UNRESTRICTED NILPOTENT PRODUCTS

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Let $A_1, A_2, \ldots, A_i, \ldots$ be a countable sequence of infinite cycles and $\prod_{i=1}^{\infty HX} A_i$ denote their unrestricted direct product. Then the following are well known theorems, due in main to Specker [22]:

THEOREM OF SPECKER. *Every countable subgroup of* $\prod_{i=1}^{\infty HX} A_i$ *is a free abelian group.*

THEOREM OF SPECKER AND ŁOS. *Let* ψ *be a homomorphism of* $\prod_{i=1}^{\infty HX} A_i$ *into a free abelian group. Then there exists a positive integer* m *such that*

$$\psi\Big(\prod_{i=m+1}^{\infty HX} A_i\Big) = 1.$$

Our aim is to investigate the corresponding situation in the case of the nilpotent product of infinite cycles. In a similar way one can derive results for the unrestricted soluble product and for the unrestricted third Burnside product both of infinite cycles and of cycles of order three.

Before we can give an outline of our main results, we must first introduce the following

Notation. Let v denote a typical power product of a set of power products of the letters of some fixed alphabet and their formal inverses. These power products are called words. The values of the words obtained by substituting elements from a group G for the above letters of the alphabet, in all possible ways, generate a subgroup of G—the verbal subgroup $V(G)$ of G. The verbal subgroups corresponding to the words

$$[[\cdot\cdot[x_1, x_2], \ldots], x_n],$$

$$d_n=[d_{n-1}(x_1,x_2,\ldots),\ d_{n-1}(y_1,y_2,\ldots)],\quad \text{where}\quad d_1(x_1,x_2)=[x_1,x_2]$$
$$\text{and}\quad x^n$$

are of particular importance and their value in G is denoted by ${}^nG$, $G^{[n]}$ and G^n respectively. They are known as the nth member of the lower central series of G, the nth member of the derived series of G and the nth Burnside subgroup of G respectively. If $\mathfrak{F}$ denotes a free group, then $\mathfrak{F}/V(\mathfrak{F})$ is known as a relatively free group (cf. [6]). In particular

$$\mathfrak{F}/{}^{n+1}\mathfrak{F},\ \mathfrak{F}/\mathfrak{F}^{[n]} \text{ and } \mathfrak{F}/\mathfrak{F}^n$$

are known as a free nth nilpotent group, a free nth soluble group and a free nth Burnside group respectively.

In Moran [16, 17] we studied the verbal product or V-product

$$\prod_{\alpha\in M}^{V} G_\alpha=F/(V(F)\cap[G_\alpha]^F)$$

of the groups $G_\alpha, \alpha\in M$, where F denotes their free product and $[G_\alpha]^F$ is the cartesian subgroup of F. In particular, we found that

$$\mathfrak{F}/V(\mathfrak{F})\cong\prod_{\alpha\in M}^{V}A_\alpha,$$

where the cardinal of M is equal to the rank of the free group $\mathfrak{F}$, and $A_\alpha, \alpha\in M$, are cyclic groups of order k. The number k is the exponent of the variety V and is given by (cf. [6])

$$V(\mathfrak{F})=\mathfrak{F}^k\cdot(V(\mathfrak{F})\cap\mathfrak{F}').$$

k is taken to be infinite if $V(\mathfrak{F})$ is contained in the commutator subgroup $\mathfrak{F}'$. Our interest will be confined to direct products, nilpotent products, soluble products, third Burnside products and free products. The exponent of the corresponding verbal subgroups in all these cases is infinite, except in the third Burnside subgroup where the exponent is three. We shall have cause to consider the third Burnside product both of infinite cycles and of cycles of order three. The latter is a free third Burnside group. It is easy to see that the former is also a relatively free group, namely, that associated with the verbal subgroup $({}^2\mathfrak{F})^3\cdot{}^4\mathfrak{F}$. Thus in both cases it is possible to speak of free generators.

Following G. Higman [8], we have in [18] defined the unrestricted verbal product $\prod_{i=1}^{\infty}{}^{HV}A_i$ as the projective limit of the verbal products

$$A_1 \leftarrow A_1 V A_2 \leftarrow \ldots \leftarrow \prod_{i=1}^{m}{}^{V} A_i \leftarrow \prod_{i=1}^{m+1}{}^{V} A_i \leftarrow \ldots$$

under the natural homomorphisms as shown by the arrows.

In the case of the unrestricted free product the situation has been fully investigated by G. Higman. In [9], he shows that every finitely generated subgroup of $\prod_{i=1}^{\infty}{}^{H*} A_i$, where as before A_i are all infinite cycles, is a free group, while in [8] he constructs a countable subgroup which is not free. Further in [8] he shows that every homomorphism of $\prod_{i=1}^{\infty}{}^{H*} A_i$ into a free group maps the unrestricted free product of all but a finite number of the factors A_i onto the unit element.

The unrestricted nth nilpotent product of a countable number of infinite cycles will, for the moment, be denoted by G. We show that every countable subgroup H of G, for which $H/z(H)$ is finitely(1) generated, can be mapped isomorphically into a free nth nilpotent group. It is not known to the author whether this is true for every countable subgroup of G. However, we are able to state the following three results concerning such subgroups. In [19] (2) Theorems 3.4 and 3.7, we gave a partial characterization of subgroups of free nth nilpotent groups. Every countable subgroup of G satisfies this characterization. Associated with every torsion-free nilpotent group A, Mal'cev has defined a torsion-free nilpotent group $M(A)$ of the same nilpotency class as A, which has the following properties:

(a) $M(A)$ is a divisible group,

(b) Some positive power of every element of $M(A)$ is contained in A.

$M(A)$ is called the *Mal'cev completion* of A. For the existence and properties of Mal'cev completions we refer the reader to the elegant paper of Lazard [11]. We show that the Mal'cev completion of G can be mapped isomorphically into the Mal'cev completion of a free nth nilpotent group. This is deduced from the result, which is of independent interest, that the unrestricted free Lie algebra over a field is a free Lie algebra over the same field. Thirdly we note that if we proceed with a similar construction in the unrestricted second nilpotent product, as G. Higman [8] did to construct a countable subgroup which is not free, then the resulting countable subgroup is a free second nilpotent group.

Every homomorphism of an unrestricted nilpotent product of a countably infinite number of infinite cycles into a free nilpotent group maps the unrestricted nilpotent

(1) $z(H)$ denotes the centre of H.

(2) The reader is assumed to have some acquaintance with the notation and results of this paper.

product of all but a finite number of the factors onto the unit element.(1) A similar result holds for a homomorphism of an unrestricted soluble product into a free soluble group. As far as the countable subgroups of an unrestricted soluble product of infinite cycles are concerned, we have only been able to prove that every countable abelian subgroup is free abelian.

The unrestricted third Burnside product of cycles of order three can be mapped isomorphically into a free third Burnside group. This follows from a Subgroup Theorem for free third Burnside groups, similar to that given for free nilpotent groups in Moran [19]. On the other hand, in the unrestricted third Burnside product B of infinite cycles, every countable subgroup can be mapped isomorphically into a third Burnside product of infinite cycles. This is also true for every abelian subgroup of B. For every abelian subgroup of B is a subgroup of the direct product of an infinite cycle with an elementary abelian group of exponent three. An investigation into the nature of the homomorphisms of B onto a nonabelian subgroup of a third Burnside product of infinite cycles, shows that B cannot be isomorphic to such a subgroup.

Finally we note that our results extend to the unrestricted products of an arbitrary number of factors. The only exception to this is a curious one which occurs even in the case of the unrestricted direct product. In the analogues of the Theorem of Specker and Łos, we must take the set of infinite cycles $A_\alpha, \alpha \in M$, to be such that the cardinal of M has measure zero. If the cardinal of M is not of measure zero, then, as shown by Łos in [24], the Theorem of Specker and Łos no longer holds for $\prod_{\alpha\in M}^{HX} A_\alpha$.

