POTENTIAL TECHNIQUES FOR BOUNDARY VALUE PROBLEMS ON C^1-DOMAINS

BY

E. B. FABES, M. JODEIT JR. and N. M. RIVIÈRE

University of Minnesota, Minneapolis, Minn., U.S.A.

Introduction

In this work we consider the Dirichlet and Neumann problems for Laplace's equation in a bounded domain, D, of $\mathbf{R}^n$, $n \geq 3$. Assuming the boundary, ∂D, to be of class C^1 and the boundary data in $L^p(\partial D)$, $1 < p < \infty$, we resolve the above problems in the form of classical double and single layer potentials respectively. More precisely, given $g \in L^p(\partial D)$ we find a solution to the Dirichlet problem,

$$\Delta u = 0 \text{ in } D, \quad u|_{\partial D} = g,$$

in the form

$$u(X) = \frac{1}{\omega_n} \int_{\partial D} \frac{\langle X - Q, N_Q \rangle}{|X - Q|^n} (Tg)(Q)\, dQ, \quad X \in D$$

where T is a continuous operator from $L^p(\partial D)$ to $L^p(\partial D)$. Here N_Q denotes the unit inner normal to ∂D at Q, $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbf{R}^n$, and ω_n is the area of the surface of the unit ball in $\mathbf{R}^n$. (See Theorem 2.3.) Using the form of our solution and properties of the operator T we are able to obtain gradient estimates near the boundary when the data, g, has a derivative in $L^p(\partial D)$. (Theorem 2.4.) For the Neumann problem,

$$\Delta u = 0 \text{ in } D, \quad \frac{\partial u}{\partial N_Q} = g \text{ on } \partial D, \left(\int_{\partial D} g = 0 \right),$$

our solution is written in the form

$$u(X) = \frac{-1}{(n-2)\,\omega_n} \int_{\partial D} \frac{Sg(Q)}{|X - Q|^{n-2}}\, dQ \quad (n \geq 3),$$

where S is also a continuous map on the subspace of $L^p(\partial D)$ consisting of functions with integral or mean value zero. (Theorem 2.6.)

Recently Björn Dahlberg through a very careful study of the Poisson kernel of D resolved the Dirichlet problem in the case of C^1 domains for data in $L^p(\partial D)$, $1<p<\infty$, and in the case of Lipschitz domains for data in $L^p(\partial D)$, $2\leq p<\infty$. (See [3].) While Dahlberg's results did not cover the Neumann problem nor give the regularity mentioned above for the Dirichlet problem, there remains along the lines of this work the very open question of the use of the double and single layer potentials in the case of Lipschitz domains.

Throughout this work D will denote a bounded domain of $\mathbf{R}^n$. Points of D will generally be denoted by the capital letters X and Y, and points on the boundary of D, ∂D, will be represented by the capital letters P and Q.

Definition. $D\in C^1$ (or $\partial D\in C^1$) means that corresponding to each point $Q\in\partial D$ there is a system of coordinates of $\mathbf{R}^n$ with origin Q and a sphere, $B(Q,\delta)$, with center Q and radius $\delta>0$, such that with respect to this coordinate system

$$D\cap B(Q,\delta)=\{(x,t)\colon x\in\mathbf{R}^{n-1},\ t>\varphi(x)\}\cap B(Q,\delta)$$

where $\varphi\in C_0^1(\mathbf{R}^{n-1})$, the space of functions in $C^1(\mathbf{R}^{n-1})$ with compact support, and $\varphi(0)=(\partial\varphi/\partial x_i)(0)=0$, $i=1,\ldots,n-1$.

Remark. If $D\in C^1$ and $\varepsilon>0$ is given we can find a finite number of spheres, $\{B(Q_j,\delta_j)\}_{j=1}^m$, $Q_j\in\partial D$, such that $\partial D\subset\bigcup_{j=1}^m B(Q_j,\delta_j)$ and

$$D\cap B(Q_j,\delta_j)=\{(x,t)\colon t>\varphi_j(x)\}\cap B(Q_j,\delta_j)$$

with

$$\varphi_j\in C_0^1(R^{n-1}),\ \varphi_j(0)=\frac{\partial\varphi_j}{\partial x_i}(0)=0,\quad i=1,\ldots,n-1,$$

and

$$\max_x|\nabla\varphi_j(x)|\leq\varepsilon.\quad\left(\nabla\varphi_j(x)=\left(\frac{\partial\varphi_j}{\partial x_1}(x),\ldots,\frac{\partial\varphi_j}{\partial x_{n-1}}(x)\right)\right).$$

1. The double and single layer potentials over a C^1-domain

We begin this section with a discussion in local coordinates of the integral part of the trace of the double layer potential on the boundary of a C^1-domain.

Suppose $\varphi(x)\in C_0^1(\mathbf{R}^{n-1})$. For $x, z\in\mathbf{R}^{n-1}$, $x\neq z$,

set

$$k(x,z)=\frac{\varphi(x)-\varphi(z)-\langle\nabla\varphi(z),x-z\rangle}{[|x-z|^2+(\varphi(x)-\varphi(z))^2]^{n/2}}$$

and

$$\hat{K}_\varepsilon f(x) = \int_{|x-z|>\varepsilon} k(x,z) f(z)\,dz, \quad \varepsilon > 0.$$

LEMMA 1.1. *There exists* $m_0 > 0$ *such that if* $m = \max|\nabla\varphi| < m_0$ and $1 < p < \infty$ *then*

(a) *the operator* $\hat{K}_* f(x) = \sup_{\varepsilon>0}|\hat{K}_\varepsilon f(x)|$ *is bounded on* $L^p(\mathbf{R}^{n-1})$ *and* $\|\hat{K}_* f\|_{L^p} \leq C_m \|f\|_{L^p}$ *where* C_m *depends only on* m, p, n *and tends to zero when* $m \to 0+$,

(b) $\hat{K} f \equiv \lim_{\varepsilon\to 0+} \hat{K}_\varepsilon f$ *exists in* $L^p(\mathbf{R}^{n-1})$ *and pointwise almost everywhere* (*a.e. Lebesgue*).

Proof. Part (a) is a very special case of Theorem 4 in [2], whose proof will appear elsewhere. We will present here an argument for this case.

For $\varepsilon > 0$ and fixed

$$\hat{K}_\varepsilon f(x) = \tfrac{1}{2}\int_{|z|>\varepsilon} (k(x, x-z) f(x-z) + k(x, x+z) f(x+z))\,dz$$

$$= \tfrac{1}{2}\int_\Sigma T_{\sigma,\varepsilon} f(x)\,d\sigma$$

where $\Sigma = \{\sigma \in \mathbf{R}^{n-1}\colon |\sigma| = 1\}$, $d\sigma$ = usual surface measure on Σ,

$$T_{\sigma,\varepsilon} f(x) = \int_\varepsilon^\infty [k(x, x-r\sigma) f(x-r\sigma) + k(x, x+r\sigma) f(x+r\sigma)]\, r^{n-2} dr.$$

Setting $T_\sigma f(x) = \sup_{\varepsilon>0} |T_{\sigma,\varepsilon} f(x)|$ it is immediate that

$$\|\hat{K}_* f\|_{L^p(\mathbf{R}^{n-1})} \leq \tfrac{1}{2}\int_\Sigma \|T_\sigma f\|_{L^p(\mathbf{R}^{n-1})}\,d\sigma.$$

However for σ fixed each $x \in \mathbf{R}^{n-1}$ is uniquely written as $x = t\sigma + w$ where $t \in (-\infty, \infty)$ and $\langle \sigma, w\rangle = 0$. Hence

$$\int |T_\sigma f(x)|^p dx = \int \left(\int_{-\infty}^{\infty} |T_\sigma f(t\sigma + w)|^p dt \right) dw,$$

and it is easy to see that

$$T_\sigma f(t\sigma + w) = \sup_{\varepsilon>0} \left| \int_{|t-r|>\varepsilon} \frac{\varphi(t\sigma + w) - \varphi(r\sigma + w) - \langle \nabla\varphi(r\sigma + w), \sigma\rangle (t-r)}{[1 + \{(\varphi(t\sigma + w) - \varphi(r\sigma + w))/(t-r)\}^2]^{n/2}} \frac{f(r\sigma + w)}{(t-r)^2}\,dr \right|.$$

From A. P. Calderón's result ([1], Theorem 2) it follows that there is a number $m_0 > 0$ such that if $\max|\nabla\varphi| \equiv m < m_0$, then for $1 < p < \infty$,

$$\int |T_\sigma f(t\sigma + w)|^p dt \leq C_{p,m} \int |f(t\sigma + w)|^p dt,$$

where $C_{p,m}$ depends only on p and m and tends to zero with m. Since

$$\iint |f(t\sigma + w)|^p dt\, dw = \int_{\mathbf{R}^{n-1}} |f(x)|^p dx$$

part (a) of Lemma 1.1 follows.

