Numerical determination of long-range stress history from strain history in concrete

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The paper presents an efficient and highly accurate step-by-step numerical algorithm for computation of stress history from any prescribed strain history in a linear age-dependent viscoelastic material. The method is applicable for any form of the creep function, including the typical case when the slope of the creep curve in the logarithmic time scale is significant over many orders of magnitude of the elapsed time period (*i.e.* retardation spectrum is broad). The time division must be in geometric progression or nearly so. The creep function may be defined by formulas or by a table of values. A FORTRAN program is presented which allows quick and economical computer solution. Numerical examples are given and excellent convergence is demonstrated. For the special case of strains varying linearly with the creep coefficient a useful new theorem is proved.

NOTATIONS

$E(t) = J(t, t)$	= Young's modulus (instantaneous);
$E_R(t, t')$	= relaxation function (Eq. 12);
E''	= fictitious elastic modulus in Eq. (4) or (14);
$J(t, t')$	= creep function (Eq. 1);
t, t', t_0	= time, time as integration variable, and time of first stress introduction (all in days);
ε	= normal strain;
ε^0	= prescribed stress-independent inelastic strain, *e.g.* thermal dilatation, shrinkage (Eq. 1);
ε''	= fictitious inelastic strain in Eq. (5 a) or (15 a);
ε_0, ε_1	= constants in Eq. (16);
$\varphi(t, t')$	= creep coefficient (Eq. 6);
σ	= normal stress;

Subscripts r, s stand for discrete times t_r, t_s.

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INTRODUCTION

The conversion of creep data into relaxation data for concrete is a problem of considerable importance. It is needed for determination of the stress response to prescribed deformation history of any concrete structure which can be assumed as homogeneous (*i.e.* having the same creep properties in all points), for evaluation of stresses in concrete from measured strain histories, for computation of shrinkage stresses, etc. However, analytical solution to this problem is rather complicated because of the fact that creep properties of concrete are strongly age-dependent (aging). To make an analytical solution feasible, various simplifications of the creep law have been introduced in the past, among which the effective modulus method [1, 2, 3], the rate-of-creep method due to Glanville [3] (called Dischinger's law in Germany), the creep laws of McHenry [4], Arutyunyan and Maslov [5] and Levi [6], and the methods of Hansen [7] and Klug and Wittmann [8] could be named [*cf.* 9, 10]. Such simplifications, however, introduce a substantial error and are unnecessary because the solution can be determined accurately by some of the numerical methods.

The simplest numerical method, which has been used by Raphael [11], England and Illston [12] and others, is to assume the stress history as a series of

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C   * RELAX * STRESS RESPONSE TO A PRESCRIBED STRAIN HISTORY (BAZANT 1967)
      DIMENSION DSTR(195), DDEF(195), T(195)
      DATA  T/0.,0.,.1/,DDEF/0.,1.,193*0./,AGE1/10./,NINT/83/,AR/16./,
     1  NP/8/,DSTR/0./
      DTR = 10.**(1./AR)
      DO 5  I = 4, NINT
    5 T(I) = T(I-1) * DTR
C       ALTERNATIVE  -   TO GET RESPONSE FOR VARIOUS         DO 50 K = 1,4
C       VALUES OF AGE1, INCLUDE THESE TWO STATEMENTS         AGE1 = 10.**K
      STRESS = 0.
      DO 50  I = 2, NINT
      TIME = AGE1 + T(I)
      I1 = I - 1
      Z = 0.
      DO 30  J = 2, I1
   30 Z  =  Z   +   (DSTR(J)*.5)*(CREEP(TIME,AGE1+T(J))-CREEP(AGE1+T(I1)
     1,AGE1+T(J))+CREEP(TIME,AGE1+T(J-1))-CREEP(AGE1+T(I1),AGE1+T(J-1)))
      DSTR(I)=((DDEF(I)-Z)*2.)/(CREEP(TIME,TIME)+CREEP(TIME,AGE1+T(I1)))
      IF( I.EQ.2) DSTR(2) = DDEF(2) / CREEP( AGE1+T(2), AGE1+T(1) )
      STRESS = STRESS + DSTR(I)
   50 IF(I .LE. 3 .OR. ((I-3)/NP)*NP .EQ. I-3) PRINT 81, I, T(I), STRESS
   81 FORMAT(3H I= I3, 8H  T(I) = E13.4, 10H  STRESS = E13.4)
      STOP
      END