§ 1. Countable subgroups of unrestricted nth nilpotent product of infinite cycles

Let G be the unrestricted nth nilpotent product of infinite cyclic groups A_i with generator $a_i (i=1,2,\ldots)$. By [18] Theorem 3.7,

$$z(G)=C({}^nG)=C({}^n\mathrm{F})/C({}^{n+1}\mathrm{F}),$$

where F is the unrestricted free product of the infinite cycles. Hence the upper central series of G is given by

$$z_l(G)=C({}^{n-l+1}G)$$

for $l=1,2,\ldots,n$. Now as a direct consequence of Hall's Basis Theorem (see e.g. [7]) for free nilpotent groups and the procedure of [18] Theorem 3.7, we have

(1) In fact our result states a little more than this. Cf. Theorem 4.4 and the example following Lemma 1.4.

LEMMA 1.1. *Let G_k denote the free n-th nilpotent group ot rank k. Then*

$$C(^lG)/C(^{l+1}G) = IL(^lG_k/^{l+1}G_k)$$

and hence
$$C(^lG)/C(^{l+1}G) = \prod_{i=1}^{\infty}{}^{HX}\{b_i(l)\cdot C(^{l+1}G)\},$$

where $b_i(l)$ $(i=1,2,\ldots)$ are the basic commutators of weight l on the elements $a_1, a_2, \ldots$.

COROLLARY 1.1.1. *Every element of G can be represented uniquely as an ordered product of the form*

$$\prod_{l=1}^{n}\left(\prod_{i=1}^{\infty}(b_i(l))^{\alpha_{il}}\right),$$

where α_{il} $(i=1,2,\ldots;\, l=1,2,\ldots,n)$ takes any integer value or zero.

Note 1.2. For each l,
$$\prod_{i=1}^{\infty}(b_i(l))^{\alpha_{il}}$$

is convergent. For from [19] Theorem 1.1 it follows that this element belongs to the unrestricted tth nilpotent product

$$\prod_{i=1}^{\infty}{}^{H(t)}\{b_i(l)\},$$

where (1)
$$t=\left[\frac{n}{l}\right].$$

Let $\Phi^{(k)}$ denote the natural homomorphism of G onto G_k which is obtained by mapping

$$a_{k+1}, a_{k+2}, \ldots$$

onto the unit element. We can now see, from the construction of the above infinite product, that the basic products of weight l can be so ordered that if

$$\Phi^{(k)}(b_i(l)) = b_i(l)$$

while
$$\Phi^{(k)}(b_j(l)) = 1$$

for some positive integer k, then $b_i(l)$ appears before $b_j(l)$ in the above infinite product. For convenience we often write $x^{(k)}$ instead of $\Phi^{(k)}(x)$.

(1) If r is a positive integer, then $[r]$ denotes the integral part of r.

The following fundamental lemma was proved for the unrestricted second nilpotent product in [18] Lemma 4.3.

LEMMA 1.3. *Let $h_1, h_2, \ldots, h_p$ $(p \geqslant 2)$ be elements of G, which are linearly independent modulo $z_{n-1}(G)$. Then the subgroup H generated by these elements is a free n-th nilpotent group and these elements are free generators.*

Proof. This is by induction on n. It is true for $n=1$, by Specker [22] (1). Suppose that the result holds for $\prod_{i=1}^{\infty}{}^{H(m)} A_i$ with $m<n$. By the induction hypothesis, $H \cdot z(G)/z(G)$ is isomorphic to

$$\prod_{i=1}^{p}{}^{(n-1)}\{h_i \cdot z(G)\} = \Big(\prod_{i=1}^{p}{}^{(n-1)}\{h_i\}\Big) \cdot z(G)/z(G) \tag{1}$$

in $G/z(G)$. Also by Specker [22], there exist infinite cyclic subgroups D_i/z_{n-1} $(i=1,2,\ldots,N)$ of G/z_{n-1} such that

$$\prod_{i=1}^{\infty}{}^{HX}(A_i \cdot z_{n-1}/z_{n-1}) = \Big(\prod_{i=1}^{N}{}^{X}(D_i/z_{n-1})\Big) \times \Big(\prod_{i>N}{}^{HX}(A_i \cdot z_{n-1}/z_{n-1})\Big) \tag{2}$$

and

$$H \cdot z_{n-1}/z_{n-1} \leqslant \prod_{i=1}^{N}{}^{X}(D_i/z_{n-1}) = D/z_{n-1}. \tag{3}$$

Because of (1), it is sufficient to show that

$$\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_s,$$

which denote the basic commutators of weight n in the elements $h_1, h_2, \ldots, h_p$, are linearly independent. Now suppose to the countrary that $\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_s$ are linearly dependent, then there exist integers ε_i (not all zero) such that

$$\bar{h}_1^{\varepsilon_1} \bar{h}_2^{\varepsilon_2} \ldots \bar{h}_s^{\varepsilon_s} = 1. \tag{4}$$

Apply $\Phi^{(N)}$ to (4) giving that

$$(\bar{h}_1^{(N)})^{\varepsilon_1} (\bar{h}_2^{(N)})^{\varepsilon_2} \ldots (\bar{h}_s^{(N)})^{\varepsilon_s} = 1, \tag{5}$$

where N is given by (2), which is a relation between the basic commutators of weight n in the elements $h_1^{(N)}, h_2^{(N)}, \ldots, h_p^{(N)}$. Hence $h_1^{(N)}, h_2^{(N)}, \ldots, h_p^{(N)}$ are not linearly independent modulo $z_{n-1}(G)$. For otherwise, by the Theorem of Mal'cev [14], they would freely

(1) Cf. also Fuchs [1] Theorem 47.1.

generate a free nth nilpotent group and this contradicts (5). Hence there exist integers α_i (not all zero) such that

$$(h_1^{(N)})^{\alpha_1} (h_2^{(N)})^{\alpha_2} \dots (h_p^{(N)})^{\alpha_p} \equiv 1 \text{ modulo } z_{n-1}(G). \quad (6)$$

We have shown that the element

$$h \cdot z_{n-1} = (h_1 \cdot z_{n-1})^{\alpha_1} (h_2 \cdot z_{n-1})^{\alpha_2} \dots (h_p \cdot z_{n-1})^{\alpha_p}$$

has the following properties:

(a) $h \cdot z_{n-1}$ is not the unit element, by assumption;
(b) $h \cdot z_{n-1}$ belongs to D/z_{n-1}, by (3);
(c) $h \cdot z_{n-1}$ belongs to $\prod_{i>N}^{HX}(A_i \cdot z_{n-1}/z_{n-1})$, by (6).

These facts, however, are inconsistent with the direct decomposition (2). Hence our assumption that

$$\bar{h}_1 \bar{h}_2, \dots, \bar{h}_s$$

are linearly dependent is false. Thus the required result follows from the induction hypothesis (1).

COROLLARY 1.3.1. *Let h_1, h_2, ... be a countable sequence of elements of G which is linearly independent modulo $z_{n-1}(G)$. Then the subgroup generated by these elements is a free n-th nilpotent group and the elements h_i are its free generators.*

LEMMA 1.4. *Let h_1, h_2, ... be a countable sequence of elements of $C(^lG)$ which is linearly independent modulo $C(^{l+1}G)$. Then the subgroup generated by these elements is a free t-th nilpotent group and the elements h_i are its free generators, where*

$$t = \left[\frac{n}{l}\right],$$

any fixed $l = 1, 2, \dots, n$.

Proof. Suppose that in the representation of the elements of G in the form given by Corollary 1.1.1, the element h_i has the representation

$$h_i = h_i(l) \cdot h_i',$$

where $h_i(l)$ and h_i' belong to $C(^lG)$ and $C(^{l+1}G)$ respectively, and h_i' belongs to $C(^lG)$ only if it is the unit element, for all i. Now using Hall's commutator collecting process [5], it follows that the required result holds for the elements h_i if and only if it holds for the elements $h_1(l), h_2(l), \ldots$. By [19] Theorem 1.1, the basic commutators of weight l on the elements $a_1, a_2, \ldots$, namely, $b_1(l)$, $b_2(l)$, ... are such that one can form the group

$$C(G_l) = \prod_{i=1}^{\infty}{}^{H(t)}\{b_i(l)\}$$

in G. Now the subgroup generated by the elements $h_1(l), h_2(l), \ldots$ is a subgroup of $C(G_l)$. As $h_1(l)$, $h_2(l)\ldots$, are linearly independent modulo $C({}^{l+1}G)$ they are also linearly independent modulo $z_{t-1}(C(G_l))$. Hence the required result follows from Corollary 1.3.1.