Because of (a) and Lebesgue's dominated convergence theorem, to show part (b) it is sufficient to prove the existence of the pointwise limit almost everywhere in $\mathbf{R}^{n-1}$.

Since $\varphi \in C_0^1$ there exists a sequence $\{\psi_j\} \subset C_0^\infty$, the space of infinitely differentiable functions with compact support, such that

$$\psi_j \to \varphi, \quad \text{and} \quad \nabla\psi_j \to \nabla\varphi \quad \text{uniformly.}$$

Set

$$k_j(x, z) = \frac{\psi_j(x) - \psi_j(z) - \langle \nabla\psi_j(z), x - z\rangle}{[|x-z|^2 + (\varphi(x) - \varphi(z))^2]^{n/2}}$$

and

$$\hat{K}_{j,\varepsilon} f(x) = \int_{|x-z|>\varepsilon} k_j(x, z)\, f(z)\, dz.$$

For j fixed it is easy to see that $\lim_{\varepsilon\to 0+} \hat{K}_{j,\varepsilon} f(x)$ exists pointwise a.e. for $f \in L^p(\mathbf{R}^{n-1})$.

For a meaaurable set $E \subset \mathbf{R}^{n-1}$ let $|E|$ denote the Lebesgue measure of E. Also for $f \in L^p(\mathbf{R}^{n-1})$ set

$$\Delta(x) = \limsup_{\varepsilon\to 0} \hat{K}_\varepsilon f(x) - \liminf_{\varepsilon\to 0} \hat{K}_\varepsilon f(x).$$

and

$$\Delta_j(x) = \limsup_{\varepsilon\to 0} \hat{K}_{j,\varepsilon} f(x) - \liminf_{\varepsilon\to 0} \hat{K}_{j,\varepsilon} f(x).$$

Since $\Delta_j(x) = 0$ a.e.

$$|\{x: \Delta(x) > \lambda > 0\}| = |\{x: \Delta(x) - \Delta_j(x) > \lambda\}|$$
$$\leqslant |\{x: \sup_{\varepsilon>0} |(\hat{K}_\varepsilon - \hat{K}_{j,\varepsilon}) f(x)| > \lambda/2\}|.$$

As a consequence of the argument of Theorem 1 in [1] the measure of the last set is $e_j \lambda^{-p} \int |f|^p dx$ where $e_j \to 0$ as $j \to \infty$. Hence $|\{x: \Delta(x) > \lambda > 0\}| = 0$, and this implies $\Delta(x) = 0$ a.e.

Now set ω_n equal to the area of the unit sphere in $\mathbf{R}^n$. For $P \in \partial D$ we will let

$$K_\varepsilon f(P) = \frac{1}{\omega_n} \int_{|P-Q|>\varepsilon} \frac{\langle P-Q, N_Q\rangle}{|P-Q|^n} f(Q)\, dQ, \quad \varepsilon > 0,$$

and

$$K_* f(P) = \sup_{\varepsilon>0} |K_\varepsilon f(P)|.$$

THEOREM 1.2. *If* $D \in C^1$ *and* $1<p<\infty$ *then*

(a) $\|K_* f\|_{L^p(\partial D)} \leqslant C\|f\|_{L^p(\partial D)}$ *with* C *depending only on* p, ∂D, *and* n;

(b) $$\lim_{\varepsilon\to 0} K_\varepsilon f \equiv \text{p.v.} \frac{1}{\omega_n}\int_{\partial D} \frac{\langle P-Q, N_Q\rangle}{|P-Q|^n} f(Q)\,dQ \equiv Kf(P)$$

exists in $L^p(\partial D)$ *and pointwise for a.e.* $P\in\partial D$;

(c) K *is compact.*

Proof. By the use of a partition of unity the proof a part (a) is reduced to showing that the Euclidean operator,

$$\sup_{\varepsilon>0}\left|\int_{|x-z|^2+(\varphi(x)-\varphi(z))^2>\varepsilon^2} \frac{\varphi(x)-\varphi(z)-\langle\Delta\varphi(z), x-z\rangle}{[|x-z|^2+(\varphi(x)-\varphi(z))^2]^{n/2}} f(z)\,dz\right|$$

maps $L^p(\mathbf{R}^{n-1})$ continuously into itself where $\varphi\in C_0^1(\mathbf{R}^{n-1})$ and $\max|\nabla\varphi|<m_0$, m_0 being the constant given by Lemma 1.1. If we set $m=\max|\nabla\varphi|$ and observe that

$$\{z: \sqrt{|x-z|^2+(\varphi(x)-\varphi(z))^2}>\varepsilon\}$$
$$=\left\{z: |x-z|>\frac{\varepsilon}{\sqrt{1+m^2}}\right\}\setminus\left\{z: |x-z|>\frac{\varepsilon}{\sqrt{1+m^2}} \quad\text{and}\quad \sqrt{|x-z|^2+(\varphi(x)-\varphi(z))^2}\leqslant\varepsilon\right\},$$

then the above operator is bounded by

$$\sup_{\varepsilon>0}\left|\int_{|x-z|<\varepsilon} k(x,z)f(z)\,dz\right| + 2(1+m^2)^{n-1/2}\sup_{\varepsilon>0}\frac{1}{\varepsilon^{n-1}}\int_{|x-z|<\varepsilon}|f(z)|\,dz$$

where

$$k(x,z)=\frac{\varphi(x)-\varphi(z)-\langle\nabla\varphi(z), x-z\rangle}{[|x-z|^2+(\varphi(x)-\varphi(z))^2]^{n/2}}.$$

The second term in the above sum is of course equal to a constant times the Hardy-Littlewood maximal function of f (see [6]) and, hence, is continuous on $L^p(\mathbf{R}^{n-1})$. That the first term is also continuous on $L^p(\mathbf{R}^{n-1})$ follows from Lemma 1.1.

As usual the proof of part (b) is completed once we have shown the existence of the limit, in $L^p(\partial D)$ or pointwise a.e., for a dense class of $L^p(\partial D)$, say $C^1(\partial D)$. Hence, assume $f\in C^1(\partial D)$. Now

$$K_\varepsilon f(P)=\frac{1}{\omega_n}\int_{|P-Q|>\varepsilon}\frac{\langle P-Q, N_Q\rangle}{|P-Q|^n}[f(Q)-f(P)]\,dQ + f(P)\frac{1}{\omega_n}\int_{|P-Q|>\varepsilon}\frac{\langle P-Q, N_Q\rangle}{|P-Q|^n}\,dQ.$$

It is clear that the first term is a bounded function of $P\in\partial D$ and $\varepsilon>0$, and, moreover, converges pointwise as $\varepsilon \to 0+$. For the second term we have

$$\int_{|P-Q|>\varepsilon} \frac{\langle P-Q, N_Q\rangle}{|P-Q|^n} dQ = -\int_{D\cap\partial B(P,\varepsilon)} \frac{\langle P-Q, N_Q\rangle}{|P-Q|^n} dQ$$

and from this identity follows the boundedness in $\varepsilon > 0$ and $P \in \partial D$ and the pointwise convergence of the second term to $f(P)\omega_n/2$.

We now set $Kf(P) = \lim_{\varepsilon\to 0} K_\varepsilon f(P)$. Again through the use of a partition of unity in order to prove K is compact it will suffice to show the compactness on $L^p(B)$,

$$B = \{x \in \mathbf{R}^{n-1}, |x| \leq 1\},$$

of the Euclidean operator

$$\lim_{\varepsilon\to 0} \int_{|x-z|^2+(\varphi(x)-\varphi(z))^2>\varepsilon^2} k(x,z) f(z)\, dz$$

where

$$k(x,z) = \frac{\varphi(x)-\varphi(z)-\langle\nabla\varphi(z), x-z\rangle}{[|x-z|^2+(\varphi(x)-\varphi(z))^2]^{n/2}}, \quad \varphi \in C_0^1(R^{n-1})$$

and $\max|\nabla\varphi| < m_0$, the constant of Lemma 1.1. For any $\eta > 0$ we can write the above limit as

$$\int_{|x-z|>\eta} k(x,z) f(z)\, dz + \lim_{\varepsilon\to 0} \int_{\substack{|x-z|<\eta \\ |x-z|^2+(\varphi(x)-u(z))^2>\varepsilon^2}} k(x,z) f(z)\, dx.$$

Observing that the L^p-norm of the second function tends to zero we conclude that

$$\lim_{\varepsilon\to 0+} \int_{|x-z|^2\cdot(\varphi(x)-\varphi(z))^2>\varepsilon^2} k(x,z) f(z)\, dz = \lim_{\varepsilon\to 0+} \int_{|x-z|>\varepsilon} k(x,z) f(z)\, dz \quad \text{(a.e.)}.$$

Since $\varphi \in C_0^1(\mathbf{R}^{n-1})$ there exists a sequence $\{\psi_j\} \subset C_0^\infty(\mathbf{R}^{n-1})$ with supports contained in a fixed compact subset of $\mathbf{R}^{n-1}$ such that $\psi_j \to \varphi$ and $\nabla\psi_j \to \nabla\varphi$ uniformly in $\mathbf{R}^{n-1}$. Set

$$k_j(x,z) = \frac{\psi_j(x)-\psi_j(z)-\langle\nabla\psi_j(z), x-z\rangle}{[|x-z|^2+(\varphi(x)-\varphi(z))^2]^{n/2}}.$$

The operator $\int_{\mathbf{R}^{n-1}} k_j(x,z) f(z)\, dz$ is easily seen to be compact on $L^p(B)$ and, using A. P. Calderón's result, ([1]), the operator,

$$\lim_{\varepsilon\to 0+} \int_{|x-z|>\varepsilon} (k(x,z)-k_j(x,z)) f(z)\, dz$$

has norm on $L^p(B)$ small with j large. This implies the compactness of the operator

$$\lim_{\varepsilon\to 0+}\int_{|x-z|>\varepsilon} k(x,z)f(z)\,dz.$$

In the next theorem we consider the double layer potential over ∂D with $D\in C^1$ and study its behavior near the surface when the density of the potential belongs to L^p.