      FUNCTION CREEP(X,Y)
      V = (X-Y)**.6
      CREEP = (3.E-7 *SQRT(.85+4./Y))*(1.+(2.94*Y**(-.118))*(V/(10.+V)))
      RETURN
      END
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Fig. 1. — FORTRAN IV program for stress response to a prescribed strain history. TIME = t, T = t — t_0, AGE1 = t_0, CREEP(t, t′) = J(t, t′), DDEF = $\Delta\varepsilon_r - \Delta\varepsilon_r{}^o$ = prescribed array, which is defined here as a step function, DSTR = $\Delta\sigma_r$, STRESS = σ_r, Z = $\Delta\varepsilon_r'' = \Delta\varepsilon_r{}^0$, NINT = total number of steps, AR = n = number of steps per $\log_{10} 10$ in log (t — t_0) — scale, I = r + 2, J = s + 2, NP = interval between the values to be printed, DDEF (1) must be set as 0. If relaxation is to be computed for various t_0-values, e.g., t_0 = 10, 10^2, 10^3, 10^4 days, include the two alternative statements.

sudden (discontinuous) stress increments and to solve the algebraic equations resulting from superposition of creep responses due to all individual stress increments. In relation to the formulation of the creep law in terms of the integral equation, the error of this method is of the order of the time step (first order method). More accurate second order methods have been used in the broader context of structural analyses [13-17]. However, no study of the accuracy and convergence seems to have been presented so far. Furthermore, no attention seems to have been paid to the fact that the creep curves of concrete exhibit in the logarithmic time scale a significant slope over many orders of magnitude of the elapsed time period, ranging from seconds to at least 50 years. (In other words, concrete has a very broad retardation spectrum.) Consequently, for accurate results, the first time step ought to be of the order of 0.0001 to 0.01 day, and if the long range response (for 30 years, *e.g.*) is to be computed, the time step cannot be kept constant but must be gradually increased during computation (according to time division in a geometric progression or nearly so) if the number of steps should not become unacceptably large. Thus, the time step at the end of computation is inevitably quite large. The numerical implications of this fact have not yet been examined (1) and in the past practice the creep has been considered to occur only within a limited time range, such as from 5 to 500 days, to allow keeping the time step about constant and still reaching the final values with no more than about 100 steps.

A study of the above aspects, and especially the presentation of an algorithm admitting time division in a geometric series, is the main objective of this paper. In addition, a fast and economical FORTRAN IV program will be presented (which, on such a computer as CDC 6400, converts any concrete creep function to the relaxation function with a three-digit accuracy in less than 20 seconds and for a cost of less than eight dollars). Finally, a useful new theorem for the case of strain histories depending linearly on the creep coefficient will be given.

CREEP LAW

If one neglects the complex dependence of strain on the histories of water content and temperature [18] (which is at present insufficiently understood), the strain of concrete is a functional of stress history. Rejecting the possibility that there could be some abrupt changes of microstructure involved in the creep process, this functional must be continuous.

(1) Time division in a geometric progression has been utilized already in a previous author's paper [17].