G' is not a closed subgroup in G, as the element

$$c = [a_1, a_2] \cdot [a_3, a_4] \ldots [a_{2m-1}, a_{2m}] \ldots$$

does not belong to G' (cf. [8]). This implies that it is impossible to find elements x_1, $x_2, \ldots, x_q$ of G, which are not contained in $C(G')$, such that (for $n > 1$)

$$\{c\} < \prod_{i=1}^{q}{}^{(n)}\{x_i\} < G.$$

This is in contrast to the case of the unrestricted direct product. However, in general we can state the following

THEOREM 1.5. *Let H be a finitely generated subgroup of G. Then there exists a positive integer N such that the natural endomorphism $\Phi^{(N)}$ induces an isomorphism of H onto a subgroup of the free n-th nilpotent group $\Phi^{(N)}(G)$ of rank N.*

Proof. As H is finitely generated,

$$(H \cap C({}^lG)) \cdot C({}^{l+1}G)/C({}^{l+1}G)$$

is a finitely generated subgroup of $C({}^lG)/C({}^{l+1}G)$. Hence, by Lemma 1.1 and Specker [22], there exist positive integers $N(l)$ and elements

$$d_1(l),\ d_2(l),\ \ldots,\ d_{N(l)}(l)$$

of $C({}^lG)$ such that

$$(H \cap C({}^lG)) \cdot C({}^{l+1}G)/C({}^{l+1}G) \leqslant \prod_{i=1}^{N(l)}{}^{X}\{d_i(l) \cdot C({}^{l+1}G)\}$$

and $$C({}^lG)/C({}^{l+1}G) = \Big(\prod_{i=1}^{N(l)}{}^X \{d_i(l)\cdot C({}^{l+1}G)\}\Big) \times \Big(\prod_{i>N(l)}{}^{HX}\{b_i(l)\cdot C({}^{l+1}G)\}\Big) \tag{7}$$

for all l. Let N be the minimum positive integer such that

$$\Phi^{(N)}(b_i(l)) = b_i(l)$$

for all $i \leqslant N(l)$ and all $l = 1, 2, \ldots, n$. This exists by the ordering given in Note 1.2.

We now show that N will serve as the number given in the statement of our theorem. Apply the natural endomorphism $\Phi^{(N)}$ to G. This induces an isomorphism of

$$\prod_{i=1}^{N(l)}{}^X \{d_i(l)\cdot C({}^{l+1}G)\} \text{ into } {}^lG_N/{}^{l+1}G_N \tag{8}$$

for all l, where $G_N = \Phi^{(N)}(G)$. For if

$$d_1(l)^{(N)},\ d_2(l)^{(N)},\ \ldots,\ d_{N(l)}(l)^{(N)}$$

are linearly dependent modulo ${}^{l+1}G_N$, then

$$d_1(l)^{\alpha_1} d_2(l)^{\alpha_2} \ldots d_{N(l)}^{\alpha_{N(l)}} \cdot C({}^{l+1}G)$$

belongs to $$\prod_{i>N}{}^{HX}\{b_i(l)\cdot C({}^{l+1}G)\}$$

which contradicts the decomposition (7), where $\alpha_1, \alpha_2, \ldots, \alpha_{N(l)}$ are integers (not all zero). Now let

$$d = \prod_{l=1}^{n}\prod_{i=1}^{N(l)} d_i(l)^{\alpha_i}$$

be an element of H which is mapped by $\Phi^{(N)}$ onto the unit element. It follows from repeated application of (8) for $l = 1, 2, \ldots, n$, that $d = 1$. Hence $\Phi^{(N)}$ induces a one-to-one mapping of H onto a subgroup of $\Phi^{(N)}(G)$ and hence is an isomorphism on H. Thus $\Phi^{(N)}$ is the required endomorphism.

THEOREM 1.6. *Let H be a countable subgroup of G such that $H/z(H)$ is finitely generated. Then H is isomorphic to a subgroup of a free n-th nilpotent group.*

Proof. Let $h_1, h_2, \ldots, h_p$ be a finite set of elements of H whose images form a set of generators of $H/z(H)$. Further let $H(1)$ be the subgroup of H generated by $h_1, h_2, \ldots, h_p$ and $H(1, n)$ be the subgroup of H generated by $H(1)$ and the isolator (1) of $H(1) \cap z(H)$

(1) Also known as servicing subgroup (cf. Kuroš [10] § 30).

in $z(H)$. Now $H(1, n)$ is a finitely generated subgroup of G and

$$H = H(1, n) \times A,$$

where A is a countable subgroup of $z(H)$. The required result will now follow from the previous theorem if we can show that A is free abelian. In fact, we state

LEMMA 1.7. *Every countable abelian subgroup of G is a free abelian group.*

Proof. This by induction on the class n. For $n = 1$, this is the content of Specker [22] Theorem 1. Suppose that the result holds for all unrestricted kth nilpotent products of infinite cycles, where $k < n$. Let A be a countable subgroup of G. Now by [18] Theorem 3.7,

$$G/z(G) = \prod_{i=1}^{\infty}{}^{H(n-1)} (A_i \cdot z/z)$$

and $A \cdot z/z$ is a countable subgroup of G/z. Hence, by the induction hypothesis,

$$A \cdot z/z \cong A/(A \cap z)$$

is a free abelian group. Further, by [18] Theorem 3.7, $A \cap z$ is a countable subgroup of the unrestricted direct product of infinite cycles and hence, by Specker, is free abelian. Finally we have that the abelian group A is the extension of a free abelian group by a free abelian group and hence must itself be free abelian.

In general, we have the following information concerning the countable subgroups of G.

THEOREM 1.8. *Let H be a countable subgroup of the unrestricted n-th nilpotent product G of a countable number of infinite cycles. Then H has a set of subgroups $H_1, H_2, \ldots, H_n$ which generate H, where*

(i) *H_l is a free nilpotent group of class $\left[\frac{n}{l}\right]$ $(l = 1, 2, \ldots, n)$;*

(ii) *$[H_i, H_j] \leqslant \{H_{i+j}, \ldots, H_n\}$ if $i + j \leqslant n$, and*

$$[H_i, H_j] = 1 \quad \textit{if} \quad i + j > n;$$

(iii)
$$H_l \cdot \{H_{m+1}, \ldots, H_n\} / \{H_{m+1}, \ldots, H_n\}$$

is a free nilpotent group of class $\left[\frac{m}{l}\right]$, freely generated by the images of the free generators of H_l, for $m = l, l+1, \ldots, n-1$ and $l = 1, 2, \ldots, n-1$;

(iv) *There exists a set of free generators of H_i $(i=1, 2, \ldots, n-1)$ that has the following properties. A subset of these free generators can be taken to be the maximal set of original commutators of weight i in H and. the images of the non-original commutators of weight l form a basis of the vector space*

$$\{[M(H_i), M(H_j)];\, i+j=l\}\cdot\{M(H_{l+1}), \ldots, M(H_n)\}/\{M(H_{l+1}), \ldots, M(H_n)\}$$

for $l=2, \ldots, n$.

Remarks. By [19] Theorem 3.4, every subgroup of a free nth nilpotent group has the above properties. On the other hand, if a group H satisfies the above properties and the torsion subgroup of

$$B_l\{B_{l+1}, \ldots, B_n\}/(\{[B_i, B_j];\, i+j=l\}\cdot\{B_{l+1}, \ldots, B_n\})$$

has finite exponent for $l=2, 3, \ldots, n$, then, by [19] Theorem 3.7, H is isomorphic to a subgroup of a free nth nilpotent group. Our use of the phrase "the torsion subgroup has finite exponent" does not conform to the standard usage. Note that this is to mean that the abelian group is the extension of an abelian group of bounded order by a free abelian group (either or both of which may be trivial). $M(H_i)$ denotes the Mal'cev completion of H_i. For the concepts of orginal and nonoriginal commutators see [19] Definition 3.2. However, we shall not explicity use these concepts here except the fact that they give a basis for the above vector space.

