PARTIAL ORDERINGS OF PERMUTATIONS AND MONOTONICITY OF A RANK CORRELATION STATISTIC

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1. Summary and introduction

The purpose of this paper which is a continuation of a preliminary report [7] of the authors is to find a condition for a rank correlation statistic based on a random sample from a bivariate distribution to be monotone with respect to the parameter involved in the underlying distribution which represents correlation between two variates.

As a tool for this aim a partial ordering w is introduced in the set Π of all permutations of $(1, 2, \cdots, n)$ for a fixed positive integer n as follows. For two elements $\pi=(r_1, \cdots, r_n)$ and $\pi'=(s_1, \cdots, s_n)$ of Π we write $\pi\overset{w}{\to}\pi'$ iff (if and only if) there exists a positive integer $i<n$ such that $r_i=s_{i+1}<r_{i+1}=s_i$ and $r_k=s_k$ for $k\neq i$, $i+1$. We define $\pi\overset{w}{\geqq}\pi'$ iff there exists a chain $\pi=\pi_0\overset{w}{\to}\pi_1\overset{w}{\to}\cdots\overset{w}{\to}\pi_m=\pi'$. Then, the space Π is seen to be a lattice with respect to this partial ordering, which answers affirmatively Savage's question 6 [5].

Next, concepts of 'positively regression dependent' and 'positively quadrant dependent' introduced respectively by Tukey [6] and Lehmann [3] are generalized to those of 'larger regression dependence' and 'larger quadrant dependence', respectively, which enable us to compare the degree of correlation between two variates for two bivariate distributions with common marginal distributions. These notions can be further led to those of 'monotone regression dependence' and 'monotone quadrant dependence'.

Now one of the main results of this paper is Theorem 6.1 which implies (Corollary 6.1) that if a family of bivariate distributions $\{F_\alpha(x, y)\}$ has monotone regression dependence on x and if a rank statistic is nondecreasing, then the statistic is stochastically nondecreasing with respect to α. Theorem 6.2 gives some sufficient conditions for a rank statistic to be nondecreasing. Several examples of nondecreasing rank statistics including two rank correlations, Kendall's τ and Spearman's ρ, and also of families of distributions with monotone dependence are given.

In Section 8 the Blomqvist rank statistic Q [1] is shown to be stochastically nondecreasing with respect to α, provided the family $\{F_\alpha(x, y)\}$ has monotone quadrant dependence. In testing the hypothesis of independence for the family $\mathscr{F}_1$ of distributions the Blomqvist test with a critical region given by $Q \geqq c$ is proved to be of Neyman-Pearson type.

2. A partial ordering in the space of permutations

Let Π be the space consisting of all permutations of $(1, 2, \cdots, n)$ for a fixed positive integer n. We shall define a partial ordering in this space as follows.

DEFINITION 2.1. Let $\pi = (r_1, \cdots, r_n)$ and $\pi' = (s_1, \cdots, s_n)$ be two elements of Π. We write $\pi \overset{w}{\rightarrow} \pi'$ iff there exists a positive integer $i < n$ such that

$$(2.1) \qquad r_i = s_{i+1} < r_{i+1} = s_i \quad \text{and} \quad r_k = s_k \quad \text{for } k \neq i,\ i+1 .$$

The element π will be said to be not smaller than π' in the sense w (weak sense) and denoted by $\pi \overset{w}{\geqq} \pi'$ iff there exists a set of elements π_0, $\pi_1, \cdots, \pi_m$ of Π for an integer $m \geqq 0$ such that

$$(2.2) \qquad \pi = \pi_0 \overset{w}{\rightarrow} \pi_1 \overset{w}{\rightarrow} \pi_2 \overset{w}{\rightarrow} \cdots \overset{w}{\rightarrow} \pi_m = \pi' .$$

The partial ordering w mentioned briefly by Lehmann [3] will be illustrated by Figures. 1 and 2 for two simple cases when $n=3$ and 4, respectively.

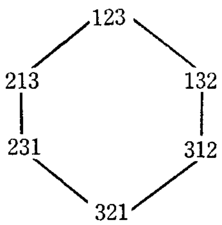

Fig. 1. Partial ordering w when $n=3$.

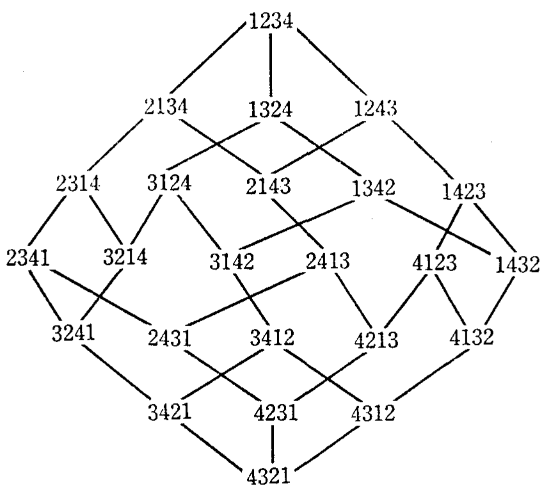

Fig. 2. Partial ordering w when $n=4$.

DEFINITION 2.2. Let $\mathcal{B}$ be the set of all pairs (k, l) of integers satisfying $1 \leqq k < l \leqq n$. For any element $\pi = (r_1, \cdots, r_n)$ of Π let $B(\pi)$ denote the set of all pairs (r_i, r_j) such that $i < j$ and $r_i < r_j$. Clearly $B(\pi)$ is a subset of $\mathcal{B}$.

PROPOSITION 2.1. Let π and π' be any elements of Π. Then a necessary and sufficient condition that $\pi \overset{w}{\geqq} \pi'$ is that

$$B(\pi) \supset B(\pi') . \tag{2.3}$$

PROOF. The necessity is obvious, since the condition $\pi \overset{w}{\longrightarrow} \pi'$ implies (2.3).