Then the functional admits a generalized Taylor series expansion whose first term, which is linear and expresses the linear principle of superposition, must be a sufficiently good approximation of the material behavior for sufficiently small stresses and their changes, and sufficiently short time histories. Practically, the linear principle of superposition has been found acceptable for stresses less than about 0.5 of the strength, provided the strain reversals are excluded. In fact, no better general stress-strain law for this range is known today (1). According to this principle, the uniaxial creep law may then be expressed in the form :

$$\varepsilon(t) - \varepsilon^0(t) = \int_0^t J(t, t')\, d\sigma(t') \quad \text{(Stieltjes integral [19])} \qquad (1)$$

where t = time from casting of concrete; σ, ε = normal stress and strain; ε^0 = prescribed stress-independent inelastic strain, for example shrinkage or thermal dilatation; $J(t, t')$ = creep function = strain in time t caused by a constant unit stress applied in time t'. (Note that $1/J(t, t) = E(t)$ = Young's modulus in time t.)

NUMERICAL ALGORITHM

When the history of strain $\varepsilon(t)$ is prescribed, Eq. (1) represents a linear Volterra's integral equation for $\sigma(t)$. The most straightforward method for its numerical solution may be based on replacement of the Stieltjes hereditary integral in Eq. (1) by a finite sum [19]. For this purpose time t may be subdivided by discrete times $t_0, t_1, t_2, \ldots, t_N$ into subintervals $\Delta t_r = t_r - t_{r-1}$ ($r = 1, 2, \ldots, N$), which will be considered as unequal. Applying the trapezoidal rule, whose error is of order Δt^2 (second order method), Eq. (1) yields :

$$\varepsilon_r - \varepsilon_r^0 = \sum_{s=1}^{r} \tfrac{1}{2}(J_{r,s} + J_{r,s-1})\,\Delta\sigma_s \qquad (2)$$

where $\Delta\sigma_s = \sigma_s - \sigma_{s-1}$ and subscript r refers to time t_r, *e.g.* $\varepsilon_r = \varepsilon(t_r)$, $J_{r,s} = J(t_r, t_s)$ etc. It should be noted that Eq. (2) is valid even for instantaneous changes of ε and σ; if an instantaneous change is to be considered at a certain time, say t_m, one simply puts $\Delta t_{m+1} = 0$ or $t_{m+1} = t_m$.

Rewriting Eq. (2) for ε_{r-1},

$$\varepsilon_{r-1} - \varepsilon_{r-1}^0 = \sum_{s=1}^{r-1} \tfrac{1}{2}(J_{r-1,s} + J_{r-1,s-1})\,\Delta\sigma_s \qquad (2\text{ a})$$

(for $r > 1$)

(1) For certain special situations, other stress-strain laws than Eq. (1) may give better prediction. This can be said, *e.g.*, of the rate-of-flow method of England and Illston [12] when creep recovery is considered. However, it must be kept in mind that this creep law is also linear and implies, therefore, the superposition (as in Eq. 1) of certain unit creep curves, although not the actual ones but distorted ones, which would necessarily give poor prediction of creep in most other situations, *e.g.* for concrete loaded at a high age. Correctly, deviations from Eq. (1) must be regarded as nonlinear effects and cannot thus be generally formulated by some modified linear stress-strain laws.

TABLE I. — Exact values of the stress relaxation ratios $\sigma(t)/\sigma(t')$ obtained with the program in figure 1, and their comparison with the predictions of the effective modulus method and the rate-of-creep method, giving values $1/(1 + \varphi(t, t_0))$ and $e^{-\varphi(t,t_0)}$, respectively, and the exact solution when the variation of E is neglected. (Creep function (7)-(10) with $\varphi(\infty, 7) = 2.5$.)