THEOREM 1.3. *For $D\in C^1$ and $f\in L^p(\partial D)$, $1<p<\infty$, set*

$$u(X)=\frac{1}{\omega_n}\int_{\partial D}\frac{\langle X-Q, N_Q\rangle}{|X-Q|^n}f(Q)\,dQ, \quad x\in D.$$

Given α, $0<\alpha<1$, there exists a constant $\delta\equiv\delta_{\alpha,D}$ such that the non-tangential maximal function of u, i.e.

$$u^*(P)=\sup\{|u(X)|:\ X\in D,\ |X-P|<\delta,\ \langle X-P, N_p\rangle>\alpha|X-P|\},$$

belongs to $L^p(\partial D)$ and $\|u^*\|_{L^p(\partial D)}\leqslant C\|f\|_{L^p(\partial D)}$ with C independent of f. As a consequence

$$u(X)\to\tfrac{1}{2}f(P)+Kf(P) \text{ pointwise for almost every } P\in\partial D$$

as $X\to P$, $X\in D$, $\langle X-P, N_p\rangle>\alpha|X-P|$.

Proof. We cover ∂D with a finite number of balls, $B_j\equiv B(P_j,\delta_j)$, $j=1,\ldots,l$, with center $P_j\in\partial D$ and radius δ_j so that

$$B(P_j,4\delta_j)\cap D=B(P_j,4\delta_j)\cap\{(x,t): x\in\mathbf{R}^{n-1}, t>\varphi(x)\} \quad\text{and}\quad |\nabla\varphi|<\frac{\alpha}{6}.$$

Using a partition of unity subordinate to the cover B_j, $j=1,\ldots,l$ we may assume the support of f is contained in B_j.

Set $\delta=\min\{\delta_j,\ j=1,\ldots,l\}$. If $P\notin B(P_j,3\delta_j)$ and $|X-P|<\delta$ then for all $Q\in B_j$, $|X-Q|\geqslant\delta_j$ and

$$u^*(P)\leqslant\frac{C}{\delta_j^{n-1}}\int_{B_j}|f|\leqslant C\|f\|_{L^p(\partial D)}.$$

In the case $P\in B(P_j,3\delta_j)$ we have $X\in B(P_j,4\delta_j)$. Using then the coordinate system described above, we see that the inequality

$$\|u^*\|_{L^p(\partial D)}\leqslant C\|f\|_{L^p(\partial D)}$$

will be valid once we can show that the operator

$$Tf(x_0)=\sup\left\{\left|\int\frac{t-\varphi(z)-\langle\nabla\varphi(z),x-z\rangle}{[|x-z|^2+(t-\varphi(z))^2]^{n/2}}f(z)\,dz\right|:(x,t),t>\varphi(x)\quad\text{and}\right.$$
$$\left.t-\varphi(x_0)-\langle\nabla\varphi(x_0),x-x_0\rangle>\alpha\sqrt{1+|\nabla\varphi(x_0)|^2}\sqrt{|x-x_0|^2+(\varphi(x)-\varphi(x_0))^2}\right\}$$

is a bounded map from $L^p(\mathbf{R}^{n-1})$ to itself.

We set $k(t, x; z)$ equal to the kernel of the above integral. We observe that the conditions on (x, t) and $\max|\nabla\varphi|$ imply the inequality

$$t-\varphi(x_0)>(\alpha-|\nabla\varphi(x_0)|)|x-x_0|\geqslant\tfrac{5}{6}\alpha|x-x_0|.$$

Hence,

$$\int_{|x_0-z|\leqslant\max(3|x-x_0|,\,t-\varphi(x_0))}|k(t,x;z)|\,|f(z)|\,dz\leqslant CMf(x_0)$$

where

$$Mf(x_0)=\sup_{r>0}\frac{1}{r^{n-1}}\int_{|z-x_0|<r}|f(z)|\,dz,$$

the classical Hardy-Littlewood maximal function.

Now set $\lambda=\max(3|x-x_0|,\ t-\varphi(x_0))$. We have

$$\int_{|z-x_0|>\lambda}k(t,x;z)f(z)\,dz=\int_{|z-x_0|>\lambda}k(t,x_0;z)f(z)\,dz+\int_{|z-x_0|>\lambda}(k(t,x;z)-k(t,x_0;z))f(z)\,dz.$$

Since

$$|k(t,x;z)-k(t,x_0;z)|\leqslant C\frac{|x-x_0|}{|x_0-z|^n},$$

the second integral is majorized by $CMf(x_0)$. Finally

$$\left|\int_{|z-x_0|>\lambda}k(t;x_0;z)f(z)\,dz\right|\leqslant\int_{|z-x_0|>\lambda}|k(t,x_0;z)-k(\varphi(x_0),x_0;z)|\,|f(z)|\,dz+\hat{K}_*f(x_0)$$

where $\hat{K}_*$ is the operator introduced in Lemma 1.1. Using the fact that

$$|k(t,x_0;z)-k(\varphi(x_0),x_0;z)|\leqslant C\frac{t-\varphi(x_0)}{|x_0-z|^n}$$

the first term on the right side above is bounded by $CMf(x_0)$. We have finally shown that $Tf(x_0)\leqslant C(Mf(x_0)+\hat{K}_*f(x_0))$ and, hence,

$$\|Tf\|_{L^p(\mathbf{R}^{n-1})}\leqslant C\|f\|_{L^p(\mathbf{R}^{n-1})},\ 1<p<\infty.$$

Since the map $f\to u^*$ is bounded on $L^p(\partial D)$, to show the nontangential pointwise limit of

u a.e. on ∂D it is sufficient to show the pointwise limit for $f \in C^1(\partial D)$, a dense subspace of $L^p(\partial D)$. For if we let $u_f(X)$ denote the double layer potential with density f and

$$\Delta_f(P) = \limsup_{\substack{X\to P,\, X\in D \\ \langle X-P, N_P\rangle > \alpha|X-P|}} u_f(X) - \liminf_{\substack{X\to P,\, X\in D \\ \langle X-P, N_P\rangle > \alpha|X-P|}} u_f(X),$$

then $|\{P: \Delta_f(P) > \lambda > 0\}| = |\{P: \Delta_{f-g}(\mathrm{P}) > \lambda > 0\}|$ for any $g \in C^1(\partial D)$, where $|E|$ denotes the (surface) measure of the set $E \subset \partial D$. Hence

$$|\{P: \Delta_f(P) > \lambda\}| \leqslant |\{P: u^*_{f-g}(P) > \lambda/2\}| \leqslant \frac{C}{\lambda^p} \|f-g\|^p_{L^p(\partial D)}$$

for any $g \in C^1(\partial D)$. This immediately implies that $|\{P: \Delta_f(P) > \lambda > 0\}| = 0$ and the almost everywhere nontangential limit of u_f for $f \in L^p$.

