$ satisfy the conditions in the theorem. The verification of this may be left to the reader. Taking K to be the Poisson kernel P we obtain estimates of the Littlewood-Paley function. The estimates for the Lusin function follow if we take $K(x, t) = P(x_1 - t_1, \ldots, x_n - t_n, t_{n+1})$ and for T the cone $|t_1| + \cdots + |t_n| \leqslant C t_{n+1}$, where C is a constant.

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