Sufficiency. Since $B(\pi) = B(\pi')$ is equivalent to $\pi = \pi'$, suppose that (2.3) holds with a proper inclusion. Let $\pi = (r_1, \cdots, r_n)$ and $\pi' = (s_1, \cdots, s_n)$. Let j be the minimum value of i satisfying $r_i \neq s_i$. Define k by $r_k = s_j$. Then the assumption (2.3) implies that $r_j < s_j$, $r_{k-1} < r_k$ and also that

$$(r_{k-1}, r_k) \in B(\pi) - B(\pi') .$$

Let π_1 denote the element of Π obtained by interchanging the two components r_{k-1} and r_k of π. Then it follows that

$$\pi \overset{w}{\longrightarrow} \pi_1 \quad \text{and} \quad B(\pi_1) = B(\pi) - (r_{k-1}, r_k) \supset B(\pi') .$$

By repeating this argument we can construct a chain $\pi_1, \cdots, \pi_m$ such that

$$\pi \overset{w}{\longrightarrow} \pi_1 \overset{w}{\longrightarrow} \cdots \overset{w}{\longrightarrow} \pi_m \quad \text{and} \quad B(\pi_m) = B(\pi') ,$$

which completes the proof.

PROPOSITION 2.2. Let B be a subset of $\mathcal{B}$. A necessary and sufficient condition that there exists $\pi_0 \in \Pi$ satisfying $B = B(\pi_0)$ is that both of the following two conditions are satisfied:
(i) If (r, s) and $(s, t) \in B$, then $(r, t) \in B$,
(ii) If $(r, t) \in B$ and $r < s < t$, then $(r, s) \in B$ or $(s, t) \in B$.

PROOF. The necessity may be obvious. It is easily seen that sufficiency follows from the following fact: If B satisfies the conditions (i) and (ii) and is included by $B(\pi)$ properly, then there exists $\pi' \in \Pi$ such that

$$\pi \overset{w}{\longrightarrow} \pi' \quad \text{and} \quad B(\pi') \supset B . \tag{2.4}$$

To prove this, let $\pi = (r_1, \cdots, r_n)$ and let (k, l) be any element of B which minimizes $l - k$ subject to the condition that $(r_k, r_l) \in B(\pi) - B$. Then $l = k + 1$, since the assumption $k < j < l$ leads us to a contradiction as is seen below.

Since $r_k < r_l$ by definition, we shall consider three cases: Case 1 when $r_k < r_j < r_l$, Case 2 when $r_j < r_k < r_l$ and Case 3 when $r_k < r_l < r_j$. In Case 1 it holds that (r_k, r_j), $(r_j, r_l) \in B(\pi)$. Since $l-k > j-k$ and $l-j$, we can conclude that (r_k, r_j), $(r_j, r_l) \in B$, and hence $(r_k, r_l) \in B$ by the assumption (i), which is a contradiction. In Case 2 we know $(r_j, r_l) \in B(\pi)$ and $l-k > l-j$, which implies by the definition of (k, l) that $(r_j, r_l) \in B$. Now the assumption (ii) implies that $(r_j, r_k) \in B$ or $(r_k, r_l) \in B$. The latter is incompatible with the definition of (k, l). From the former relation it follows that $(r_j, r_k) \in B(\pi)$, which contradicts with the assumption $k < j$. Case 3 can be treated similarly.

Thus we know $l = k+1$. Define π' by interchanging two components r_k and r_{k+1} of π. Then $\pi \overset{w}{\to} \pi'$ and

$$B(\pi') = B(\pi) - (r_k, r_{k+1}) \supset B .$$

This proves (2.4) and the proof of the proposition is complete.

The following theorem is an answer to an open problem presented by Savage [5], Question 6.

THEOREM 2.1. *The space Π is a lattice by defining the join $\pi_1 \vee \pi_2$ of two elements π_1 and π_2 of Π as the smallest (in the sense of the partial ordering w) element π satisfying $\pi \overset{w}{\geqq} \pi_i$ for $i=1, 2$, while defining the meet $\pi_1 \wedge \pi_2$ dually.*

PROOF. Let π_i, $i=1, 2$, be any elements of Π. It is sufficient to show that both the join and the meet of them are uniquely determined. We shall show the existence of $\pi_0 \in \Pi$ which satisfies that $\pi_0 \overset{w}{\geqq} \pi_i$ for $i= 1, 2$ and also that

$$(2.5) \qquad \pi \overset{w}{\geqq} \pi_i \quad \text{for } i=1, 2 \implies \pi \overset{w}{\geqq} \pi_0 .$$

Let $B(\pi_1, \pi_2)$ be the set of all $(r, s) \in \mathscr{B}$ for which there exists a chain of integers $r = r_0 < r_1 < \cdots < r_m = s$ for some m, where

$$(r_i, r_{i+1}) \in B(\pi_1) \cup B(\pi_2) \qquad \text{for } i=0, 1, \cdots, m-1 .$$

Obviously it holds that

$$(2.6) \qquad B(\pi_1, \pi_2) \supset B(\pi_i) \qquad \text{for } i=1, 2 .$$

Now it can be shown that $B = B(\pi_1, \pi_2)$ satisfies the conditions (i) and (ii) of Proposition 2.2, and hence there exists $\pi_0 \in \Pi$ which satisfies $B(\pi_0) = B(\pi_1, \pi_2)$. By (2.6) we find that $B(\pi_0) \supset B(\pi_i)$ for $i=1, 2$, which is equivalent to $\pi_0 \overset{w}{\geqq} \pi_i$ for $i=1, 2$ because of Proposition 2.1.

Suppose now that $\pi \overset{w}{\geqq} \pi_i$ for $i=1, 2$, or equivalently, $B(\pi)\supset B(\pi_i)$ for $i=1, 2$. Then it can be easily shown that $B(\pi)\supset B(\pi_1, \pi_2)=B(\pi_0)$, and hence $\pi \overset{w}{\geqq} \pi_0$.