So now assume $f \in C^1(\partial D)$. Then

$$u(X) = \frac{1}{\omega_n} \int_{\partial D} \frac{\langle X-Q, N_Q\rangle}{|X-Q|^n} (f(Q) - f(P))\, dQ + f(P).$$

It is easy to see that when $X \to P$ we can pass the limit inside the integral sign and, therefore,

$$\begin{aligned} \lim_{X\to P} u(X) &= \frac{1}{\omega_n} \int_{\partial D} \frac{\langle P-Q, N_Q\rangle}{|P-Q|^n} (f(Q) - f(P))\, dQ + f(P) \\ &= \lim_{\varepsilon\to 0+} \frac{1}{\omega_n} \int_{|P-Q|>\varepsilon} \frac{\langle P-Q, N_Q\rangle}{|P-Q|^n} (f(Q) - f(P))\, dQ + f(P) \\ &= \tfrac{1}{2} f(P) + Kf(P). \end{aligned}$$

Remark. In the case $f \in L^1(\partial D)$ the double layer potential, $u(X)$, with density f has the property that u^* belongs to weak-$L^1(\partial D)$ i.e. there exists a constant C such that for each $\lambda > 0$,

$$|\{P: u^*(P) > \lambda\}| \leqslant \frac{C}{\lambda} \int_{\partial D} |f|\, dQ.$$

This inequality is valid because the operator $\hat{K}$ of Lemma 1.1 is bounded from $L^1(\mathbf{R}^{n-1})$ into weak-$L^1(\mathbf{R}^{n-1})$ and this in turn implies that the operator K_* and, therefore, K are bounded from $L^1(\partial D)$ into weak-$L^1(\partial D)$. Exactly as in the proof of Theorem 1.3 we conclude that $u(X) \to \frac{1}{2} f(P) + Kf(P)$ nontangentially for a.e. $P \in \partial D$ when $f \in L^1(\partial D)$. (Specifically we mean, as before, that $u(X) \to \frac{1}{2} f(P) + Kf(P)$ for a.e. $P \in \partial D$ as $X \to P$, $X \in D$, $\langle X-P, N_p\rangle > \alpha|X-P|$.)

We now turn to the study of the regularity of the double layer potential when the density is regular. We begin by studying the behavior of the Euclidean operator, $\hat{K}$, on smooth functions.

We will denote by $L_1^p(\mathbf{R}^{n-1})$ the space of functions $f(x)\in L^p(\mathbf{R}^{n-1})$ whose gradient, ∇f, also belongs to $L^p(\mathbf{R}^{n-1})$. We set

$$\|f\|_{L_1^p(\mathbf{R}^{n-1})} = \|f\|_{L^p(\mathbf{R}^{n-1})} + \|\nabla f\|_{L^p(\mathbf{R}^{n-1})}.$$

LEMMA 1.4. *Suppose* $\varphi\in C_0^\infty(\mathbf{R}^{n-1})$. *For* $f\in L_1^p(\mathbf{R}^{n-1})$, $1<p<\infty$, *set*

$$\hat{K}f(x) = \int_{\mathbf{R}^{n-1}} k(x,z)\,f(z)\,dz$$

where once again

$$k(x,z) = \frac{\varphi(x)-\varphi(z)-\langle\nabla\varphi(z),x-z\rangle}{[|x-z|^2+(\varphi(x)-\varphi(z))^2]^{n/2}}.$$

Then

$$\frac{\partial}{\partial x_i}\hat{K}f(x) = \int_{\mathbf{R}^{n-1}} \frac{\partial k}{\partial x_i}(x,z)\,(f(z)-f(x))\,dz, \quad i=1,\dots,n-1.$$

Proof We first establish the formula for $f\in C_0^\infty(\mathbf{R}^{n-1})$. Let $e_1, \dots, e_{n-1}$ denote the standard basis of $\mathbf{R}^{n-1}$. Since $\int_{\mathbf{R}^{n-1}} k(x,z)\,dz$ is constant we can write

$$\begin{aligned}\frac{\hat{K}f(x+he_i)-\hat{K}f(x)}{h} &= \int_{\mathbf{R}^{n-1}} \frac{(k(x+he_i,z)-k(x,z))}{h}\,(f(z)-f(x))\,dz \\ &= \int_{|x-z|>2|h|} + \int_{|x-z|\leqslant 2h} \frac{(k(x+he_i,z)-k(x,z))}{h}\,(f(z)-f(x))\,dz \\ &\equiv A_h(x)+B_h(x).\end{aligned}$$

It is easy to see that $A_h(x)$ converges to

$$\int_{\mathbf{R}^{n-1}} \frac{\partial k}{\partial x_i}(x,z)\,(f(z)-f(x))\,dz$$

and $B_h(x)\to 0$ when $h\to 0$.

To obtain the formula for $f\in L_1^p(R^{n-1})$ we first note that when $\varphi\in C_0^\infty(\mathbf{R}^{n-1})$ the operator

$$\int \frac{\partial k}{\partial x_i}(x,z)\,(f(z)-f(x))\,dz$$

is continuous from $L_1^p(\mathbf{R}^{n-1})$ to $L^p(\mathbf{R}^{n-1})$, $1<p<\infty$. This implies that $\hat{K}$ maps $L_1^p(\mathbf{R}^{n-1})$ into itself continuously, and we also obtain the formula,

$$\frac{\partial}{\partial x_i}\hat{K}f(x)=\int\frac{\partial k}{\partial x_i}(x,z)\,(f(z)-f(x))\,dz.$$

THEOREM 1.5. *Suppose $\varphi(x)\in C_0^1(R^{n-1})$ and $|\nabla\varphi(x)|<m_0$, the constant of Lemma 1.1. Then for $1<p<\infty$ the operator*

$$\hat{K}f(x)=\lim_{\varepsilon\to 0}\int_{|x-z|>\varepsilon}k(x,z)f(z)\,dz$$

maps $L_1^p(\mathbf{R}^{n-1})$ continuously into itself. Moreover

$$\frac{\partial}{\partial x_i}\hat{K}f(x)=\lim_{\varepsilon\to 0}\int_{|x-z|>\varepsilon}\frac{\partial k}{\partial x_i}(x,z)\,(f(z)-f(x))\,dz.$$

All of the above limits exist in $L^p(\mathbf{R}^{n-1})$ and pointwise almost everywhere. (See [2].)

Proof. Since $\varphi\in C_0^1(\mathbf{R}^{n-1})$ there exists $\{\varphi_j\}\subset C_0^\infty(\mathbf{R}^{n-1})$ such that $\varphi_j\to\varphi$ and $\nabla\varphi_j\to\nabla\varphi$ uniformly. We may then suppose that $\max|\nabla\varphi_j|<m_0$ for all j.
Set

$$k_j(x,z)=\frac{\varphi_j(x)-\varphi_j(z)-\langle\nabla\varphi_j(z),x-z\rangle}{[|x-z|^2+(\varphi_j(x)-\varphi_j(z))^2]^{n/2}}$$

and

$$\hat{K}_jf(x)=\int_{\mathbf{R}^{n-1}}k_j(x,z)f(z)\,dz.$$

From Lemma 1.4

$$\frac{\partial}{\partial x_i}\hat{K}_jf(x)=\int_{\mathbf{R}^{n-1}}\frac{\partial}{\partial x_i}k_j(x,z)\,(f(z)-f(x))\,dz$$

and as a special case of Theorem 4 in [2] we have

$$\left\|\frac{\partial}{\partial x_i}\hat{K}_jf\right\|_{L^p(\mathbf{R}^{n-1})}\leqslant C\|f\|_{L_1^p(\mathbf{R}^{n-1})}$$

where C is independent of j. Hence $\hat{K}_j\colon L_1^p(\mathbf{R}^{n-1})\to L_1^p(\mathbf{R}^{n-1})$ continuously with norm bounded independently of j. Since $\hat{K}_j f \to \hat{K} f$ in $L^p(\mathbf{R}^{n-1})$ we conclude that $\hat{K} f \in L_1^p(\mathbf{R}^{n-1})$ whenever $f \in L_1^p(\mathbf{R}^{n-1})$ and $\|\hat{K} f\|_{L_1^p} \leqslant C \|f\|_{L_1^p}$. Also, in the sense of distributions,

$$\frac{\partial}{\partial x_i} \hat{K} f = \lim_j \frac{\partial}{\partial x_i} \hat{K}_j f = \lim_j \int \frac{\partial}{\partial x_i} k_j(x,z)\,(f(z)-f(x))\,dz$$

$$= \lim_{\varepsilon \to 0} \int_{|x-z|>\varepsilon} \frac{\partial}{\partial x_i} k(x,z)\,(f(z)-f(x))\,dz.$$

Definition. For $1 \leqslant p \leqslant \infty$, $L_1^p(\partial D)$ will denote the space of functions $f \in L^p(\partial D)$ with the property that for any covering, $\{B_j\}_{j=1}^l$, of ∂D with the properties described in the definition of a C^1 domain, D, and for any $\psi \in C_0^1(B_j)$ the function $\psi(x, \varphi_j(x)) f(x, \varphi_j(x)) \equiv \widetilde{\psi f}$ has partial derivatives, in the sense of distributions, given by functions in $L^p(\mathbf{R}^{n-1})$. If we fix a covering $\{B_j\}_{j=1}^l$ and a partition of unity, $\{\psi_j\}$, of ∂D subordinate to this cover we can define.

$$\|f\|_{L_1^p(\partial D)} = \|f\|_{L^p(\partial D)} + \sum \|\nabla \widetilde{\psi_j f}\|_{L^p(\mathbf{R}^{n-1})}.$$

(We are assuming that each $\psi_j \in C_0^1(\mathbf{R}^n)$.) It is not difficult to see that using a different covering and a different partition of unity subordinate to the cover will give rise to a norm equivalent to the one we have defined. (See [4].)