Method	t_0 in days	$t - t_0$ (in days)			
		10^1	10^2	10^3	10^4
present method (variable E)	10^1	.555	.328	.220	.179
	10^2	.640	.455	.373	.343
	10^3	.704	.529	.451	.425
	10^4	.760	.598	.521	.496
present method (constant E)	10^1	.568	.369	.278	.244
	10^2	.640	.456	.375	.346
	10^3	.704	.529	.451	.425
	10^4	.760	.598	.521	.496
effective modulus method	10^1	.596	.406	.327	.304
	10^2	.659	.473	.390	.364
	10^3	.717	.541	.456	.429
	10^4	.769	.607	.524	.497
rate-of-creep method	10^1	.508	.232	.128	.101
	10^2	.596	.329	.209	.175
	10^3	.674	.428	.303	.264
	10^4	.741	.524	.403	.363

and subtracting this equation from (2), one can obtain (see also [13]) :

$$\Delta\sigma_r = E_r''(\Delta\varepsilon_r - \Delta\varepsilon_r'') \qquad (3)$$

where

$$E_r'' = 2/(J_{r,r} + J_{r,r-1}) \qquad (4)$$

$$\Delta\varepsilon_r'' = \sum_{s=1}^{r-1} \Delta\sigma_s \tfrac{1}{2}(J_{r,s} + J_{r,s-1} - J_{r-1,s} - J_{r-1,s-1}) + \Delta\varepsilon_r^0 \qquad (5\text{ a})$$

(for $r > 1$)

$$\Delta\varepsilon_r'' = \Delta\varepsilon_r^0 \qquad (5\text{ b})$$

(for $r = 1$)

Equations (3), (4), (5 a), (5 b) are recurrent algebraic equations which allow a step-by-step computation of the values $\Delta\sigma_r$ when $\Delta\varepsilon_r$- values are given. A FORTRAN IV program based on these equations is presented in Figure 1.

APPLICATION, EXAMPLES AND DISCUSSION

For computation of the long-range response of concrete it is necessary to restrict admissible strain histories to those in which the prescribed strain history exhibits an immediate (discontinuous) change (if any) only in time of loading, t_0, and is followed by a continuous change at a gradually decreasing rate, such that in log $(t - t_0)$- scale the slope of the ε- curve is nowhere too large. The variation of strain in structures under steady load, shrinkage and steady support conditions for $t > t_0$ is practically always of this type. (If $\varepsilon(t)$ is prescribed with sudden changes at several times, the linearity of Eq. (1) allows the strain history to be decomposed into several components, each of which is of the type defined above.)

The restriction to strain histories just introduced is necessary to allow gradual increase of the time interval Δt_r with r, which is needed for reaching final values of interest, *e.g.* the values for $t - t_0 = 10{,}000$ days, with an acceptable number of steps. The steps Δt_r are best chosen in such a way that $J(t_r, t_0) - J(t_{r-1}, t_0)$ is nearly constant and equal, for high accuracy, about 0.02 of $J(t_r, t_0)$ or less.

For practical computation it is most convenient to choose the discrete times t_r in a geometric progression, that is t_r/t_{r-1} = constant. In the log $(t - t_0)$ time scale the time step then appears as constant and putting $t_r = 10^{1/n}\, t_{r-1}$ or $\log t_r = 1/n + \log t_{r-1}$, n represents the number of steps per decade log 10. (For the creep law (7) - (10) below one should chose $\Delta t_1 \leqslant 0.1$ day and $n \geqslant 8$ if good accuracy is desired.)

As an example, the computer results are shown in Table I for the following creep function :

$$J(t, t') = (1 + \varphi(t, t'))/E(t') \tag{6}$$

$$E(t') = E(28)\,[t'/(4 + 0.85\,t')]^{\frac{1}{2}} \tag{7}$$

$$\varphi(t, t') = \varphi_u(t')\, f(t - t') \tag{8}$$

$$\varphi_u(t') = \varphi(\infty, 7)\; 1.25\, t'^{-0.118} \tag{9}$$

$$f(t - t') = (t - t')^{0.6}/[10 + (t - t')^{0.6}] \tag{10}$$

where t is expressed in days, φ = creep coefficient. (Equations (7)-(9) have been recently recommended by ACI Committee 209 [20], along with a method of determination of constants $\varphi(\infty, 7)$ and $E(28)$.) Practical demonstration of convergence of the numerical algorithm is given in Table II. Comparison in Table I with the prediction of the old simplified methods for concrete creep indicates the importance of an accurate numerical analysis. It is also seen from Table I that a neglect of the variation of elastic modulus E results in a serious error. In Ref [21] one can find application of the program to shrinkage induced stresses. It should be noted that the program works for any creep function, including functions without a bounded final value of creep (which is a more realistic assumption than a bounded final value).