3. Other partial orderings of permutations

Two kinds of partial orderings besides w can be defined in the space Π as follows, the first of which is the same as that introduced by Lehmann [3] as 'better ordered than' and the second was considered by Savage [4], [5]. Though they will not be used in the following sections of this article, they are included here partly because of their possible applicability and partly because of their close relationship to the partial ordering w which will play an important role starting with Section 4. All propositions in this section will be stated without proof.

DEFINITION 3.1. Let $\pi=(r_1, \cdots, r_n)$ and $\pi'=(s_1, \cdots, s_n)$ be elements of Π. We write $\pi \overset{w}{\rightarrow} \pi'$ iff there exist two subscripts i, j $(i<j)$ such that

$$r_i=s_j=r_j-1=s_i-1 \qquad \text{and} \qquad r_k=s_k \quad \text{for } k\neq i, j, \tag{3.1}$$

while we write $\pi \overset{s}{\rightarrow} \pi$ iff there exist two subscripts i, j $(i<j)$ such that

$$r_i=s_j<r_j=s_i \qquad \text{and} \qquad r_k=s_k \quad \text{for } k\neq i, j. \tag{3.2}$$

The element π will be said to be larger than π' in the sense w' (another weak sense) or s (strong sense) and denoted by $\pi \overset{w'}{\geqq} \pi'$ or $\pi \overset{s}{\geqq} \pi'$ according as there exists a chain $\pi=\pi_0\rightarrow\pi_1\rightarrow\cdots\rightarrow\pi_m\rightarrow\pi'$ in the sense w' or s.

The following proposition, an immediate consequence of the definition, gives a justification of the names, 'weak' and 'strong'.

PROPOSITION 3.1. If $\pi \overset{w}{\geqq} \pi'$ or $\pi \overset{w'}{\geqq} \pi'$, then $\pi \overset{s}{\geqq} \pi'$.

The following proposition exhibits duality of two orderings w and w' and self-duality of s.

PROPOSITION 3.2. The space Π is a group with multiplication

$$(r_1, \cdots, r_n)(s_1, \cdots, s_n)=(r_{s_1}, \cdots, r_{s_n}).$$

Denoting by π^{-1} the inverse element of π, it holds that

$$\pi \overset{w}{\geqq} \pi' \Longleftrightarrow \pi^{-1} \overset{w'}{\geqq} \pi'^{-1}, \qquad \pi \overset{s}{\geqq} \pi' \Longleftrightarrow \pi^{-1} \overset{s}{\geqq} \pi'^{-1}.$$

The following two propositions are w'-analogues of Proposition 2.1 and Theorem 2.1, respectively.

PROPOSITION 3.3. For any $\pi=(r_1, \cdots, r_n) \in \Pi$ let $B'(\pi)$ be the set of all pairs (i, j) of positive integers such that $i<j$ and $r_i<r_j$. Then

$$\pi \overset{w'}{\geqq} \pi' \iff B'(\pi) \supset B'(\pi') .$$

PROPOSITION 3.4. The space Π is a lattice by defining the join and the meet in the fashion of Theorem 2.1 with the ordering w' replacing w.

As a characterization of the partial ordering s we shall state the following proposition which is a generalization of Lehmann's Theorem 5 [3].

PROPOSITION 3.5. Let $\pi=(r_1, \cdots, r_n)$ and $\pi'=(s_1, \cdots, s_n)$. For any positive integer $m<n$ let $r_{1m}<\cdots<r_{mm}$ be the rearrangement of $r_1, \cdots, r_m$ in order of magnitude, while $s_{1m}<\cdots<s_{mm}$ be the rearrangement of $s_1, \cdots, s_m$. Then a necessary and sufficient condition that $\pi \overset{s}{\geqq} \pi'$ is that

$$r_{km} \leqq s_{km} \qquad \text{for any } k \text{ and } m \text{ such that } 1 \leqq k \leqq m \leqq n .$$

4. A property of wedge sets

As noted before only the partial ordering w will be treated henceforth in this paper. For simplicity, therefore, we shall drop the superscript w in the notation $\pi \overset{w}{\rightarrow} \pi'$ or $\pi \overset{w}{\geqq} \pi'$.

DEFINITION 4.1. For any positive integers k and l let $[y_k \geqq y_l]$ denote the subset of R^n consisting of the point $(y_1, \cdots, y_n)$ which satisfies $y_k \geqq y_l$. A subset $[y_{s_1} \geqq \cdots \geqq y_{s_n}]$ can be defined similarly. These sets may be called wedge sets in R^n.

The following proposition will play a key role in the proof of Theorem 6.1.

PROPOSITION 4.1. For any $\pi \in \Pi$ it holds that

$$\bigcup_{(s_1, \cdots, s_n) \geqq \pi} [y_{s_1} \geqq \cdots \geqq y_{s_n}] = \bigcap_{(k, l) \in B(\pi)} [y_k \geqq y_l] . \tag{4.1}$$

PROOF. Let $y^0=(y_1^0, \cdots, y_n^0)$ be any member of the left-hand side of (4.1). By definition there exists $\pi'=(s_1, \cdots, s_n) \geqq \pi$ satisfying $y_{s_1}^0 \geqq \cdots \geqq y_{s_n}^0$. Take any $(k, l) \in B(\pi)$. Then Proposition 2.1 implies that $(k, l) \in B(\pi')$, and hence $y_k^0 \geqq y_l^0$. Thus the point y^0 belongs to the right-hand side of (4.1).