As a consequence of Theorem 1.5 we have

THEOREM 1.6. *For* $1 < p < \infty$ *the operator*

$$Kf(P) \equiv \text{p.v.} \frac{1}{\omega_n} \int_{\partial D} \frac{\langle P-Q, N_Q \rangle}{|P-Q|^n} f(Q)\,dQ$$

maps $L_1^p(\partial D) \to L_1^p(\partial D)$ *continuously and, moreover,* K *is compact on* $L_1^p(\partial D)$.

Proof. As stated above the continuity of K on $L_1^p(\partial D)$ follows immediately from Theorem 1.5. Concerning the compactness of K it is enough to show the compactness on $L_1^p(\mathbf{R}^{n-1})$ of the Euclidean operator

$$[(\psi \hat{K}) f](x) = \psi(x) \hat{K}(f)(x)$$

where $\psi(x) \in C_0^1(\mathbf{R}^{n-1})$ and $\hat{K}$ is the operator of Theorem 1.5. Again let $\{\varphi_j\} \subset C_0^\infty(\mathbf{R}^{n-1})$ be a sequence of function such that $\varphi_j \to \varphi$ and $\nabla \varphi_j \to \nabla \varphi$ uniformly. From Theorem 1.5

$$\psi(x)\nabla\hat{K}f(x)=\psi(x)\,\text{p.v.}\int\frac{\nabla(\varphi-\varphi_j)(x)-\nabla(\varphi-\varphi_j)(z)}{[|x-z|^2+(\varphi(x)-\varphi(z))^2]^{n/2}}(f(z)-f(x))\,dz$$

$$-n\psi(x)\,\text{p.v.}\int(f(z)-f(x))\frac{[(\varphi-\varphi_j)(x)-(\varphi-\varphi_j)(z)-\langle\nabla(\varphi-\varphi_j)(z),x-z\rangle]}{[|x-z|^2+(\varphi(x)-\varphi(z))^2]^{n/2+1}}$$

$$\times(x-z+(\varphi(x)-\varphi(z))\nabla\varphi(x))\,dz$$

$$+\psi(x)\int\frac{\nabla\varphi_j(x)-\nabla\varphi_j(z)}{[|x-z|^2+(\varphi(x)-\varphi(z))^2]^{n/2}}(f(z)-f(x))\,dz$$

$$-n\psi(x)\int(f(z)-f(x))\frac{[\varphi_j(x)-\varphi_j(z)-\langle\nabla\varphi_j(z),x-z\rangle]}{[|x-z|^2+(\varphi(x)-\varphi(z))^2]^{n/2+1}}$$

$$\times(x-z+(\varphi(x)-\varphi(z))\nabla\varphi(x))\,dz.$$

The first two operators on the right hand side of the above equality as operators from $L_1^p(\mathbf{R}^{n-1})$ into $L^p(\mathbf{R}^{n-1})$ have noims tending to zero as j tends to ∞. (Again use Theorem 4 in [2].) For j fixed the final two operators are compact fiom $L_1^p(\mathbf{R}^{n-1})$ into $L^p(\mathbf{R}^{n-1})$. From these observations it easily follows that $\psi\hat{K}$ is compact on $L_1^p(\mathbf{R}^{n-1})$.

THEOREM 1.7. *Assume $f\in L_1^p(\partial D)$, $1<p<\infty$, and let*

$$u(X)=\frac{1}{\omega_n}\int_{\partial D}\frac{\langle X-Q,N_Q\rangle}{|X-Q|^n}f(Q)\,dQ,\quad X\in D.$$

Then given α, $0<\alpha<1$, there exists $\delta=\delta_{\alpha,D}$ such that the nontangential maximal function of ∇u, i.e. $(\nabla u)^*(P)=\sup\{|\nabla u(X)|: X\in D, |X-P|<\delta, \langle X-P,N_p\rangle>\alpha|X-P|\}$, belongs to $L^p(\partial D)$ and

$$\|(\nabla u)^*\|_{L^p(\partial D)}\leqslant C\,\|f\|_{L_1^p(\partial D)}\quad (C \text{ independent of } f).$$

Proof. The proof follows closely that of Theorem 1.3. We may assume for example that f is supported in $B\cap\partial D$ where B is a sphere with center on ∂D such that

$$B\cap D=B\cap\{(x,t): x\in\mathbf{R}^{n-1},\, t>\varphi(x),\, \varphi\in C_0^1(\mathbf{R}^{n-1})\}$$

and $|\nabla\varphi|\leqslant\alpha/6$. The problem is reduced to proving that the function

$$\tilde{u}(x,t)=\int_{\mathbf{R}^{n-1}}\frac{t-\varphi(z)-\langle\nabla\varphi(z),x-z\rangle}{[|x-z|^2+(t-\varphi(z))^2]^{n/2}}\tilde{f}(z)\,dz$$

has the following property:

$$(\nabla\tilde{u})^*(x_0) = \sup\Big\{ |\nabla\tilde{u}(x,t)| \,|: x\in\mathbf{R}^{n-1},\ t>\varphi(x) \quad\text{and}$$

$$\frac{t-\varphi(x_0)-\langle\nabla\varphi(x_0), x-x_0\rangle}{\sqrt{1+|\nabla\varphi(x_0)|^2}} > \alpha\sqrt{|x-x_0|^2+(\varphi(x)-\varphi(x_0))^2}\Big\}$$

belongs to $L^p(\mathbf{R}^{n-1})$ and

$$\|(\nabla\tilde{u})^*\|_{L^p(\mathbf{R}^{n-1})} \leqslant C\,\|\tilde{f}\|_{L_1^p(\mathbf{R}^{n-1})}.$$

As in Theorem 1.3 we set

$$k(t,x;z) = \frac{t-\varphi(z)-\langle\nabla\varphi(z), x-z\rangle}{[|x-z|^2+(t-\varphi(z))^2]^{n/2}}$$

and we observe that

$$\nabla\tilde{u}(x,t) = \int (\nabla_{x,t}k(x,t;z))\,(\tilde{f}(z)-\tilde{f}(x_0))\,dz.$$

Set $\nabla k(x,t;z) = \nabla_{x,t}k(x,t;z)$ and $\lambda = \max(3|x-x_0|,\ t-\varphi(x_0))$. Proceeding as in Theorem 1.3 we have

$$|\nabla\tilde{u}(x,t)| \leqslant \int_{|z-x_0|\leqslant\lambda} |\nabla k(x,t;z)|\,|\tilde{f}(z)-\tilde{f}(x_0)|\,dz$$

$$+ \int_{|z-x_0|>\lambda} |\nabla k(x,t;z)-\nabla k(x_0,\varphi(x_0);z)|\,|\tilde{f}(z)-\tilde{f}(x_0)|\,dz$$

$$+ \Big|\int_{|z-x_0|>\lambda} \nabla k(x_0,\varphi(x_0);z)\,(\tilde{f}(z)-\tilde{f}(x_0))\,dz\Big|.$$

The first and second terms on the right side of the above inequality are dominated by a constant times $\tilde{f}^*(x_0)$ where

$$\tilde{f}^*(x_0) = \sup_{r>0}\frac{1}{r^{n-1}}\int_{|x-x_0|<r} |\nabla\tilde{f}|(x)\,dx.$$

Again we have $\tilde{f}^*\in L^p(\mathbf{R}^{n-1})$ and $\|\tilde{f}^*\|_{L^p}\leqslant C\|\tilde{f}\|_{L_1^p}$. Finally from Theorem 4 in [2] the $\sup_{\lambda>0}|\int_{|z-x_0|>\lambda}\nabla k(x_0,\varphi(x_0);z)\,(\tilde{f}(z)-\tilde{f}(x_0))\,dz|$ belongs to $L^p(\mathbf{R}^{n-1})$ with norm bounded by a constant times the norm in $L^p(\mathbf{R}^{n-1})$ of $\tilde{f}$. This concludes the proof of Theorem 1.7.

We now turn our study to the behavior of the single layer potential over a C^1-domain with density in the class L^p of the boundary.

LEMMA 1.8. *For $f\in L^p(\mathbf{R}^{n-1})$, $1<p<\infty$, and $\varepsilon>0$ set*

$$\tilde{K}_\varepsilon^* f(x) = \int_{|x-z|>\varepsilon} \frac{\varphi(x)-\varphi(z)-\langle\nabla\varphi(x), x-z\rangle}{[|x-z|^2+(\varphi(x)-\varphi(z))^2]^{n/2}}\,f(z)\,dz$$

with $\varphi\in C_0^1(\mathbf{R}^{n-1})$. *There exists a constant* $m_0>0$ *such that if* $\max|\nabla\varphi|<m_0$ *then*

(a) $\|\sup_{\varepsilon>0}|\hat{K}_\varepsilon^* f|\,\|_{L^p(\mathbf{R}^{n-1})}\leqslant C\|f\|_{L^p(\mathbf{R}^{n-1})}$, *C independent of f,*

(b) $\lim_{\varepsilon\to 0}K_\varepsilon^* f\equiv K^* f$ *exists in* $L^p(\mathbf{R}^{n-1})$ *and pointwise almost everywhere.*

Proof. Both parts (a) and (b) are proved in the exact same manner as the corresponding parts of Lemma 1.1.