Proof. $(H \cap C(^lG))\cdot C(^{l+1}G)/C(^{l+1}G)$ is a countable subgroup of $C(^lG)/C(^{l+1}G)$ and hence, by Lemma 1.1 and the Theorem of Specker, is free abelian. Let $h_{\alpha(l)}$ be a typical element of a set of elements of $H \cap C(^lG)$ whose images form a basis for the free abelian group

$$(H \cap C(^lG))\cdot C(^{l+1}G)/C(^{l+1}G).$$

Let H_l be the subgroup of H generated by the elements $h_{\alpha(l)}$ $(l=1, 2, \ldots, n)$, then, by Lemma 1.4, H_l satisfies condition (i). Obviously,

$$H \cap C(^lG) = H_l \cdot (H \cap C(^{l+1}G))$$

for $l=1, 2, \ldots, n$. In particular, H is generated by the subgroups $H_1, H_2, \ldots, H_n$.

(ii) $$[H_i, H_j] \leqslant [C(^iG), C(^jG)] \cap H \leqslant C(^{i+j}G) \cap H$$

which gives the required result.

(iii) $H_l\cdot\{H_{m+1}, \dots, H_n\}/\{H_{m+1}, \dots, H_n\} = H_l\cdot(H\cap C({}^{m+1}G))/(H\cap C({}^{m+1}G))$

$$\cong H_l/(H_l\cap C({}^{m+1}G)) \cong H_l\cdot C({}^{m+1}G)/C({}^{m+1}G),$$

by repeated use of the Isomorphism theorem. Hence the latter subgroup is the image of H_l under the natural homomorphism which maps

$$\prod_{i=1}^{\infty}{}^{H(n)}A_i \quad \text{onto} \quad \prod_{i=1}^{\infty}{}^{H(m)}A_i.$$

For, by [18], the kernel of this homomorphism is

$$C({}^{m+1}\mathbf{F})/C({}^{n+1}\mathbf{F}) = C({}^{m+1}G),$$

where $\mathbf{F}$ is the unrestricted free product of the infinite cycles A_i. Thus the required result follows on using Lemma 1.4.

(iv) From the above constructed subgroup H_i, we take a finite set of generators which generate a subgroup H_i^* of finite rank for each i and satisfy the above conditions (i), (ii) and (iii). By Theorem 1.5 and [19] Theorem 3.4, there exists a set of free generators of H_i^* $(i=1, 2, \dots, n-1)$ and a subset of the free generators of H_l which can be taken to be the original commutators of weight l, such that the non-original commutators of weight l from a basis for the vector space

$$\{[M(H_i^*), M(H_j^*)]; i+j=l\} \text{ modulo } \{M(H_{l+1}^*), \dots, M(H_n^*)\}.$$

However, by [19] Lemma 3.6,

$$\{M(H_{l+1}^*), \dots, M(H_n^*)\} = M(\{H_1^*, H_2^*, \dots, H_n^*\}) \cap \{M(H_{l+1}), \dots, M(H_n)\}.$$

Hence it follows from the Isomorphism theorem that we can consider the above vector space modulo $\{M(H_{l+1}), \dots, M(H_n)\}$.

We now consider our given system as the union of an ascending sequence of systems whose generating free nilpotent subgroups all have finite ranks. B_i is the union of the corresponding B_i^*, for all i. This is, in fact, how our ascending sequence is constructed, namely, by considering successively the cases $i=1, 2, \dots, n$. Thus, by the above procedure, we have a subset of a set of free generators of H_i $(i=1, 2, \dots, n-1)$ which can be taken to be the original commutators of weight less than n, such that every finite subset of the set of non-original commutators of weight l are linearly independent modulo $\{M(H_{l+1}), \dots, M(H_n)\}$ over the rationals. Hence we have established property (iv).

§ 2. Unrestricted free Lie algebra over a field

Let L_k denote the free Lie algebra having the elements $x_1, x_2, \dots, x_k$ as its set of free generators over the field Ω $(k=1, 2, \dots)$. If $m>k$, then there exists a natural homomorphism of L_m onto L_k. We now form the projective limit $\mathbf{L}$ of these free Lie algebras under the above homomorphisms, and call $\mathbf{L}$ the *unrestricted free Lie algebra* over Ω.

We proceed as in the case of the unrestricted nilpotent product and give a convenient unique representation for the elements of $\mathbf{L}$. This is a direct consequence of Hall's Basis Theorem (see [4]) for a free Lie algebra. We first introduce the following

Notation. $\sum$ and $\sum^*$ will denote restricted and unrestricted sums of elements respectively. $\Phi^{(k)}$ denotes the natural homomorphism of $\mathbf{L}$ onto L_k which is obtained by mapping $x_{k+1}, x_{k+2}, \ldots$ onto the zero element of L_k. The image of an element x of $\mathbf{L}$ under $\Phi^{(k)}$ is denoted by $x^{(k)}$.

LEMMA 2.1. *Every element of* $\mathbf{L}$ *can be represented uniquely in the form*

$$\sum_{l=1}^{\infty}\Big(\sum_{i=1}^{\infty}{}^{*}\alpha_{il}\cdot b_i(l)\Big),$$

where $\alpha_{il}\in\Omega$ *for all values of* i *and* l. $b_i(l)$ *runs through all the basic monomials of weight* l *on the free generators* $x_1, x_2, \ldots$ *(for fixed* l*). In the unrestricted infinite sum the basic monomials of weight* l *are so ordered that if*

$$\Phi^{(k)}(b_i(l)) = b_i(l) \quad \textit{while} \quad \Phi^{(k)}(b_j(l)) = 0,$$

for some positive integer k, *then* $b_i(l)$ *appears before* $b_j(l)$ *in the unrestricted sum* $\sum^*$.

Notation. All the elements of $\mathbf{L}$ which involve only basic monomials of weight not less than l in the above representation form an ideal of $\mathbf{L}$ which we will denote by ${}_l\mathbf{L}$.

We come now to the main result of this section which will show that $\mathbf{L}$ is a free Lie algebra. We commence with

CONSTRUCTION 2.2. Firstly we notice that ${}_i\mathbf{L}/{}_{i+1}\mathbf{L}$ is a vector space over Ω, for all i, and hence it is possible to construct the following sets A_i. Let $A_1 = C_1$ be a set of elements of $\mathbf{L}$ that is linearly independent modulo ${}_2\mathbf{L}$. Suppose that the sets A_ν and C_ν have already been defined for all $\nu < n$ (where $n > 1$) and the elements of the sets $A_\nu (\nu = 1, 2, \ldots, n-1)$ have been ordered so that an element of A_ν is greater than an element of $A_{\nu'}$ if $\nu > \nu'$. We define C_n to be the set of all basic monomials on the elements of the sets $A_1, A_2, \ldots, A_{n-1}$ which belong to ${}_n\mathbf{L}$ but do not belong to ${}_{n+1}\mathbf{L}$. Finally A_n is a set(1) of elements of ${}_n\mathbf{L}$ which is linearly independent modulo the subalgebra(2) generated by ${}_{n+1}\mathbf{L}$ and the set C_n.

(1) In the language of our paper [19], A_n is a set of original monomials of weight n while C_n is a set of non-original monomials of weight n.

(2) This subalgebra is easily seen to be an ideal.