Conversely, let y^0 be any member of the right-hand side of (4.1), i.e.,

$$(k, l) \in B(\pi) \implies y_k^0 \geqq y_l^0 . \tag{4.2}$$

There exists $\pi'=(s_1, \cdots, s_n) \in \Pi$ such that $y^0_{s_1} \geqq \cdots \geqq y^0_{s_n}$. Define $\pi''=(t_1, \cdots, t_n)=\pi \vee \pi'$, the join of π and π' in the sense given in Theorem 2.1. By definition $(t_1, \cdots, t_n) \geqq \pi$, so that if we can show that $y^0_{t_1} \geqq \cdots \geqq y^0_{t_n}$, then the point y^0 proves to belong to the left-hand side of (4.1) and the proof is complete. Now we shall show that

$$y^0_{t_i} \geqq y^0_{t_{i+1}} \tag{4.3}$$

for any i. If $t_i > t_{i+1}$, then the inequality $\pi'' \geqq \pi'$ implies that the two numbers t_i and t_{i+1} appear in $(s_1, \cdots, s_n)$ in the same order, which implies (4.3) in turn. On the contrary, if $t_i < t_{i+1}$, then the pair $(t_i, t_{i+1}) \in B(\pi'')$. In fact it belongs to $B(\pi) \cup B(\pi')$, because otherwise we can construct π^* such that $\pi'' > \pi^* \geqq \pi$ and π' by interchanging the two components t_i and t_{i+1} in π'', a contradiction with the definition of π''. In the case when $(t_i, t_{i+1}) \in B(\pi)$ the relation (4.3) follows from (4.2). In the remaining case when $(t_i, t_{i+1}) \in B(\pi')$ also we obtain (4.3) easily.

5. Monotone regression dependence

DEFINITION 5.1. Assume that each of two bivariate cdf's $F(x, y)$ and $G(x, y)$ has continuous marginal distributions and that both have a common marginal distribution of x. Let $F(y \mid x)$ and $G(y \mid x)$ denote the conditional cdf's of y given x based on F and G, respectively, which are assumed to be continuous in y for any x. Let $F^{-1}(u \mid x)$ and $G^{-1}(u \mid x)$ denote the minimum values of u-points of them for $0<u<1$. Then the distribution $G(x, y)$ will be said to have larger regression dependence on x than $F(x, y)$ iff

$$F^{-1}(u \mid x') \geqq F^{-1}(v \mid x) \implies G^{-1}(u \mid x') \geqq G^{-1}(v \mid x) \tag{5.1}$$

for any $x'>x$ and any u and v.

The assumption of common marginal distribution of x is not essential, since any bivariate distribution with continuous marginal distributions can be so transformed by a suitable transformation of x as to satisfy the assumption.

DEFINITION 5.2. A family of distributions, $\{F_\alpha(x, y) \mid \alpha \in \Omega\}$, where $\Omega \subset R^1$, is said to have monotone regression dependence on x with respect to α iff the distribution $F_{\alpha'}(x, y)$ has larger regression dependence on x than $F_\alpha(x, y)$ for any $\alpha' > \alpha$.

The following proposition shows that Definition 5.1 provides a generalization of Lehmann's concept [3] of 'positively regression dependent', or the family $\mathcal{F}_2$.

PROPOSITION 5.1. Let $F_1(x)$ and $F_2(y)$ be the marginal distributions of $F(x, y)$. A necessary and sufficient condition that $F(x, y) \in \mathscr{F}_2$ is that $F(x, y)$ has larger regression dependence on x than the product distribution $F_1(x)F_2(y)$.

PROOF. *Necessity.* Assume $F \in \mathscr{F}_2$ and we shall show that

$$F_2^{-1}(u) \geqq F_2^{-1}(v) \Longrightarrow F^{-1}(u \mid x') \geqq F^{-1}(v \mid x) \tag{5.2}$$

for any $x' > x$. Since $F \in \mathscr{F}_2$ by definition,

$$F(y \mid x') \leqq F(y \mid x) \qquad \text{for any } y,$$

and hence

$$F^{-1}(v \mid x') \geqq F^{-1}(v \mid x) \qquad \text{for any } v.$$

The condition $F_2^{-1}(u) \geqq F_2^{-1}(v)$ is equivalent to $u \geqq v$, which implies

$$F^{-1}(u \mid x') \geqq F^{-1}(v \mid x').$$

Thus we have proved (5.2). The sufficiency may be proved by inverting the above argument.

PROPOSITION 5.2. Assume that both of the conditional cdf's $F(y \mid x)$ and $G(y \mid x)$ are continuous and strictly increasing in y for each x. Then a necessary and sufficient condition that $G(x, y)$ has larger regression dependence on x than $F(x, y)$ is that

$$F(y \mid x) \geqq G(y' \mid x) \quad \Longrightarrow \quad F(y \mid x') \geqq G(y' \mid x') \tag{5.3}$$

for any $x' > x$ and any y and y'.

PROOF. Only sufficiency will be proved, because necessity may be treated similarly. Assume that G does not have larger regression dependence on x than F and we shall show that we are led to a contradiction. By the assumption there exist x, x', u and v such that $x' > x$, $F^{-1}(u \mid x') \geqq F^{-1}(v \mid x)$ and $G^{-1}(u \mid x') < G^{-1}(v \mid x)$. Define $y = F^{-1}(v \mid x)$ and $y' = G^{-1}(u \mid x')$, then it follows that

$$\begin{aligned} &F(y \mid x') \leqq u, \qquad F(y \mid x) = v, \\ &G(y' \mid x') = u, \qquad G(y' \mid x) < v. \end{aligned} \tag{5.4}$$

By the second and the fourth relations together with the continuity of $F(y \mid x)$ there exists a sufficiently small $\varepsilon > 0$ such that

$$F(y - \varepsilon \mid x) > G(y' \mid x).$$

The first and the third relations of (5.4), however, implies that

$$F(y-\varepsilon \mid x')<u=G(y' \mid x')\,,$$

which together with the preceding inequality contradicts with (5.3).

Example 5.1. Let U and V be any independent real random variables each having a continuous distribution. Define

(i) $$X=U, \qquad Y_\rho=\rho U+(1-\rho^2)^{1/2}V \quad \text{for } -1\leqq\rho\leqq 1\,,$$

or

(ii) $$X=U, \qquad Y_\alpha=\alpha U+V \qquad \text{for } -\infty<\alpha<\infty\,.$$

Then each of the families of the distributions $F_\rho(x, y)$ for the case (i) and $F_\alpha(x, y)$ for (ii) has monotone regression dependence on x with respect to ρ and α, respectively. A bivariate normal distribution is a particular case of (i) when U and V are $N(0, 1)$ variates.