THEOREM 1.9. *For* $f\in L^p(\partial D)$, $1<p<\infty$, *and* $\varepsilon>0$ *set*

$$K_\varepsilon^* f(P)=\frac{-1}{\omega_n}\int_{|P-Q|>\varepsilon}\frac{\langle P-Q, N_p\rangle}{|P-Q|^n}f(Q)\,dQ.$$

Then

(a) $\|\sup_\varepsilon|K_\varepsilon^* f|\,\|_{L^p(\partial D)}\leqslant C\|f\|_{L^p(\partial D)}$ *with C independent of f,* *and*

(b) $\lim_{\varepsilon\to 0+}K_\varepsilon^* f\equiv K^* f$ *exists in* $L^p(\partial D)$ *and pointwise almost everywhere.*

(c) K^* is compact.

Proof. The proof of (a) follows the same line of argument of part (a) in Theorem 1.2 and is left to the reader. For part (b) it is again sufficient to show the existence of the limit for almost ever $P\in\partial D$. This last statement will be justified if we can show the existence of

$$\lim_{\varepsilon\to 0}\int_{|x-z|^2+(\varphi(x)-\varphi(z))^2>\varepsilon^2}\frac{\varphi(x)-\varphi(z)-\langle\nabla\varphi(x),x-z\rangle}{[|x-z|^2+(\varphi(x)-\varphi(z))^2]^{n/2}}\tilde{f}(z)\,dz$$

for almost every $x\in\mathbf{R}^{n-1}$ when $\tilde{f}\in L^p(\mathbf{R}^{n-1})$, $\varphi\in C_0^1(\mathbf{R}^{n-1})$, and $\max|\nabla\varphi|<m_0$, the constant of Lemma 1.8. We now pick a sequence $\{\psi_j\}\subset C_0^\infty(\mathbf{R}^{n-1})$ such that $\psi_j\to\varphi$ and $\nabla\psi_j\to\nabla\varphi$ uniformly, and we set

$$\hat{K}_\varepsilon^*\tilde{f}(x)=\int_{|x-z|^2+(\varphi(x)-\varphi(z))^2>\varepsilon^2}\frac{\varphi(x)-\varphi(z)-\langle\nabla\varphi(x),x-z\rangle}{[|x-z|^2+(\varphi(x)-\varphi(z))^2]^{n/2}}\tilde{f}(z)\,dz$$

and

$$\hat{K}_{\varepsilon,j}^*\tilde{f}(x)=\int_{|x-z|^2+(\varphi(x)-\varphi(z))^2>\varepsilon^2}\frac{\psi_j(x)-\psi_j(z)-\langle\nabla\psi_j(x),x-z\rangle}{[|x-z|^2+(\varphi(x)-\varphi(z)^2]^{n/2}}\tilde{f}(z)\,dz.$$

For j fixed $\lim_{\varepsilon\to 0}\hat{K}_{\varepsilon,j}^*\tilde{f}(x)$ exists for almost every $x\in\mathbf{R}^{n-1}$ and, using Theorem 4 in [2],

$$\|\sup_{\varepsilon>0}|\hat{K}_\varepsilon^*\tilde{f}(x)-\hat{K}_{\varepsilon,j}^*\tilde{f}(x)|\,\|_{L^p(\mathbf{R}^{n-1})}\leqslant C_j\|\tilde{f}\|_{L^p(\mathbf{R}^{n-1})}$$

where $C_j\to 0$ as $j\to\infty$. As in the proof of part (b) of Lemma 1.1, we now can conclude the existence almost everywhere of $\lim_{\varepsilon\to 0}\hat{K}_\varepsilon^*\tilde{f}(x)$.

In virtue of Theorem 1.2, K^* is the adjoint of an operator, K, compact on each $L^p(\partial D)$, $1<p<\infty$, hence the same holds for K^*.

THEOREM 1.10. *For $f\in L^p(\partial D)$, $1<p<\infty$, and $X\notin\partial D$, set*

$$u(X)=\frac{-1}{\omega_n(n-2)}\int_{\partial D}\frac{f(Q)}{|X-Q|^{n-2}}\,dQ.\quad \text{Then given } \alpha, 0<\alpha<1,$$

(a) *there exists $\delta\equiv\delta_\alpha>0$ such that the functions*

$$(\nabla u)_i^*(P)=\sup\{|\nabla u(X)|:\ X\in D,\ |X-P|<\delta\ \langle X-P, N_p\rangle>\alpha|X-P|\}$$

and

$$(\nabla u)_e^*(P)=\sup\{|\nabla u(X)|:\ X\in \mathbf{R}^n\backslash\bar{D},\ |X-P|<\delta,\ \langle X-P, N_p\rangle<-\alpha|X-P|\}$$

belong to $L^p(\partial D)$ and

$$\|(\nabla u)_i^*\|_{L^p(\partial D)}+\|(\nabla u)_e^*\|_{L^p(\partial D)}\leqslant C\|f\|_{L^p(\partial D)},\quad C \text{ independent of } f,$$

(b) *$\partial u/\partial N_p(X)\equiv\langle\nabla u(X), N_p\rangle\to(\frac{1}{2}I-K^*)f(P)$ pointwise for almost every $P\in\partial D$ as $X\to P$, $X\in D$, $\langle X-P, N_p\rangle>\alpha|X-P|$, and $\partial u/\partial N_p(X)\to(\frac{1}{2}+K^*)f(P)$ pointwise for almost every $P\in\partial D$ as $X\to P$, $X\in\mathbf{R}^n\backslash\bar{D}$, $\langle X-P, N_p\rangle<-\alpha|X-P|$. Here K^* is the operator of Theorem 1.9.*

Proof. The prooof of part (a) follows the exact lines of the first part of Theorem 1.3 and is again left to the reader. For part (b) it is sufficient to prove the existence of the pointwise limit for almost everk $P\in\partial D$ when $f\in C^1(\partial D)$. We will consider only the case of the interior nontangential limit, i.e. $X\in D$, the exterior limit, $X\in\mathbf{R}^n\backslash\bar{D}$, being handled analogously.

$$\langle\nabla u(X), N_p\rangle=\frac{1}{\omega_n}\int_{\partial D}\frac{\langle X-Q, N_p\rangle}{|X-Q|^n}f(Q)\,dQ$$

$$=\frac{1}{\omega_n}\int_{\partial D}\frac{\langle X-Q, N_p\rangle}{|X-Q|^n}(f(Q)-f(P))\,dQ+f(P)+\frac{f(P)}{\omega_n}\int_{\partial D}\frac{\langle X-Q, N_p-N_Q\rangle}{|X-Q|^n}\,dQ.$$

Since $f\in C^1(\partial D)$ it is clear that the limit of the first term above when $X\to P$ exists and equals

$$\frac{1}{\omega_n}\int_{\partial D}\frac{\langle P-Q, N_p\rangle}{|P-Q|^n}(f(Q)-f(P))\,dQ.$$

N_p is a continuous function on ∂D and hence we can find a sequence of (vector-valued) functions, $N_{j,p}$, belonging to $C^1(\partial D)$ such that $N_{j,p}\to N_p$ uniformly on ∂D. Hence

$$\frac{1}{\omega_n}\int_{\partial D}\frac{\langle X-Q, N_p-N_Q\rangle}{|X-Q|^n}dQ=\frac{1}{\omega_n}\left\langle N_p-N_{j,p}, \int_{\partial D}\frac{X-Q}{|X-Q|^n}dQ\right\rangle$$
$$+\frac{1}{\omega_n}\int_{\partial D}\frac{\langle X-Q, N_{j,Q}-N_Q\rangle}{|X-Q|^n}dQ+\frac{1}{\omega_n}\int\frac{\langle X-Q, N_{j,p}-N_{j,Q}\rangle}{|X-Q|^n}dQ$$

The arguments of Theorem 1.2 imply the existence of a positive number $\delta>0$ such that the functions

$$|N_p-N_{j,p}|\sup\left\{\left|\int_{\partial D}\frac{X-Q}{|X-Q|^n}dQ\right|: X\in D, |X-P|<\delta, \langle X-P, N_p\rangle>\alpha|X-P|\right\}$$

and

$$\sup\left\{\left|\int_{\partial D}\frac{\langle X-Q, N_{j,Q}-N_Q\rangle}{|X-Q|^n}dQ\right|: X\in D, |X-P|<\delta, \langle X-P, N_p\rangle>\alpha|X-P|\right\}$$

belong to $L^q(\partial D)$ for all q, $1<q<\infty$, and each L^q norm tends to zero as $j\to\infty$. For j fixed

$$\lim_{X\to P}\frac{1}{\omega_n}\int_{\partial D}\frac{\langle X-Q, N_{j,p}-N_{j,Q}\rangle}{|X-Q|^n}dQ=\frac{1}{\omega_n}\int_{\partial D}\frac{\langle P-Q, N_{j,p}-N_{j,Q}\rangle}{|P-Q|^n}dQ.$$