The following lemma is fundamental for our purposes:

LEMMA 2.3. *If $A_1, A_2, \dots, A_{n-1}$ are finite sets, then C_n is a set of linearly independent elements of ${}_n\mathbf{L}$ modulo ${}_{n+1}\mathbf{L}$ for $n = 1, 2, \dots$.*

Proof. We proceed by induction on n. The result is true, by construction, when $n = 1$. Suppose that the result is true for $C_1, C_2, \dots, C_{n-1}$. Now as these sets are finite, for every m, where $1 \leqslant m \leqslant n-1$, there exist elements $d_i(m)$ of ${}_m\mathbf{L}$ and positive integers $N(m)$ and $q(m)$ such that

$$ {}_m\mathbf{L}/{}_{m+1}\mathbf{L} = \Big(\sum_{i=1}^{q(m)} \{d_i(m) + {}_{m+1}\mathbf{L}\}\Big) + \Big(\sum_{i>N(m)}^{*} \{b_i(m) + {}_{m+1}\mathbf{L}\}\Big) $$

and
$$ (C_m \cup A_m) + {}_{m+1}\mathbf{L} \leqslant \sum_{i=1}^{q(m)} (\{d_i(m) + {}_{m+1}\mathbf{L}\}). \tag{9} $$

In the above, $\sum$ and $\sum^*$ denote restricted and unrestricted direct sums respectively, while $\sum_{i>N(m)}^*$ is to mean that those and only those basic monomials of weight m on $x_1, x_2, \dots$ occur in the unrestricted direct sum which satisfy the condition

$$ \Phi^{(N(m)+1)}(b_i(m)) = 0. $$

Suppose that contrary to our lemma, the elements of C_n are linearly dependent modulo ${}_{n+1}\mathbf{L}$, then there exist scalars γ_{n_i} (not all zero) such that

$$ c = \gamma_{n_1} c_{n_1} + \dots + \gamma_{n_k} c_{n_k} \text{ belongs to } {}_{n+1}\mathbf{L}. $$

Let N denote the maximum of $N(1), N(2), \dots, N(n-1)$. Apply the homomorphism $\Phi^{(N)}$ to c. Hence

$$ c^{(N)} = \gamma_{n_1} c_{n_1}^{(N)} + \dots + \gamma_{n_k} c_{n_k}^{(N)} \text{ belongs to } {}^{n+1}L_N. $$

This implies that for some $l (\leqslant n-1)$,

$$ C_l^{(N)} \cup A_l^{(N)}, $$

where the superscripts have their obvious meaning, must be a set of linearly dependent elements of ${}_l\mathbf{L}$ modulo ${}_{l+1}\mathbf{L}$. For, otherwise, by the Theorem of Širšov [21] (cf. [19] Theorem 2.3), the elements of the set

$$ A_1^{(N)} \cup A_2^{(N)} \cup \dots \cup A_{n-1}^{(N)} $$

freely generate a free Lie subalgebra of L_N and this contradicts the fact that $c^{(N)}$ belongs to ${}^{n+1}L_N$. Hence there exist scalars $\varepsilon_{li}, \varepsilon'_{li}$ (not all zero) such that

$$\varepsilon_{l1} a_{l1}^{(N)} + \dots + \varepsilon_{lk} a_{lk}^{(N)} + \varepsilon'_{l1} c_{l1}^{(N)} + \dots + \varepsilon'_{lk'} c_{lk'}^{(N)}$$

belongs to ${}_{l+1}\mathbf{L}$, where $a_{li} \in A_l$ and $c_{li} \in C_l$. Thus the element

$$\bar{a}_l = \varepsilon_{l1} a_{l1} + \dots + \varepsilon_{lk} a_{lk} + \varepsilon'_{l1} c_{l1} + \dots + \varepsilon'_{lk'} c_{lk'}$$

has the following properties:

(a) $\bar{a}_l$ does not belong to ${}_{l+1}\mathbf{L}$, by the induction hypothesis and Construction 2.2;

(b) $\bar{a}_l$ belongs to $\sum\limits_{i>N}^{*} (\{b_i(l) + {}_{l+1}\mathbf{L}\})$ modulo ${}_{l+1}\mathbf{L}$;

(c) $\bar{a}_l$ belongs to $\sum\limits_{i=1}^{q(l)} (\{d_i(l) + {}_{l+1}\mathbf{L}\})$ modulo ${}_{l+1}\mathbf{L}$.

The above three properties of the element $\bar{a}_l + {}_{l+1}\mathbf{L}$ contradict the direct decomposition (9) for ${}_l\mathbf{L}/{}_{l+1}\mathbf{L}$.

THEOREM 2.4. $\mathbf{L}$ *is a free Lie algebra over* Ω.

Proof. Let $A_1, A_2, \dots, A_n, \dots$ be maximal sets satisfying the conditions of Construction 2.2. The elements of C_n are linearly independent modulo ${}_{n+1}\mathbf{L}$. For, by Lemma 2.3, every finite subset of C_n is linearly independent modulo ${}_{n+1}\mathbf{L}$. Hence the elements of the set

$$\bigcup_{n=1}^{\infty} A_n$$

are a set of free generators for $\mathbf{L}$.

Notice that the above set of free generators for $\mathbf{L}$ does not coincide with the set A_1. For the element

$$(x_1, x_2) + (x_3, x_4) + \dots + (x_{2m-1}, x_{2m}) + \dots$$

is an element of ${}_2\mathbf{L}$ which is not contained in $(\mathbf{L}, \mathbf{L})$.

We can now see that the same situation holds for the *unrestricted free n-th nilpotent Lie algebra* $\mathfrak{L}$ over a field Ω. The latter is defined as the projective limit of the free n-th nilpotent Lie algebras $\mathfrak{L}_k$ on the free generators $x_1, x_2, \dots, x_k$ $(k = 1, 2, \dots)$ under the natural homorphisms defined as above. Reducing $\mathbf{L}$ modulo ${}_{n+1}\mathbf{L}$, we obtain the following consequence of Theorem 2.4.

THEOREM 2.5. *The unrestricted free n-th nilpotent Lie algebra over a field* Ω *is isomorphic to a subalgebra of a free n-th nilpotent Lie algebra over* Ω.

§ 3. The Mal'cev completion of the unrestricted nth nilpotent product of a countable number of infinite cycles

LEMMA 3.1. *Let $G^{(\alpha)}$, $\alpha \in M$, be an inverse system of torsion-free nilpotent groups under some set of homorphisms, such that their projective limit is a nilpotent group. Then*

$$M(IL(G^{(\alpha)})) \leqslant IL(M(G^{(\alpha)})).$$

Proof. Firstly we need to show that the groups $M(G^{(\alpha)})$ form an inverse system of groups under the appropriate homorphisms. Suppose $\pi_{\alpha\beta}: G^{(\alpha)} \to G^{(\beta)}$ is the given homomorphism of $G^{(\alpha)}$ into $G^{(\beta)}$, where $\alpha > \beta$. By Lazard [11] Theorem 4.10, there exists a unique homomorphism $\pi^*_{\alpha\beta}: M(G^{(\alpha)}) \to M(G^{(\beta)})$, which extends the homomorphism $\pi_{\alpha\beta}$. If $\alpha > \beta > \gamma$, then $\pi^*_{\alpha\beta}\,\pi^*_{\beta\gamma}$ is a homomorphism of $M(G^{(\alpha)})$ into $M(G^{(\gamma)})$, which extends $\pi_{\alpha\gamma}$. As such a homomorphism is unique, it follows that

$$\pi^*_{\alpha\beta}\ \pi^*_{\beta\gamma} = \pi^*_{\alpha\gamma}.$$

Let $g = \langle g^{(\alpha)} \rangle$ be an element of $IL(G^{(\alpha)})$ and n be any positive integer. Then $\langle (g^{(\alpha)})^{1/n} \rangle$ is an element of $IL(M(G^{(\alpha)}))$, by the construction(1) of $\pi^*_{\alpha\beta}$ from $\pi_{\alpha\beta}$. Hence

$$x^n = g$$

has a solution $\qquad x = \langle (g^{(\alpha)})^{1/n} \rangle$ belonging to $IL(M(G^{(\alpha)}))$,

which is unique as $IL(M(G^{(\alpha)}))$ is torsion-free. This shows that the Mal'cev completion of $IL(G^{(\alpha)})$ is contained in the inverse limit of the groups $M(G^{(\alpha)})$.

It is easy to see that, in general, it is not possible to turn the above inequality into an equality even in the case when all the groups are abelian.