Example 5.2. Let U and V be any independent real random variables each having a continuous distribution, U being distributed on the interval $(0, 1)$, while V on $(0, \infty)$. Define

$$X=U, \qquad Y_\alpha=(1+\alpha U)V \quad \text{for } \alpha>-1\,.$$

Then the family of the distributions $F_\alpha(x, y)$ of (X, Y_α) has monotone regression dependence on x with respect to α.

6. Monotonicity of rank correlation

Let (X, Y) be a bivariate random variable having a cdf $F(x, y)$ with continuous marginal distributions and let $(x_1, y_1), \cdots, (x_n, y_n)$ be n independent observations on (X, Y). Let r_i be the rank of x_i (from the largest on) among $\{x_1, \cdots, x_n\}$ and s_j the rank of y_j among $\{y_1, \cdots, y_n\}$. Define t_k for $k=1, \cdots, n$ by the two relations, $t_k=r_j$ and $k=s_j$, or equivalently by

(6.1) $$(t_1, \cdots, t_n)=(r_1, \cdots, r_n)(s_1, \cdots, s_n)^{-1}\,,$$

where the inverse is to be interpreted in the sense stated in Proposition 3.2. It is noted here that $(t_1, \cdots, t_n)$ is a function of the random sample (x_i, y_i), $i=1, \cdots, n$, or a statistic.

DEFINITION 6.1. A statistic R will be called a rank statistic iff

(6.2) $$R(x_1, y_1, \cdots, x_n, y_n)=\varphi(r_1, s_1, \cdots, r_n, s_n)=\psi(t_1, \cdots, t_n)$$

for some functions φ and ψ. It is nondecreasing iff ψ is nondecreasing, i.e., (6.3) $$\pi \geqq \pi' \implies \phi(\pi) \geqq \phi(\pi') .$$

Any rank statistic is desirable to be nondecreasing in order to serve as a measure of correlation between X and Y, or a rank correlation.

THEOREM 6.1. *If*
(i) *$G(x, y)$ has larger regression dependence on x than $F(x, y)$, and*
(ii) *R is a nondecreasing rank statistic,*
then R is stochastically not smaller under G than under F, i.e.,

(6.4) $$P_G\{R \geqq c\} \geqq P_F\{R \geqq c\} \quad \textit{for any real } c .$$

PROOF. Using the conditional probability given x, we can write

(6.5) $$P_F\{R \geqq c\} = E^x[P_F\{R \geqq c \mid x\}] ,$$

where $x=(x_1, \cdots, x_n)$. Being a rank statistic, R is invariant under any permutation of $(x_1, \cdots, x_n)$, and hence we may suppose that $x_1 > \cdots > x_n$ in considering the conditional probability given x. For a fixed x such that $x_1 > \cdots > x_n$ it holds that

(6.6) $$\{(y_1, \cdots, y_n) \in R^n \mid R \geqq c\} = \bigcup_{\phi(\pi') \geqq c} [y_{t_1} \geqq \cdots \geqq y_{t_n}] ,$$

where the right-hand side is the union with respect to $\pi'=(t_1, \cdots, t_n)$ satisfying $\phi(\pi') \geqq c$. Since ϕ is nondecreasing, we find that symbolically

(6.7) $$\bigcup_{\phi(\pi') \geqq c} = \bigcup_{\phi(\pi) \geqq c} \bigcup_{\pi' \geqq \pi} .$$

From (6.6), (6.7) and Proposition 4.1 it follows that

$$P_F\{R \geqq c \mid x\} = P_F\left\{ \bigcup_{\phi(\pi) \geqq c} \bigcap_{(k,l) \in B(\pi)} [y_k \geqq y_l] \mid x \right\} .$$

If we define $\eta_i = F(y_i \mid x_i)$ or equivalently $y_i = F^{-1}(\eta_i \mid x_i)$ for $i=1, \cdots, n$, then conditionally for given x the random variables $\eta_1, \cdots, \eta_n$ are distributed independently and identically according to a uniform distribution over the interval $(0, 1)$. Thus we have

$$P_F\{R \geqq c \mid x\} = P^n\left\{ \bigcup_{\phi(\pi) \geqq c} \bigcap_{(k,l) \in B(\pi)} [F^{-1}(\eta_k \mid x_k) \geqq F^{-1}(\eta_l \mid x_l)] \right\} .$$

Since $x_k > x_l$ and the cdf G has larger regression dependence on x than F the right-hand side does not exceed

$$P^n\left\{ \bigcup_{\phi(\pi) \geqq c} \bigcap_{(k,l) \in B(\pi)} [G^{-1}(\eta_k \mid x_k) \geqq G^{-1}(\eta_l \mid x_l)] \right\} ,$$

which is identical with $P_G\{R \geqq c \mid x\}$. Consequently, we find that

$$P_G\{R \geqq c \mid x\} \geqq P_F\{R \geqq c \mid x\} ,$$

which implies (6.4) because of (6.5).

COROLLARY 6.1. *If a family of distributions* $\{F_\alpha(x, y) \mid \alpha \in \Omega\}$, *where* $\Omega \subset R^1$, *has monotone regression dependence on* x *and if* R *is a nondecreasing rank statistic, then* R *is stochastically nondecreasing with respect to* α, *i.e.,* $P_{F_\alpha}\{R \geqq c\}$ *is a nondecreasing function of* $\alpha \in \Omega$ *for any real* c.