Combining together the above observations we conclude that for almost every $P\in\partial D$

$$\frac{1}{\omega_n}\int_{\partial D}\frac{\langle X-Q, N_p-N_Q\rangle}{|X-Q|^n}dQ$$

converges as $X\to P$ nontangentially, $X\in D$, to

$$\frac{1}{\omega_n}\lim_{\varepsilon\to 0}\int_{|P-Q|>\varepsilon}\frac{\langle P-Q, N_p-N_Q\rangle}{|P-Q|^n}dQ,$$

and, therefore, for almost every $P\in\partial D$

$$\frac{\partial u}{\partial N_p}(X)\to\tfrac{1}{2}f(P)-K^*f(P)\quad\text{as}\quad X\to P, X\in D, \langle X-P, N_p\rangle>\alpha|X-P|.$$

2. The Dirichlet and Neumann problems

We recall the operator

$$Kf(P)=\frac{1}{\omega_n}\text{p.v.}\int_{\partial D}\frac{\langle P-Q, N_Q\rangle}{|P-Q|^n}f(Q)\,dQ.$$

THEOREM 2.1. *Assume $D\in C^1$ is bounded and $\mathbf{R}^n\setminus\bar{D}$ is connected. Then $\frac{1}{2}I+K$ is invertible on $L^p(\partial D)$ for each p, $1<p<\infty$.*

Proof. We will in fact show that the adjoint of $\frac{1}{2}I+K$, namely $\frac{1}{2}I+K^*$ is invertible on each $L^p(\partial D)$, $1<p<\infty$. Since K^* is compact it is enough to prove that $\frac{1}{2}I+K^*$ is injective. So let's assume that $f\in L^p(\partial D)$ and $(\frac{1}{2}I+K^*)f=0$.

We first observe that $f\in L^q(\partial D)$ for every q, $1<q<\infty$. To see this we take a sphere, $B=B(P_0,\delta)$, with center on ∂D such that

$$B\cap D=\{(x,t): t>\varphi(x),\ \varphi\in C_0^1(\mathbf{R}^{n-1}),\ \max|\nabla\varphi|<\varepsilon\}$$

where ε is a fixed small positive number. We also take two functions ψ, $\theta\in C_0^\infty(B)$ with the properties

$$\theta=\begin{cases}1 & \text{in } B(P_0,\delta/3)\\ 0 & \text{in } \mathbf{R}^n\setminus B(P_0,\frac{2}{3}\delta),\end{cases}\qquad \psi\equiv 1 \quad\text{in}\quad B(P_0,\tfrac{3}{4}\delta).$$

Then $\theta f=-2\theta K^*f=-2(\theta K^*-K^*\theta)f-2K^*(\theta f)$ and finally

$$\theta f+2(\psi K^*\psi)\theta f=-2\psi(\theta K^*-K^*\theta)f\equiv g.$$

The function, g, satisfies the inequality

$$|g(P)|\leqslant C_\delta\int_{\partial D}\frac{|f(Q)|}{|P-Q|^{n-2}}\,dQ.$$

If $1/p-1/(n-1)\equiv 1/q>0$, $g\in L^q(\partial D)$; if $1/p-1/(n-1)\leqslant 0$, $g\in L^q(\partial D)$ for all q, $1<q<\infty$. Since the norm of $\psi K^*\psi$ is small on L^q we conclude that θf, and, hence, $f\in L^q(\partial D)$. Continuing in this manner we prove the observation that $f\in L^q(\partial D)$ for each q, $1<q<\infty$.

We now introduce the single layer potential over ∂D of the function f, i.e.

$$u(X)=-\frac{1}{\omega_n(n-2)}\int_{\partial D}\frac{1}{|X-Q|^{n-2}}f(Q)\,dQ,\quad X\in\mathbf{R}^n.$$

Since $f\in L^p(\partial D)$ for each p, $1<p<\infty$, we can integrate by parts in the integral,

$$\int_{\mathbf{R}^n\setminus\bar{D}}|\nabla u(X)|^2\,dX$$

and obtain that it is equal to

$$\int_{\partial D}u\frac{\partial u}{\partial N_Q}\,dQ=0\quad\text{since}\quad\frac{\partial u}{\partial N_Q}=-(\tfrac{1}{2}I+K^*)f=0.$$

Therefore $u(X)$ is identically constant in $\mathbf{R}^n\setminus\bar{D}$. Since $\lim_{|X|\to\infty}u(X)=0$ and $\mathbf{R}^n\setminus\bar{D}$ is connected, $u(X)\equiv 0$ in $\mathbf{R}^n\setminus\bar{D}$. $u(X)$ is a continuous function on $\mathbf{R}^n$, and, hence, in D, u is harmonic, and $u|_{\partial D}=0$.

From the classical uniqueness theorem on harmonic functions, $u(X)\equiv 0$ on $\mathbf{R}^n$. Using now part (b) of Theorem 1.10,

$$(\tfrac{1}{2}I-K^*)f=0 \quad \text{and} \quad -(\tfrac{1}{2}I+K^*)f=0.$$

It follows immediately that $f\equiv 0$ on ∂D, and we have then proved that the null space of $\frac{1}{2}I+K^*$ is $\{0\}$.

COROLLARY 2.2. *For $1<p<\infty$, $\frac{1}{2}I+K$ is invertible on $L_1^p(\partial D)$.*

Proof. From Theorem 1.6, K is compact on $L_1^p(\partial D)$ and since $\frac{1}{2}I+K$ is one-to-one on L^p it is a fortiori injective on $L_1^p(\partial D)$. The Fredholm theory of compact operators implies the invertibility on $L_1^p(\partial D)$.

THEOREM 2.3. *Suppose $D\in C^1$ is bounded and $\mathbf{R}^n\setminus\bar{D}$ is connected. Given $g\in L^p(\partial D)$, $1<p<\infty$, there exists a unique harmonic function $u(X)$, defined in D such that for each α, $0<\alpha<1$,* there exists $\delta>0$ *for which*

(i) *the nontangential maximal function of u, namely,*

$$u_\alpha^*(P)=\sup\{|u(X)|:|X-P|<\delta,\ \langle X-P,N_p\rangle>\alpha|X-P|\},$$

belongs to $L^p(\partial D)$ and $\|u_\alpha^*\|_{L^p(\partial D)}\leqslant C\|g\|_{L^p(\partial D)}$,

(ii) *$u(X)\to g(P)$ for almost every $P\in\partial D$ as $X\to P$*

$$\langle X-P,N_p\rangle>\alpha|X-P|.$$

In fact u has the form of the double layer potential

$$u(X)=\frac{1}{\omega_n}\int_{\partial D}\frac{\langle X-Q,N_Q\rangle}{|X-Q|^n}\,Tg(Q)\,dQ \quad \text{where} \quad T=(\tfrac{1}{2}+K)^{-1}.$$

Proof. It is immediate from Theorems 1.3 and 2.1 that the double layer potential of Tg satisfies (i) and (ii).

To begin the proof of uniqueness we introduce the (Green's) function

$$G(X,Y)=\frac{1}{|X-Y|^{n-2}}-\frac{1}{\omega_n}\int_{\partial D}\frac{\langle Y-Q,N_Q\rangle}{|Y-Q|^n}\,T\left(\frac{1}{|X-\cdot|^{n-2}}\right)(Q)\,dQ.$$

Fixing $\varepsilon>0$ we take $\psi_\varepsilon(Y)\in C_0^\infty(D)$ satisfying $0\leqslant\psi_\varepsilon\leqslant 1$, $\psi_\varepsilon\equiv 1$ on

$$\{Y\in D:\operatorname{dist}(Y,\partial D)\leqslant\varepsilon\},\quad \left|\frac{\partial^\alpha}{\partial Y^\alpha}\psi_\varepsilon\right|\leqslant\frac{C_\alpha}{\varepsilon^{|\alpha|}}.$$

Fixing now also $X \in D$ we have for small ε,

$$u(X) = (u\psi_\varepsilon)(X) = \int_D G(X, Y)\, \Delta_Y(\psi_\varepsilon u)(Y)\, dY.$$

If u is harmonic in D integrating by parts we have

$$u(X) = -2\int_D \langle \nabla_Y G(X, Y), \nabla_Y \psi_\varepsilon(Y)\rangle u(Y)\, dY - \int_D G(X, Y)\, \Delta\psi_\varepsilon(Y)\, u(Y)\, dY.$$