THEOREM 3.2. *The Mal'cev completion of "the unrestricted n-th nilpotent product G of a countable number of infinite cycles" is isomorphic to the Mal'cev completion of "a subgroup of a free n-th nilpotent group".*

Proof. Let G_k denote the free nth nilpotent group of rank k and

$$\pi_{k+1}: G_{k+1} \to G_k \qquad (k = 1, 2, \ldots)$$

denote the natural homomorphism of G_{k+1} onto G_k. By Lazard [11] Theorem 4.10, there exists a unique homomorphism π^*_{k+1} of $M(G_{k+1})$ onto $M(G_k)$, which extends π_{k+1}. By a similar argument as that given in the previous lemma, we have that

(1) Cf. construction of π^* from π in Lazard [11].

$$\pi_{k+1}^*: M(G_{k+1}) \to M(G_k) \qquad (k=1, 2, \dots)$$

is an inverse system of groups and homomorphisms. By Lazard [11] § 4 especially Theorem 4.15 and the remark afterwards, π_{k+1}^* is the natural homomorphism of the free nth nilpotent Lie algebra $M(G_{k+1})$ of rank $k+1$ over the rationals onto the free nth nilpotent free Lie algebra $M(G_k)$ of rank k over the rationals. Now

$$M(IL_\pi(G_k)) \leqslant IL_{\pi^*}(M(G_k)).$$

by Lemma 3.1, and $IL_{\pi^*}(M(G_k))$ is the unrestricted free nth nilpotent Lie algebra $\mathfrak{L}$ over the rationals. By Theorem 2.5, $\mathfrak{L}$ is isomomorphic to a subalgebra K of a free nth nilpotent Lie algebra H over the rationals. By Lazard [11] § 4, H can also be considered as the Mal'cev completion of a free nth nilpotent group. Further, by Lazard [11] § 4, K is a divisible subgroup of the Mal'cev completion of a free nth nilpotent group P.

$$K = M(P \cap K)$$

and $P \cap K$ is a subgroup of P. For

$$K \geqslant M(P \cap K) \text{ as } K \geqslant P \cap K,$$

by Lazard [11] Theorem 4.10.

On the other hand, if k is an arbitrary element of K, then there exists a positive integer n such that k^n belongs to P. Hence, by Lazard [11] Theorem 4.9,

$$K \leqslant M(P \cap K).$$

Thus we have that $IL_{\pi^*}(M(G_k))$ is isomorphic to $M(P \cap K)$. Similarly we obtain the required result for the subalgebra $M(IL_\pi(G_k))$.

Because of the results of § 1, the following corollary to the above theorem is of interest.

COROLLARY 3.2.1. *The Mal'cev completion of "a countable subgroup of G" is isomorphic to the Mal'cev completion of "a countable subgroup of a free n-th nilpotent group".*

§ 4. Homomorphisms of unrestricted nth nilpotent product of infinite cycles into a free nilpotent group

We show that, as in the case of the unrestricted direct product, the only way of obtaining a subgroup of a free nilpotent group from the unrestricted nilpotent product of infinite cycles, by means of a homomorphism, is to map the unrestricted nilpotent product of most of the cycles onto the unit element. Throughout this section $A_i (i=1,2,\dots)$ denote infinite cycles.

Lemma 4.1. *If ψ is a homomorphism of $\prod_{i=1}^{\infty}{}^{H(n)}A_i$ into a free nilpotent group such that*

$$\psi\Big(\prod_{i=1}^{\infty}{}^{(n)}A_i\Big)=1,$$

then

$$\psi\Big(\prod_{i=1}^{\infty}{}^{H(n)}A_i\Big)=1.$$

Proof. We proceed by induction on n. By the Theorem of Specker and Łos (cf. [1], Theorems 47.2–47.4), the result is true for $n=1$. Actually we also need to use [19] Theorem 1.5, which states that every abelian subgroup of a free nilpotent group is free abelian. Suppose that the result is true for all unrestricted mth nilpotent products, where $m<n$. From

$$\psi\Big(\prod_{i=1}^{\infty}{}^{(n)}A_i\Big)=1$$

follows that (¹)

$$\psi\Big(\prod_{i=1}^{\infty}{}^{(t)}\{b_i(l)\}\Big)=1$$

for $l=1,2,\dots,n$, where $b_i(l)$ $(i=1,2,\dots)$ are the basic commutators of weight l on the elements $a_1, a_2, \dots$, by [19] Theorem 3.1. By Corollary 1.1.1 and [19] Theorem 3.1,

$$G=\prod_{i=1}^{\infty}{}^{H(n)}A_i=\Big\{\Big(\prod_{i=1}^{\infty}{}^{H(t)}\{b_i(l)\}\Big);\ \ l=1,2,\dots,n\Big\}$$

and

$$C([G,G])=\Big\{\Big(\prod_{i=1}^{\infty}{}^{H(t)}\{b_i(l)\};\ \ l=2,3,\dots,n\Big\}.$$

Hence, by the induction hypothesis, $\psi(C([G,G]))-1$. Thus ψ defines a homomorphism of $G/C(G')$ into A, where A is an abelian subgroup of a free nilpotent group. However, by [19] Theorem 1.5, A is free abelian. By [18] Corollary 1.6.1,

$$G/C(G')\cong\prod_{i=1}^{\infty}{}^{HX}A_i.$$

Hence ψ maps G onto the unit element, by the Theorem of Specker and Łos.

(¹) $t=\left[\dfrac{n}{l}\right]$.

LEMMA 4.2. *Let ψ be a homomorphism of $G=\prod_{i=1}^{\infty H(n)} A_i$ into a free abelian group. Then there exists some positive integer m such that*

$$g^{(m)}=1 \text{ implies that } \psi(g)=1$$

for all elements g of G.

Proof. We proceed by induction on n. The result is true for $n=1$, by the Theorem of Specker and Łos. Suppose that the result is true for all unrestricted mth nilpotent products, where $m<n$. ψ defines a homomorphism of $G/C(G')$ into a free abelian group A. For G' belongs to the kernel of ψ and this, by the argument given in the previous proof, shows that $C(G')$ also belongs to the kernel of ψ. Now, by the Theorem of Specker and Łos, there exists a positive integer m such that

$$g^{(m)} \text{ belongs to } C(G') \text{ implies that } \psi(g)=1,$$

which gives the required result. One has to use the fact that

$$G/C(G') \cong \prod_{i=1}^{\infty HX} A_i.$$

Before proving the main theorem of this section, we state the following simple consequence of [19] Theorem 1.6.

LEMMA 4.3. *If B is a nonabelian subgroup of a free n-th nilpotent group A, then*

$$z(B)=z_s(A) \cap B,$$

where s is some positive integer less than $(\frac{1}{2}n+1)$.

THEOREM 4.4. *Let ψ be a homomorphism of $G=\prod_{i=1}^{\infty H(n)} A_i$ into a free nilpotent group. Then there exists a positive integer m such that*

$$g^{(m)}=1 \text{ implies that } \psi(g)=1$$

for all elements g of C.

Proof. We proceed by induction on n. Let B denote the image of G under the homomorphism ψ. If B is abelian, then the required result follows from [19] Theorem 1.5 and Lemma 4.2. Hence we may assume that B is nonabelian. Then, by Lemma 4.3, there exists a positive integer s such that

$$z(B)=z_s(A) \cap B,$$

where A is the free nilpotent group of which B is a subgroup. Thus

$$z(G)\psi \leqslant z(G\psi) = z(B) = z_s(A) \cap B$$

and ψ induces a homomorphism of $G/z(G)$ onto

$$B/z(B) = B/(z_s(A) \cap B) \cong B \cdot z_s(A)/z_s(A).$$

By [18] Theorems 2.2 and 3.7,

$$G/z(G) \cong \prod_{i=1}^{\infty}{}^{H(n-1)} A_i.$$

By a result of Witt [23], $B \cdot z_s(A)/z_s(A)$ is isomorphic to a subgroup of the free nilpotent group $A/z_s(A)$. Hence, by the induction hypothesis, there exists a positive integer p such that

$$g^{(p)} = 1 \text{ implies that } \psi(g) \text{ belongs to } z_s(A).$$

In particular,

$$\psi\Big(\prod_{i=p+1}^{\infty}{}^{H(n)} A_i\Big) \leqslant z_s(A),$$

where $z_s(A)$ is free abelian. By Lemma 4.2, there exists a positive integer m ($\geqslant p+1$) which has the following property:

$$g^{(m)} = 1 \text{ implies that } \psi(g) = 1$$

for all elements g of G.