THEOREM 6.2. *Each of the following four conditions is sufficient for a rank statistic* R *to be nondecreasing*:

(i) $$R = \sum_{i=1}^{n} f(r_i) g(s_i) ,$$

where both f *and* g *are nondecreasing real-valued functions,*

(ii) $$R = \sum_{i=1}^{n} \varphi(r_i, s_i) ,$$

where φ *is a real-valued function for which*

$$\varphi(r, s) + \varphi(r', s') \geqq \varphi(r', s) + \varphi(r, s') ,$$

whenever $r' > r$ *and* $s' > s$,

(iii) $$R = \sum_{i \neq j = 1}^{n} f(r_i, r_j) g(s_i, s_j) ,$$

where both functions $f(r, r')$ *and* $g(r, r')$ *are nondecreasing in* r *and nonincreasing in* r',

(iv) $$R = \sum_{i \neq j, k}^{n} f(r_i, r_j) g(s_i, s_k) ,$$

where both functions $f(r, r')$ *and* $g(r, r')$ *are nondecreasing in* r.

It will be noted that Theorem 6.1 combined with the part (i) which is a special case of each of (ii), (iii) or (iv), contains Lehmann's Theorem 4 [3] as a special case and also that a statistic of the form (iii) was studied by Daniels [2].

PROOF. We shall deal with the cases (ii), (iii) and (iv), where R can be written as $\sum_k \varphi(t_k, k)$, $\sum_{k \neq l} f(t_k, t_l) g(k, l)$ and $\sum_{k \neq l, m} f(t_k, t_l) g(k, m)$, respectively. Denoting these functions by $\phi(\pi)$, $\pi = (t_1, \cdots, t_n)$, let us show (6.3). Assume $\pi \xrightarrow{w} \pi' = (s_1, \cdots, s_n)$. Then by definition there exists a subscript i such that

$$t_i = s_{i+1} < t_{i+1} = s_i \quad \text{and} \quad t_k = s_k \quad \text{for } k \neq i,\ i+1 .$$

In Case (ii) it holds that

$$\psi(\pi)=\sum_{k=1}^{n}\varphi(t_k, k)\ ,$$

while

$$\psi(\pi')=\sum_{k\neq i, i+1}\varphi(t_k, k)+\varphi(t_{i+1}, i)+\varphi(t_i, i+1)\ .$$

Taking the difference, we have

$$\psi(\pi)-\psi(\pi')=\varphi(t_i, i)+\varphi(t_{i+1}, i+1)-\varphi(t_{i+1}, i)-\varphi(t_i, i+1)\geqq 0\ .$$

Case (iii). It holds that

$$\begin{aligned}\psi(\pi)=&\sum_{k, l\neq i, i+1} f(t_k, t_l)g(k, l)\\ &+\sum_{l\neq i, i+1}[f(t_i, t_l)g(i, l)+f(t_{i+1}, t_l)g(i+1, l)]\\ &+\sum_{k\neq i, i+1}[f(t_k, t_i)g(k, i)+f(t_k, t_{i+1})g(k, i+1)]\\ &\quad+f(t_i, t_{i+1})g(i, i+1)+f(t_{i+1}, t_i)g(i+1, i)\ ,\end{aligned}$$

while $\psi(\pi')$ can be obtained by interchanging t_i and t_{i+1} in the above expression. Therefore,

$$\begin{aligned}\psi(\pi)-\psi(\pi')=&\sum_{l\neq i, i+1}[f(t_{i+1}, t_l)-f(t_i, t_l)][g(i+1, l)-g(i, l)]\\ &+\sum_{k\neq i, i+1}[f(t_k, t_i)-f(t_k, t_{i+1})][g(k, i)-g(k, i+1)]\\ &\quad+[f(t_{i+1}, t_i)-f(t_i, t_{i+1})][g(i+1, i)-g(i, i+1)]\geqq 0\ .\end{aligned}$$

Case (iv). After a straightforward calculation we have

$$\psi(\pi)-\psi(\pi')=\sum_{l=1}^{n}[f(t_{i+1}, t_l)-f(t_i, t_l)]\sum_{m=1}^{n}[g(i+1, m)-g(i, m)]\geqq 0\ .$$

Example 6.1. The following five rank statistics are all nondecreasing:

$$R_1=\sum_{i\neq j=1}^{n}\mathrm{sgn}\,(r_i-r_j)(s_i-s_j)/n(n-1)\ ,$$

which is Kendall's rank correlation τ;

$$R_2=3\sum_{i\neq j, k}^{n}\mathrm{sgn}\,(r_i-r_j)(s_i-s_k)/n(n^2-1)\ ,$$

which is Spearman's rank correlation ρ:

$$R_3=\frac{1}{4}\sum_{i=1}^{n}\left[\mathrm{sgn}\left(r_i-\frac{n+1}{2}\right)+1\right]\left[\mathrm{sgn}\left(s_i-\frac{n+1}{2}\right)+1\right],$$

the number of observations falling in the first quadrant with respect to the sample median;

$$R_4 = \sum_{i=1}^{n} \xi_{r_i n} \eta_{s_i n} ,$$

where ξ_{in} (or η_{in}) stands for the expected value of the ith order statistic based on a random sample of size n from some fixed distribution U (or V). If both U and V are $N(0, 1)$, we get the normal score correlation coefficient. Finally,

$$R_5 = -\sum_{i=1}^{n} |r_i - s_i|^p \qquad \text{with } p \geqq 1 ,$$

which becomes equivalent to Spearman's ρ when $p=2$.

7. Monotone quadrant dependence

DEFINITION 7.1. Assume that cdf's $F(x, y)$ and $G(x, y)$ have common continuous marginal distributions. Then the distribution G is said to have larger quadrant dependence than F iff

$$G(x, y) \geqq F(x, y) \qquad \text{for all } x \text{ and } y.$$

A family of distributions $\{F_\alpha(x, y) \mid \alpha \in \Omega\}$, where $\Omega \subset R^1$, is said to have monotone quadrant dependence iff the distribution $F_{\alpha'}$ has larger quadrant dependence than F_α for any $\alpha' > \alpha$.