Let $\{\psi_j(Y)\}_{j=1}^m$ be a finite set of functions such that $\psi_j \in C_0^\infty(\mathbf{R}^n)$, $\sum_{j=1}^m \psi_j \equiv 1$ on $\{Y \in \mathbf{R}^n\colon \operatorname{dist}(Y, \partial D) \leqslant \delta\}$ and the support $\psi^q \subset B_j$ where $B_j \cap D = B_j \cap \{(x, t)\colon t > \varphi_j(x),$ $\varphi_j(x) \in C_0^1(\mathbf{R}^{n-1})\}$. Then

$$\int_D |\nabla_Y G(X, Y)|\, |\nabla\psi_\varepsilon(Y)|\, |u(Y)|\, \psi_j(Y)\, dY$$

$$\leqslant \frac{C}{\varepsilon} \int_{|z| \leqslant c} \int_0^\varepsilon |\nabla_Y G(X; z, t + \varphi(z))|\, |u(z, t + \varphi(z))|\, dt\, dz$$

$$\leqslant C \int_{|z| \leqslant c} \sup_{0 \leqslant r \leqslant \varepsilon} |\nabla_Y G(X, z, r + \varphi(z))|\, \frac{1}{\varepsilon} \int_0^\varepsilon |u(z, t + \varphi(z))|\, dt\, dz.$$

Since $G(X, Q) \in L_1^q(\partial D)$ for each q, $1 < q < \infty$,

$$\sup_{0 \leqslant r \leqslant \varepsilon} |\nabla_Y G(X, z + r\varphi(z))| \leqslant (\nabla_Y G(X))^*(z, \varphi(z)) \in L^q(\{z\colon |z| \leqslant c\}).$$

Here we have used the result and notation of Theorem 1.7. It is easy to see that there is an α, $0 < \alpha < 1$, such that $\sup_{0 < t < \varepsilon} |u(z, t + \varphi(z))| \leqslant u_\alpha^*(z, \varphi(z))$. If in addition to being harmonic, $u_\alpha^*(z, \varphi(z)) \in L^p(\{z\colon |z| \leqslant c\})$, and $u(z, t + \varphi(z)) \to 0$ as $t \to 0$ for almost every z, $|z| \leqslant c$, then under these conditions on u we have shown that

$$\int_D |\nabla_Y G(X, Y)|\, |\nabla\psi_\varepsilon(Y)|\, |u(Y)|\, |\psi_j(Y)|\, dY \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

In a very similar manner and with these same conditions on u we have

$$\int_D |G(X, Y)|\, |\Delta\psi_\varepsilon(Y)|\, |u(Y)|\, \psi_j(Y)\, dY \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

An immediate consequence of Theorems 1.7, 2.3 and Corollary 2.2 is

THEOREM 2.4. *Assume the hypotheses of Theorem 2.3 on D. If $g \in L_1^p(\partial D)$, $1 < p < \infty$, then the solution of the Dirichlet problem given by Theorem 2.3 has the additional property*

(iii) $(\nabla u)^*(P) = \sup\{|\nabla u(X)|: X \in D, |X-P| < \delta, \langle X-P, N_p\rangle > \alpha|X-P|\}$ *belongs to* $L^p(\partial D)$ *and*

$$\|(\nabla u)^*\|_{L^p(\partial D)} \leq C\|g\|_{L_1^p(\partial D)}, \quad C \text{ independent of } g.$$

We turn now to the Neumann problem.

THEOREM 2.5. *If $D \in C^1$ is bounded and connected then $(\frac{1}{2}I - K^*)$ is invertible on the subspace of $L^p(\partial D)$, $1 < p < \infty$, consisting of those functions f such that $\int_{\partial D} f dQ = 0$. Here*

$$K^*f(P) = \lim_{\varepsilon\to 0} -\frac{1}{\omega_n}\int_{|P-Q|>\varepsilon} \frac{\langle P-Q, N_p\rangle}{|P-Q|^n} f(Q)\, dQ.$$

Proof. Since K^* is compact on $L^p(\partial D)$ it is enough to show that $\frac{1}{2}I - K^*$ is one-to-one. So assume $f = 2K^*f$ and $\int_{\partial D} f = 0$. Exactly as in Theorem 2.1 we conclude that $f \in L^q(\partial D)$ for every q, $1 < q < \infty$. Consider now the single layer potential of f over ∂D, namely

$$u(X) = -\frac{1}{(n-2)\,\omega_n}\int_{\partial D} \frac{f(Q)}{|X-Q|^{n-2}}\, dQ.$$

An integration by parts shows that

$$\int_D |\nabla u(X)|^2 dX = \int_{\partial D} u \frac{\partial u}{\partial N_Q}\, dQ = \int_{\partial D} u(\tfrac{1}{2}f - K^*f)\, dQ = 0.$$

Hence $u(X) \equiv$ constant in $\bar{D}$. In $\mathbf{R}^n \setminus \bar{D}$, $u(X)$ is harmonic and $\lim_{|X|\to\infty} u(X) = 0$. As noted $u|_{\partial D} = c$, a constant. Since the maximum or minimum of u in $\mathbf{R}^n \setminus D$ must occur on ∂D we conclude that the maximum or the minimum occurs at every point $P \in \partial D$ and, therefore the limit of $(\partial u/\partial N_p)(X)$ as $X \to P$ nontangentially, $X \in \mathbf{R}^n \setminus \bar{D}$ is of constant sign. But this limit equals $-\frac{1}{2}f - K^* = -f$. Hence f is of constant sign. But since $\int_{\partial D} f = 0$ we must have $f = 0$ on ∂D.

THEOREM 2.6. *Suppose $D \in C^1$ is bounded and connected.*

Given $g \in L^p(\partial D)$, $1 < p < \infty$, with $\int_{\partial D} g = 0$, there exists a harmonic function, $u(X)$, defined in D such that to each α, $0 < \alpha < 1$, there corresponds a $\delta > 0$ for which

(i) *the nontangential maximal function of ∇u, namely, $(\nabla u)^*(P) = \sup\{|\nabla u(X)|: |X-P| < \delta, \langle X-P, N_p\rangle > \alpha|X-P|\}$, belongs to $L^p(\partial D)$ (and $\|(\Delta u)^*\|_{L^p(\partial D)} \leq C\|g\|_{L^p(\partial D)}$),*

(ii) $\dfrac{\partial u}{\partial N_p}(X) \equiv \langle \nabla u(X), N_p\rangle \to g(P)$ *for almost every* $P \in \partial D$ *as* $X \to P, \langle X-P, N_p\rangle > \alpha|X-P|$.

The harmonic function, $u(X)$, satisfying (i) and (ii) is uniquely determined up to a constant and can be taken in the form

$$u(X) = -\frac{1}{(n-2)\,\omega_n}\int_{\partial D}\frac{Sg(Q)}{|X-Q|^{n-2}}\,dQ$$

where $S = (\frac{1}{2}I - K^*)^{-1}$ on the subspace of $L^p(\partial D)$ consisting of functions with integral zero.

Proof. We immediately conclude from Theorem 1.10 that the above single layer potential of Sg has properties (i) and (ii).

For the uniqueness we consider the Neumann function,

$$N(X, Y) = \frac{1}{(n-2)|X-Y|^{n-2}} + \frac{1}{\omega_n(n-2)}\int_{\partial D}\frac{1}{|Y-Q|^{n-2}}\,S\left(\frac{\langle X-\cdot\,, N.\rangle}{|X-\cdot|^n} - \omega_n\right)(Q)\,dQ.$$

An integration by parts shows that

$$\sum_{i=1}^{n}\int_D \frac{\partial N}{\partial Y_i}(X, Y)\,\frac{\partial u}{\partial Y_i}(Y)\,dY = u(X) + c$$

where c is a constant. However, if $(\nabla u)^* \in L^p(\partial D)$ and $(\partial u/\partial N_p)(X) \to 0$ as $X \to P$ nontangentially then the left-hand side of the above equality is zero. Hence $u(X) \equiv$ constant in D.

References

[1]. Calderón, A. P., Cauchy integrals on Lipschitz curves and related operators. *Proc. Nat, Acad. Sci. U.S.A.*, 74 (1977), 1324–1327.

[2]. Calderón, A. P., Calderón, C. P., Fabes, E. B., Jodeit, M. & Riviére, N. M., Applications of the Cauchy integral along Lipschitz cvrves. To appear in *Bull. Amer. Math. Soc.*

[3]. Dahlberg, B. E. J., On the Poisson Integral for Lipschitz and C^1-domains. Technical Report, Department of Mathematics, Chalmers University of Technology and the University of Göteborg, No. 1977–78.

[4]. Seeley, R. T., Singular integrals on compact manifolds. *Amer. J. Math. Journal,* 81 (1959), 658–690.

[5]. Stein, E. M., *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, Princeton, New Jersey, 1970.

Received November 7, 1977