It is no complication if the creep function is given in a tabular form rather than an analytical expression. In this case the subroutine for $J(t, t_0)$ may be modified as is shown in Figure 2. (This subroutine can be somewhat simplified if $\varphi(t, t')$ is assumed as a product of two functions of one variable as in Eq. (8).) This is the case of the CEB creep function, for which the values of relaxation function obtained with the above program can be found in Ref. 21.

The subroutine in Figure 2 is useful especially for direct conversion of measured creep data into relaxation data. Application of a program such as presented is inevitable when stresses in concrete structures are to be determined from strain histories measured by strain gages.

Somewhat simpler expressions are obtained if the Stieltjes integral in Eq. (1) is replaced with the sum $\sum_{s=1}^{r} J_{r,s}\, \Delta\sigma_s$ whose error is of the order Δt (first order method). Then Eq. (3) is again obtained but instead of Eqs. (4)-(5 a).

TABLE II. — Computed stress relaxation ratio $\sigma(t_N)/\sigma(t_0)$ for various numbers N of subdivision of time interval (t_0, t_N), using the second order method (Eqs. (3), (4), (5a), (5b), program in figure 1) and the first order method (Eqs. (3), (11)); $t_N - t_0 = 1{,}000$ days; beginning of relaxation at $t_0 = 35$ days; $\Delta t_1 = 0.1$ day; constant step in log $(t - t_0)$ scale; creep function (7)-(10) with $\varphi(\infty, 7) = 2.35$.

number of steps N	2^{nd} order method	1^{st} order method
8	.3532	.3311
16	.3592	.3468
32	.3618	.3556
64	.3630	.3599
128	.3635	.3619
256	.3637	.3629

$$E''_r = E_r, \quad \Delta\varepsilon''_r = \sum_{s=1}^{r-1} \Delta\sigma_s\,(J_{r,s} - J_{r-1,s}) + \Delta\varepsilon^0_r \tag{11}$$

Accuracy of the first order method is lower, as is seen from Table II.

The uniaxial creep law is often expressed in a different form which is obtained from Eq. (1) on integration by parts. If a replacement by finite sum is carried out in this formulation, a somewhat different, although analogous, algorithm is obtained (*Cf.* [13], [15], [16]). It has been found, however, that this algorithm does not allow the time step to be increased arbitrarily and becomes numerically unstable when the time step significantly exceeds the shortest retardation time needed for the approximation of the creep curves by a generalized Kelvin model. In the case of creep function (8)-(10) it would thus be necessary to keep Δt_r smaller than about 1 day, and so at least 10,000 steps would be needed to determine the 30-year response.

FURTHER EXTENSIONS

The relaxation data can be converted into creep data with a similar algorithm. In this case the creep law is defined as follows

$$\sigma(t) = \int_0^t E_R(t, t')\,(d\varepsilon(t') - d\varepsilon^0(t')) \tag{12}$$

(Stieltjes integral)

in which $E_R(t, t')$ = relaxation function = stress in time t caused by a constant unit strain introduced in time t'. Replacing the Stieltjes hereditary integral in Eq. (12) with a finite sum according to the trapezoidal rule, one can derive