The assumption of common marginal distributions is again not essential here as in Definition 5.1. Obviously the above definition generalizes Lehmann's notion [3] of 'positively quadrant dependent' or the family $\mathcal{F}_1$, which is stated as the following

PROPOSITION 7.1. Let $F_1(x)$ and $F_2(y)$ be the marginal distributions of $F(x, y)$. A necessary and sufficient condition that $F \in \mathcal{F}_1$ is that $F(x, y)$ has larger quadrant dependence than the product distribution $F_1(x)F_2(y)$.

PROPOSITION 7.2. If a distribution $G(x, y)$ has larger regression dependence on x than another distribution $F(x, y)$, then G has larger quadrant dependence than F.

PROOF. Assume that G has larger regression dependence on x than F and that G does not have larger quadrant dependence than F. Then there exists a point (x, y) at which $G(x, y) < F(x, y)$. It will be readily seen that this implies the existence of two values x' and x'' such that $x'' < x < x'$ and

$$G(y \mid x') > F(y \mid x') \quad \text{and} \quad G(y \mid x'') < F(y \mid x'') .$$

Taking any numbers u and v which satisfy

$$G(y \mid x') > u > F(y \mid x') \quad \text{and} \quad G(y \mid x'') < v < F(y \mid x''),$$

we find that

$$F^{-1}(u \mid x') > y, \qquad G^{-1}(u \mid x') \leqq y,$$
$$F^{-1}(v \mid x'') \leqq y, \qquad G^{-1}(v \mid x'') > y.$$

Hence it follows that

$$F^{-1}(u \mid x') > F^{-1}(v \mid x'') \quad \text{and} \quad G^{-1}(u \mid x') < G^{-1}(v \mid x''),$$

which contradicts with the assumption or (5.1).

PROPOSITION 7.3. Suppose a distribution $G(x, y)$ has larger quadrant dependence than $F(x, y)$. If two real-valued functions $f(x)$ and $g(y)$ are concordant, i.e., both are nondecreasing or both are nonincreasing, then

$$\mathrm{Cov}_G(f(X), g(Y)) \geqq \mathrm{Cov}_F(f(X), g(Y)),$$

provided both covariances exist. In particular,

$$\mathrm{Cov}_G(X, Y) \geqq \mathrm{Cov}_F(X, Y).$$

This proposition, a generalization of Lehmann's Lemma 3 [3], can be proved similarly by using Hoeffding's lemma ([3], p. 1139):

$$\mathrm{Cov}_F(X, Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [F(x, y) - F_1(x)F_2(y)]\,dx\,dy.$$

As a consequence we can obtain the following proposition, where the part (i) is a special case of the part (ii).

PROPOSITION 7.4. Suppose $G(x, y)$ has larger quadrant dependence than $F(x, y)$ and let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be a random sample from either of these distributions.

(i) Let $X = f(X_1, \cdots, X_n)$ and $Y = g(Y_1, \cdots, Y_n)$, where the two functions f and g are concordant with respect to the ith argument for each i when other arguments are fixed, then it holds that

$$\mathrm{Cov}_G(X, Y) \geqq \mathrm{Cov}_F(X, Y),$$

provided both covariances exist.

(ii) Let (U, V) be a random variable distributed according to a bivariate distribution $H(u, v)$ independently of the sample (X_i, Y_i), $i=1, \cdots, n$. Let $X = f(X_1, \cdots, X_n, U)$ and $Y = g(Y_1, \cdots, Y_n, V)$, where f and g are concordant with respect to the ith argument for each $i \leqq n$ when others are fixed, then

$$\mathrm{Cov}_{G,H}(X, Y) \geqq \mathrm{Cov}_{F,H}(X, Y),$$

provided both sides exist.

COROLLARY 7.1. *Assume that $G(x, y)$ has larger quadrant dependence than $F(x, y)$. Then each of the following correlation measures:*

Kendall's $\tau=\mathrm{Cov}\,(\mathrm{sgn}\,(X_1-X_2), \mathrm{sgn}\,(Y_1-Y_2))$,

Spearman's $\rho=\mathrm{Cov}\,(\mathrm{sgn}\,(X_1-X_2), \mathrm{sgn}\,(Y_1-Y_3))$,

Blomqvist's $q=\mathrm{Cov}\,(\mathrm{sgn}\,(X-\text{median}\,X), \mathrm{sgn}\,(Y-\text{median}\,Y))$,

is not smaller under G than under F, i.e.,

$$\tau_G\geqq\tau_F\,, \qquad \rho_G\geqq\rho_F \quad \text{and} \quad q_G\geqq q_F\,.$$

8. Monotonicity of the Blomqvist rank statistic

Let $(x_1, y_1), \cdots, (x_n, y_n)$ be a random sample from a bivariate distribution and $(r_1, s_1), \cdots, (r_n, s_n)$ be the ranks defined in Section 6. Using the notation $\#[i \mid C(i)]$ to mean the number of i which satisfies the condition $C(i)$, we define

$$Q=\mathrm{Max}\left\{\begin{array}{l} \#[i \mid r_i \text{ and } s_i<(n+1)/2]\,, \\ \#[i \mid r_i \text{ and } s_i>(n+1)/2] \end{array}\right\}. \tag{8.1}$$

It is easy to see that the difference of $\#[i \mid r_i$ and $s_i<(n+1)/2]$ and $\#[i \mid r_i$ and $s_i>(n+1)/2]$ is zero for even n and ±1 or zero for odd n. The Blomqvist rank statistic $(n_1-n_2)/(n_1+n_2)=Q'$ (say) [1] is equivalent to Q, since

$$Q'=\left\{\begin{array}{ll} 4Q/n-1 & \text{for even } n\,, \\ 4Q/(n-1)-1 & \text{for odd } n\,, \end{array}\right. \tag{8.2}$$

and hence Q itself may be called the Blomqvist rank statistic.

Easily we have the following

PROPOSITION 8.1. The Blomqvist statistic Q is nondecreasing.