§ 5. Unrestricted soluble products of infinite cycles

Let G denote a free soluble group of derived length n, namely, $\mathfrak{F}/\mathfrak{F}^{[n]}$, where $\mathfrak{F}$ is a free group. Then the last member $G^{[n-1]}$ of the derived series of G has been characterized, by Mal'cev [15] Theorem 1, as the maximal normal abelian subgroup of G. It can also be considered to be the set of all left Engel elements or alternatively as the maximal normal locally nilpotent subgroup (1) of G. For our purposes, it is convenient to consider the set $R(X)$ of all left Engel elements of the group X. g is said to be a left Engel element of the group X if for each x of X there exists a positive integer h such that

$$[x, \underbrace{g, \ldots, g}_{h}] = 1.$$

(1) These two subgroups, in fact, coincide in a soluble group. Cf. for instance Gruenberg [3].

Note that $R(X)$ is a normal subgroup of the soluble group X and

$$R(X)\Phi \leqslant R(X\Phi)$$

for all homomorphisms Φ of the soluble group X.

As in free nilpotent groups we have the following result.

LEMMA 5.1. *Every abelian subgroup of a free soluble group is free abelian.*

Proof. This proceeds by induction on the derived length n of the free soluble group G. The result is true for $n=1$. Suppose that the result is true for all free soluble groups of derived length less than n. If A is an abelian subgroup of G, then

$$A\cdot G^{[n-1]}/G^{[n-1]} \cong A/(A \cap G^{[n-1]})$$

is an abelian subgroup of $G/G^{[n-1]}$ and hence, by the induction hypothesis, is free abelian. Thus A as an abelian group that is an extension of a free abelian group by a free abelian group is itself free abelian.

THEOREM 5.2. *The radical of the unrestricted n-th soluble product of a countable number of infinite cycles is*

$$C(\mathbf{F}^{[n-1]})/C(\mathbf{F}^{[n]}),$$

which is the unrestricted direct product of a countable number of infinite cycles(¹).

Proof. According to the above mentioned result of Mal'cev,

$$R(F_k/F_k^{[n]}) = F_k^{[n-1]}/F_k^{[n]},$$

which is a free abelian group of countably infinite rank, for $k=2, 3, \dots$. As previously F_k denotes the free group of rank k. The above system is an inverse system of groups and homomorphisms (induced by the natural homomorphisms of the groups $F_k/F_k^{[n]}$). It is easy to see from the definition of left Engel elements, that

$$R(IL(F_k/F_k^{[n]})) = IL(R(F_k/F_k^{[n]})) = IL(F_k^{[n-1]}/F_k^{[n]}) = C(\mathbf{F}^{[n-1]})/C(\mathbf{F}^{[n]})$$

by a similar argument to that given in [18] Theorem 3.7. The inverse limit is, in fact, the unrestricted direct product of the infinite cycles.

Now using the fundamental Lemma 1 of A. W. Mostowski [20] one can establish the following result by means of an induction argument on the derived length of a free soluble group.

(¹) $\mathbf{F}$ denotes the unrestricted free product of the cycles.

LEMMA 5.3. *Let G be a free n-th soluble group. A set of elements of $G^{[m]}$ which is linearly independent modulo $G^{[m+1]}$ freely generates a free suluble group of derived length $(n-m)$.*

The following subgroup theorem for free soluble groups is sufficient for our purposes (1).

THEOREM 5.4. *Every subgroup H of a free n-th soluble group is generated by a set of subgroups*

$$H_0, H_1, H_2, \dots, H_m, \dots, H_{n-1}$$

which are free soluble groups of derived length

$$n, n-1, n-2, \dots, n-m, \dots, 1$$

respectively. Moreover

$$[H_i, H_j] \leqslant \{H_{k+1}, \dots, H_{n-1}\},$$

where k is the minimum of i and j, if $k+1 \leqslant n-1$. While $[H_i, H_j]=1$ if i and j both exceed $n-1$. Further

$$H_i \cdot \{H_m, H_{m+1}, \dots, H_{n-1}\}/\{H_m, H_{m+1}, \dots, H_{n-1}\}$$

is a free $(m-i)$-th soluble group with the images of the free generators of H_i as a set of free generators, for $i=0, 1, \dots, m-1$ and $m=1, 2, \dots, n-1$.

Proof. It is sufficient to make the following observation. If we take a set of elements of $H \cap G^{[m]}$ whose images form a basis for the free abelian group

$$(H \cap G^{[m]}) \cdot G^{[m+1]}/G^{[m+1]},$$

then, by Lemma 5.3, they freely generate a free $(n-m)$th soluble group H_m in G. It is easy to see that the subgroups $H_0, H_1, H_2, \dots, H_{n-1}$ satisfy the above conditions. Of course, it is possible that some of these subgroups may have rank zero, that is, are trivial subgroups.

It is not difficult to verify that there exists a unique representation for every element of a free soluble group with free generators a_i in terms of the obvious complex commutators of the form (2)

$$a_1^{\alpha_1} a_2^{\alpha_2} \dots [a_i^{\beta_i}, a_j^{\beta_j}]^{\gamma_{ij}} \dots [[a_i^{\lambda_i}, a_j^{\lambda_j}]^{\mu_{ij}}, [a_k^{\lambda_k}, a_l^{\lambda_l}]^{\mu_{kl}}]^{\nu_{ijkl}} \dots .$$

(1) It is not difficult to verify that every countable subgroup of the unrestricted nth soluble product of infinite cycles is a group of this type. This follows from a result corresponding to Lemma 1.4.

(2) The powers occurring in the form are integers and only a finite number of them are nonzero.

Using this representation instead of representation by basic commutators and the radical instead of the centre, we can establish as above the following results for unrestricted soluble products.

THEOREM 5.5. *Every countable abelian subgroup of* $\prod_{i=1}^{\infty}{}^{H[n]} A_i$ *is a free abelian group.*

THEOREM 5.6. *If* ψ *is a homomorphism of* $\prod_{i=1}^{\infty}{}^{H[n]} A_i$ *into a free soluble group so that*

$$\psi\Big(\prod_{i=1}^{\infty}{}^{[n]}A_i\Big)=1,$$

then

$$\psi\Big(\prod_{i=1}^{\infty}{}^{H[n]}A_i\Big)=1.$$

THEOREM 5.7. *Let* ψ *be a homomorphism of* $G=\prod_{i=1}^{\infty}{}^{H[n]}A_i$ *into a free soluble group. Then there exists a positive integer* m *such that*

$$g^{(m)}=1 \ \textit{ implies that } \ \psi(g)=1$$

for all elements g *of* G.

In order to prove the above theorem we need the following simple result corresponding to Lemma 4.3.

LEMMA 5.8. *If* B *is a nonabelian subgroup of a free* n*-th soluble group* A, *then*

$$R(B)=R(A)\cap B.$$

§ 6. Unrestricted third Burnside product

The results in this section will only be briefly outlined as their proof proceeds in a similar way to that previously given for unrestricted nilpotent products. We will consider side by side the cases of the unrestricted third Burnside product of (a) infinite cycles (b) cycles of order three.

Every element of $\prod_{i=1}^{\infty}{}^{H3}A_i$ can be represented uniquely as an ordered product of the form

$$\prod_{l=1}^{3}\Big(\prod_{i=1}^{\infty}(b_i(l))^{\alpha_{il}}\Big),$$

where $b_i(l)$ $(i=1,2,\ldots)$ are the basic commutators of weight l on the elements $a_1, a_2, \ldots$. If the factors A_i are infinite cycles, then α_{i1} takes any integer value or zero. If the factors A_i are cycles of order three, then α_{i1} takes values 0, 1, 2. For $l=2, 3$, the values of α_{il} range over 0, 1, 2.

See Levi [12] for the unique representation of elements of a free third Burnside group and [17] Theorem 6.6 gives the corresponding representation for the third Burnside product of infinite cycles.