$$\Delta\varepsilon_r = \Delta\sigma_r/E''_r + \Delta\varepsilon''_r \tag{13}$$

```
      FUNCTION CREEP(X,Y)
      COMMON /REL/  AJ(16,8), TL(16), AL(8)
      ALL = ALOG10(Y)
      L = 1
    3 L = L + 1
      A = AL(L) - ALL
      IF(A .LT. 0.) GO TO 3
      AA = ALL - AL(L-1)
      IF(X .GT. Y) GO TO 5
      CREEP = (AJ(1,L-1)*A + AJ(1,L)*AA)/(AL(L)-AL(L-1))
      RETURN
    5 TLL = ALOG10(X - Y)
      K = 1
    6 K = K + 1
      B = TL(K) - TLL
      IF(B .LT. 0.) GO TO 6
      BB = TLL - TL(K-1)
      CREEP = ((AJ(K-1,L-1)*A + AJ(K-1,L)*AA)*B +
     1 (AJ(K,L-1)*A + AJ(K,L)*AA)*BB)/((TL(K)-TL(K-1))*(AL(L)-AL(L-1)))
      RETURN
      END
```

Fig. 2. — Alternative subroutine for program in Fig. 1, for the case when J(t, t′) is defined by a table of discrete values AJ(k, l) = J(τ_l + θ_k, τ_l) for arbitrarily selected discrete times τ_l, l = 1, 2, ... (ages at loading) and θ_k, k = 1, 2, ... (times elapsed from loading). The COMMON statement must also be inserted in Fig. 1, and the values AJ(k, l), TL (k) (= $\log_{10} \theta_k$) and AL(l) (= $\log_{10} \tau_l$) must be assigned in Fig. 1. The J(t, t′)-values are determined by double linear interpolation in log t′- and log (t — t′) - scales. The values θ_k = t′ — t and τ_l = t′ must be selected sufficiently close to each other and must cover the whole field of values t′ — t and t′ called from the main program. AJ (1, l) = 1/E(τ_l) and θ_1 should be set equal to the time to which the E-value pertains, usually about 0.001 day. The largest τ_l and θ_k must be greater than the largest t′ and t — t′ called.

where

$$E''_r = \tfrac{1}{2}\,(E_{R_{r,r}} + E_{R_{r,r-1}}) \tag{14}$$

$$\Delta\varepsilon''_r = \frac{1}{E''_r} \sum_{s=1}^{r-1} \Delta\varepsilon_s \, \tfrac{1}{2}\,(E_{R_{r,s}} + E_{R_{r,s-1}} - E_{R_{r-1,s}} - E_{R_{r-1,s-1}}) + \Delta\varepsilon^0_r \tag{15 a}$$

(for $r > 1$)

$$\Delta\varepsilon''_r = \Delta\varepsilon^0_r \tag{15 b}$$

(for $r = 1$).

The above algorithm can be easily generalized to multiaxial stress states. Then one has instead of $J\,(t,\,t')$ two creep functions, one for the volumetric and one for the deviatoric components [17]. However, in view of the present poor knowledge of multiaxial creep, no more sophisticated assumption than a constant Poisson ratio can be justified. Then both creep functions are proportional to $J\,(t,\,t')$ and the response is also fully characterized by the uniaxial relaxation function.

It should be pointed out that Eqs. (3)-(5b) or (13)-(15 b) may be also used as basis for the analysis of creep effects in a general nonhomogeneous structure. Namely, Eq. (3) or (13) can be regarded as a fictitious elastic stress-strain law with pseudo-instantaneous elastic modulus E''_r and pseudo-inelastic strain $\Delta\varepsilon''_r$ because both E''_r and $\Delta\varepsilon''_r$ can be computed. The values $\Delta\sigma_r$ and $\Delta\varepsilon_r$ may thus be determined as the elastic solution for elastic moduli E''_r, inelastic strains $\Delta\varepsilon''$ and given changes of loads and boundary displacements during time interval Δt_r. The creep problem is thus converted to a series of elasticity problems [10, 9]. (For details and examples References [15] and [16] may be consulted.) Nevertheless, in practical application this method is not feasible if the structural system involves too many unknowns, as in three-dimensional finite element analyses, because the requirement for storing the complete history of stress for all elements of the structure overtaxes the capacity of computers presently available. A modification of the method which avoids this difficulty has been derived in Ref. [17].