In fact, $Q=[R_3]$, where [ ] denotes Gauss' symbol and R_3 is defined in Example 6.1. Hence we find by Theorem 6.1 that Q is stochastically not smaller under G than under F, provided G has larger regression dependence on x than F. If the assumption of 'larger regression dependence' is weakened to 'larger quadrant dependence', then we can state the following

THEOREM 8.1. *If a bivariate distribution $G(x, y)$ has larger quadrant* *dependence than $F(x, y)$, then the Blomqvist statistic Q is stochastically not smaller under G than under F, i.e.,*

$$(8.3) \qquad P_G\{Q \geqq c\} \geqq P_F\{Q \geqq c\} \qquad \textit{for any real } c.$$

PROOF. First suppose n is even, $n=2m$. Define two subsets of R^n,

$$S=\left\{(x_1, \cdots, x_n) \,\middle|\, \operatorname*{Max}_{1 \leqq i \leqq m} r_i = m\right\},$$

$$T=\{(y_1, \cdots, y_n) \mid \#[i \mid s_i \leqq m \text{ and } 1 \leqq i \leqq m] \geqq c\}$$

and let $I_S(x_1, \cdots, x_n)$ and $I_T(y_1, \cdots, y_n)$ denote the indicator functions of them. Then it will be readily verified that

$$P_F\{Q \geqq c\} = \binom{n}{m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} I_S(x_1, \cdots, x_n) \times I_T(y_1, \cdots, y_n) \prod_{i=1}^{n} dF(x_i, y_i).$$

A similar expression holds also for $P_G\{Q \geqq c\}$. Now both functions $I_S(x_1, \cdots, x_n)$ and $I_T(y_1, \cdots, y_n)$ are nondecreasing in each of their first m arguments and nonincreasing in each of the last m ones. Therefore, (8.3) follows from Proposition 7.4 (i).

Suppose now n is odd, $n=2m-1$. By virtue of the definition of Q it does not depend on the values taken by y_i as long as $r_i=m$. Defining

$$S=\left\{(x_1, \cdots, x_n) \,\middle|\, \operatorname*{Max}_{1 \leqq i \leqq m-1} r_i = m-1,\ r_m = m\right\},$$

$$T=\left\{(y_1, \cdots, y_n) \,\middle|\, \operatorname{Max}\begin{Bmatrix} \#[i \mid s_i < m \text{ and } 1 \leqq i < m] \\ \#[i \mid s_i > m \text{ and } m < i \leqq n] \end{Bmatrix} \geqq c\right\},$$

we find that

$$P_F\{Q \geqq c\} = \binom{n}{m-1} m \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} I_S(x_1, \cdots, x_n) \times I_T(y_1, \cdots, y_n) \prod_{i=1}^{n} dF(x_i, y_i)$$

and also that each of the two functions $I_S(x_1, \cdots, x_n)$ and $I_T(y_1, \cdots, y_n)$ is nondecreasing in each of the first $m-1$ arguments and nonincreasing in each of the last $m-1$ ones as before, whereas I_T is independent of y_m. Hence (8.3) holds by Proposition 7.4 (ii).

COROLLARY 8.1. *If a family of distributions $\{F_\alpha(x, y) \mid \alpha \in \Omega\}$, $\Omega \subset R^1$, has monotone quadrant dependence, then the Blomqvist statistic is stochastically nondecreasing with respect to α.*

THEOREM 8.2. *In testing the hypothesis of independence,*

$$H_0 : F(x, y) = F_1(x) F_2(y) \qquad \textit{for all } x \textit{ and } y\,,$$

for the family $\mathcal{F}_1$ *of distributions with continuous marginal distributions the Blomqvist test with a critical region* $Q \geqq c$ *is of Neyman-Pearson type, i.e.,* $P_F\{Q=c\}/P_{F_1F_2}\{Q=c\}$ *is nondecreasing in c, and a fortiori unbiased.*

PROOF. Suppose n is even, $n=2m$. It will be easily seen that

$$P_{F_1\cdot F_2}\{Q=c\} = \binom{m}{c}^2 \Big/ \binom{n}{m}\,,$$

$$P_F\{Q=c\} = \binom{n}{m}\binom{m}{c}^2 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} I_S(x_1, \cdots, x_n) \times I_{T_c}(y_1, \cdots, y_n) \prod_{i=1}^{n} dF(x_i, y_i)\,,$$

where

$$S = \left\{ (x_1, \cdots, x_n) \,\middle|\, \operatorname*{Max}_{i=1,\cdots,m} r_i = m \right\},$$

$$T_c = \left\{ (y_1, \cdots, y_n) \,\middle|\, \operatorname*{Max}_{\substack{i=1,\cdots,c,\\ m+c+1,\cdots,n}} s_i = m \right\}.$$

Thus we have only to show that

$$\text{(8.4)} \qquad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} I_S(x_1, \cdots, x_n) I_{T_c}(y_1, \cdots, y_n) dF(x_c, y_c) dF(x_{m+c}, y_{m+c})$$
$$\geqq \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} I_S(x_1, \cdots, x_n) I_{T_{c-1}}(y_1, \cdots, y_n) dF(x_c, y_c) dF(x_{m+c}, y_{m+c})\,.$$

Since $F \in \mathcal{F}_1$ and the functions I_S and I_{T_c} are concordant at either the cth or the $(m+c)$th argument, the first member of the inequality (8.4) is not smaller than

$$\text{(8.5)} \qquad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} I_S(x_1, \cdots, x_n) I_{T_c}(y_1, \cdots, y_n) dF_1(x_c) dF_2(y_c) \times dF_1(x_{m+c}) dF_2(y_{m+c})$$

by virtue of Proposition 7.4 (i). On the other hand, I_S and $I_{T_{c-1}}$ are discordant at each of these arguments, and hence the second member of (8.4) is not larger than the integral which is obtained by replacing T_c in (8.5) by T_{c-1}. These two integrals, however, take the same value, since the function $I_{T_{c-1}}$ can be obtained by only interchanging two arguments y_c and y_{m+c} in the function I_{T_c}. This proves (8.4).

The case when n is odd may be treated similarly.

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