The upper central series of $\prod_{i=1}^{\infty}{}^{H3} A_i$ is given by

$$z = \prod_{i=1}^{\infty}{}^{HX} \{b_i(3)\}$$

$$z_2 = \left(\prod_{i=1}^{\infty}{}^{HX} \{b_i(2)\}\right) \times \left(\prod_{i=1}^{\infty}{}^{HX} \{b_i(3)\}\right)$$

and

$$z_3 = \prod_{i=1}^{\infty}{}^{H3} A_i.$$

z and z_2 are elementary abelian groups of exponent 3. Every finitely generated subgroup of $\prod_{i=1}^{\infty}{}^{H3} A_i$ is isomorphic to a subgroup of $\prod_{i=1}^{\infty}{}^{3} A_i$.

Every abelian subgroup of $\prod_{i=1}^{\infty}{}^{H3} A_i$ is a subgroup of the direct product of an elementary abelian group of exponent 3 and an infinite cyclic group.

In order to distinguish more precisely between the cases when A_i are cycles of order three and when A_i are infinite cycles, we must now consider a subgroup theorem for the third Burnside product in these two cases. It is obviously sufficient to consider nonabelian subgroups.

However, we first prove the following

LEMMA 6.1. *Let A be a free third Burnside group. Then any set of elements of A, which is linearly independent modulo A', freely generates a free third Burnside subgroup of A.*

Proof. As A/A' is elementary abelian of exponent 3, the given set of linearly independent elements can be expanded to a set S of elements of A whose images form a basis for A/A'. By Mal'cev [13] Theorem 5 a, S is a set of free generators for A, which gives the required result.

We also have the corresponding result for the third Burnside product of infinite cycles

LEMMA 6.2. *Let A be the third Burnside product of infinite cycles. Then any set of elements of A, which is linearly independent modulo A', freely generates a third Burnside product of infinite cycles.* (1)

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(1) Cf. Introduction for the fact that this is a relatively free group.

Proof. Let A_α, $\alpha \in M$, be infinite cyclic groups and $A = \prod_{\alpha \in M}^{3} A_\alpha$.

Then
$$A \cong (\mathfrak{F}/{}^4\mathfrak{F})/({}^2\mathfrak{F}/{}^4\mathfrak{F})^3,$$

where $\mathfrak{F}$ is the free group on the generators of the cyclic groups A_α. Now the required result follows from the corresponding theorem of Mal'cev [14] for free nilpotent groups.

THEOREM 6.3. *The following conditions are necessary and sufficient for a non-abelian group B to be isomorphic to a subgroup of a free third Burnside group.*

B is generated by subgroups B_1, B_2, and B_3 where

(i) *B_1 is a free third Burnside group, while B_2 and B_3 are elementary abelian groups of exponent 3;*

(ii) *If $i+j \leqslant 3$, then $[B_i, B_j] \leqslant \{B_{i+j}, B_3\}$, while $[B_i, B_j] = 1$ if $i+j > 3$;*

(iii) *$B_2 \cap B_3 = 1$;*

(iv) *Let d_β be a typical element of a set of elements of B_2, whose images form a basis for*
$$B_2 \cdot B_3/(B_3 \cdot [B_1, B_1]).$$

Then $[B_1, B_2]$ is freely generated by the basic commutators of weight three on the free generators of B_1 and all commutators of the form $[b_1, d_\beta]$, where b_1 and d_β traverse all the free generators of B_1 and all the above constructed elements of B_2 respectively.

Proof. First a few remarks about the necessity of the above conditions. Let B be a subgroup of a free third Burnside group G. Then, as in the Subgroup Theorem for free nilpotent groups, the subgroups B_1, B_2 and B_3 are constructed from sets of elements of B whose images form a basis for

$$B \cdot G'/G', \quad (B \cap G') \cdot {}^3G/{}^3G \quad \text{and} \quad B \cap {}^3G$$

respectively. Conditions (i) to (iii) follow from the construction of the subgroups and Lemma 6.1. Condition (iv) is a consequence of the fact that

$$B_2 = (\textstyle\prod^X \{b_2\}) \times (\prod_\beta^X \{d_\beta\}),$$

where b_2 runs over the basic commutators of weight two on the free generators of B_1. Now in order to prove the sufficiency of our conditions, we proceed as in the Subgroup Theorem for free nilpotent groups and consider the group

$$B/B_3 = (B_1 \cdot B_3/B_3) \times (\textstyle\prod_\beta^X (\{d_\beta\} \cdot B_3/B_3)).$$

From (ii) it follows that $B_1 \cdot B_3/B_3$ is the second nilpotent product of the subgroups generated by the images of the free generators of B_1. Take a free third Burnside group G with a sufficient number of free generators. B/B_3 can obviously be mapped isomorphically into $G/z(G)$. It remains to show that, as in the case of free nilpotent groups, this isomorphism can be extended to give an isomorphism of B into G. This causes no difficulties.

We state, without any further comment, the corresponding result for the third Burnside product of infinite cycles.

THEOREM 6.4. *The following conditions are necessary and sufficient for a non-abelian group B to be isomorphic to a subgroup of the third Burnside product of infinite cycles.*

B is generated by subgroups B_1, B_2 and B_3 where

(i) *B_1 is the third Burnside product of infinite cycles, while B_2 and B_3 are elementary abelian groups of exponent 3;*

(ii) *If $i+j \leqslant 3$, then $[B_i, B_j] \leqslant \{B_{i+j}, B_3\}$, while $[B_i, B_j] = 1$ if $i+j > 3$;*

(iii) *$B_2 \cap B_3 = 1$;*

(iv) *Same as* (iv) *of Theorem 6.3 with the above given different interpretation of B_1.*

As in the case of the unrestricted nilpotent product we are able to prove the following result for the unrestricted Burnside product.

LEMMA 6.5. *Every finite set of elements of $G = \prod_{i=1}^{\infty}{}^{H3} A_i$ (both when all the groups A_i are infinite cycles and when they are all cycles of order 3) which is linearly independent modulo $C(G')$ freely generates a third Burnside product of cycles.*

This enables us, with the help of the above Subgroup Theorems, to deduce the following main results.

THEOREM 6.6. *The unrestricted third Burnside product of a countable number of cycles of order three is isomorphic to a subgroup of a free third Burnside group.*

THEOREM 6.7. *Every countable subgroup of the unrestricted third Burnside product of infinite cycles is isomorphic to a subgroup of a third Burnside product of infinite cycles.*

The other main problem is solved by

THEOREM 6.8. *Let ψ be a homomorphism of the unrestricted third Burnside product*

G of a countably infinite number of infinite cycles onto a nonabelian subgroup of a third Burnside product A of infinite cycles. Then there exists a positive integer m such that

$$g^{(m)} = 1 \text{ implies that } \psi(g) \text{ belongs to } z_2(A)$$

for all g of G.

Proof. It is sufficient to point out the following facts. If B is the image of G under ψ, then

$$z_2(G)\,\psi \leqslant z_2(G\psi) = z_2(B) = z_2(A) \cap B.$$

Hence ψ induces a homomorphism of

$$G/z_2(G) \cong \prod_{i=1}^{\infty}{}^{HX} A_i,$$

where A_i are infinite cycles, onto

$$B/(z_2(A) \cap B) \cong B \cdot z_2(A)/z_2(A).$$

COROLLARY 6.8.1. *The unrestricted third Burnside product of a countably infinite number of infinite cycles is not isomorphic to a subgroup of a third Burnside product of infinite cycles.*

Generalizations. There is no difficulty in extending our above given results to the unrestricted products of an "arbitrary" number of cycles. However, we make the following relevant remarks. In the case of the above subgroup theorems, we use [18] Theorem 2.2 which gives a representation for unrestricted verbal products in terms of a factor group of the unrestricted free product of the same factors. For an arbitrary verbal product we have been able to prove this result only when the product has a countable number of factors. However, for unrestricted nilpotent soluble, and Burnside products of the cycles considered above, we have a unique representation which enables us to extend [18] Theorem 2.2 to an arbitrary number of factors. A similar situation holds for the unrestricted free nth nilpotent Lie algebra. In the theorems similar to the Theorem of Specker and Łos, we must, as mentioned in the introduction, confine our attention to a set of cycles such that the set has measure zero.

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