THE SPECIAL CASE OF STRAINS VARYING LINEARLY WITH THE CREEP COEFFICIENT

In the case that the strain history has the form

$$\varepsilon\,(t) - \varepsilon^0\,(t) = \varepsilon_0 + \varepsilon_1\,\varphi\,(t,\,t_0) \quad \text{for } t > t_0 \tag{16}$$

and also

$$\sigma\,(t) = 0 \quad \text{for } 0 < t < t_0 \tag{17}$$

where ε_0 and ε_1 are arbitrary constants, the solution of stress history can be simply obtained from the solution for $\varepsilon_1 = 0$, $\varepsilon_0 = 1$, *i.e.* from the relaxation function. The following *theorem* holds true.

If conditions (16) and (17) are satisfied, stress $\sigma\,(t)$ varies linearly with $E_R\,(t,\,t_0)$ and the stress-strain relations may be written (exactly) in the form of an incremental elastic law :

$$\Delta\sigma\,(t) = E''\,(t,\,t_0)\,(\Delta\varepsilon\,(t) - \Delta\varepsilon''\,(t)) \tag{18}$$

in which

$$\Delta\varepsilon\,(t) = \varepsilon\,(t) - \varepsilon\,(t_0),\ \Delta\sigma\,(t) = \sigma\,(t) - \sigma\,(t_0) \tag{19}$$

$$\Delta\varepsilon''\,(t) = \frac{\sigma\,(t_0)}{E\,(t_0)}\,\varphi\,(t,\,t_0) + \varepsilon^0\,(t) - \varepsilon^0\,(t_0) \tag{20}$$

$$E''\,(t,\,t_0) = \frac{E\,(t_0) - E_R\,(t,\,t_0)}{\varphi\,(t,\,t_0)} = \frac{E\,(t_0) - E_R\,(t,\,t_0)}{E\,(t_0)\,J\,(t,\,t_0) - 1} \tag{21}$$

Proof. First it is necessary to formulate the Volterra integral equation relating the creep function $J(t, t')$ and relaxation function $E_R(t, t')$. To this end, one may consider the strain history as a unit step function, that is $\varepsilon = 1$ for $t \geqslant t_0$ and $\varepsilon = 0$ for $t < t_0$, in which case the stress response is, by definition, $\sigma(t) = E_R(t, t_0)$ for $t \geqslant t_0$. Substitution in Eq. (1) with $\varepsilon^0 = 0$ then yields, after rearrangements,

$$J(t, t_0) E(t_0) + \int_{t_0}^{t} J(t, t') \frac{\partial E_R(t', t_0)}{\partial t'} dt' = 1 \qquad (t \geqslant t_0) \quad (22)$$

Now, assume that Eqs. (18)-(21) are true. If one substitutes Eq. (21) with Eqs. (16), (19) and (20) in Eq. (18) and notes that $\sigma(t_0)/E(t_0) = \varepsilon^0$, Eq. (18) becomes

$$\sigma(t) = \sigma(t_0) + (E(t_0) - E_R(t, t_0))(\varepsilon_1 - \varepsilon_0) \quad (23)$$
(for $t \geqslant t_0$)

Insertion of this expression and Eq. (16) into Eq. (1) yields :

$$\varepsilon_0 + \varepsilon_1 (E(t_0) J(t, t_0) - 1) = J(t, t_0) \sigma(t_0) - (\varepsilon_1 - \varepsilon_0) \int_{t_0}^{t} J(t, t') \frac{\partial E_R(t', t_0)}{\partial t'} dt' \quad (24)$$

or

$$\varepsilon_0 - \varepsilon_1 = E(t_0) J(t, t_0)(\varepsilon_0 - \varepsilon_1) + (\varepsilon_0 - \varepsilon_1) \int_{t_0}^{t} J(t, t') \frac{\partial E_R(t, t')}{\partial t'} dt' \quad (25)$$

If $\varepsilon_0 = \varepsilon_1$, this equation is identically satisfied, and if $\varepsilon_0 \neq \varepsilon_1$, Eq. (25) may be divided by $(\varepsilon_0 - \varepsilon_1)$, which yields identity (22). On the other hand if Eqs. (18)-(21) were false, Eq. (25) would be also false which is impossible because identity (22) would be violated. Eqs. (18)-(21) are thus proved to be true (and exact) for any ε_0 and ε_1. (This theorem with proof has been first published in Ref. [21] and is here given for reader's convenience.)

It should also be noted that the theorem just proved can serve as basis of a very efficient approximate method of analysis of creep problems for nonhomogeneous structures which has been outlined in Ref. [21]. Namely, in most of such problems (*e.g.*, prestress loss, composite cross sections, shrinkage stresses, differential creep, creep buckling) the variation of strain with time is well approximated by a function of form (16) linearly dependent on creep coefficient $\varphi(t, t_0)$ or creep function $J(t, t_0)$. Because Eq. (18) has the form of a fictitious incremental elastic law, the approximate solution of the creep problem may be obtained carrying out a single elastic analysis with fictitious elastic moduli $E''(t, t_0)$ for inelastic strains (20). Moduli E'' may be conveniently expressed in a form reminiscent of the effective modulus

$$E''(t, t_0) = \frac{E(t_0)}{1 + \chi(t, t_0)\varphi(t, t_0)} \quad (26)$$

where, according to (21),

$$\chi(t, t_0) = \left(1 - \frac{E_R(t, t_0)}{E(t_0)}\right)^{-1} - \frac{1}{\varphi(t, t_0)} \quad (27)$$

The values of χ usually lie between 0.5 and 1.0. Because in absence of aging $\chi \doteq 1$, coefficient χ introduces a correction due to aging. For this reason χ has been named aging coefficient and E'' age-adjusted effective modulus. A detailed discussion of this method, with tables of χ-values for the ACI and CEB creep functions, is given in Ref. [21], along with examples of application, showing distinct superiority of the method over the well-known simplified methods, such as the rate-of-creep method and the effective modulus method.

CONCLUSION

With the algorithm described, conversion of creep data into relaxation data becomes a simple task and can be carried out for any form of a linear age-dependent viscoelastic stress-strain law.

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RÉSUMÉ

Détermination numérique de l'évolution à long terme des contraintes à partir de l'évolution des déformations dans le béton. — *Cet article présente un algorithme numérique pas à pas propre au calcul de la variation des contraintes à partir de la variation de toute déformation déterminée dans un matériau dont les propriétés visco-élastiques sont une fonction linéaire du temps. La base de l'algorithme est une approximation par somme finie de l'intégrale de Stieljes, qui exprime la superposition des déformations en réponse à tous les accroissements de contrainte antérieurs. On présente un programme en FORTRAN IV qui permet d'obtenir par ordinateur une solution rapide et économique. Ce programme peut s'appliquer à quelque forme que ce soit de la fonction fluage, même dans le cas où la pente de la courbe de fluage sur une échelle de temps logarithmique est importante pour plusieurs ordres de grandeur du temps écoulé, c'est-à-dire lorsque le spectre de retardation est étendu. Afin de tenir compte du fait, on divise le temps selon une progression géométrique (ou presque). On peut définir la fonction de fluage par des formules, et aussi bien par une table de valeurs, propre à opérer la conversion directe des déformations mesurées en contraintes. On donne des exemples numériques et une excellente convergence apparaît. On peut entreprendre de même la conversion des données de relaxation des contraintes en données de fluage. Dans le cas particulier des déformations qui varient en fonction linéaire du coefficient de fluage, on démontre un nouveau théorème dont l'utilité est d'établir que la variation de contrainte correspondante est une fonction linéaire de la courbe de relaxation des contraintes. Ce théorème permet une généralisation de la méthode du module effectif, ce qui améliore notablement la précision lorsque les propriétés de fluage dépendent de l'âge.*

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