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THE USE OF STOKES'S FORMULA IN GEODESY

In the summary of Sir Harold JEFFREYS's paper in the *Bulletin Géodésique* No. 30, Dec., 1953, page 337 we read :

> « The use of STOKES's formula to adjust survey data in present « circumstances is definitely undesirable, unless general agreement « is first obtained on the distribution of gravity to be adopted as a « basis of computation... »

As it is indeed a main object of geodesy today to do this very thing some comment seems called for. Sir Harold does not write as a geodesist and at times his meaning may be misjudged. Moreover he does not appreciate the aims of geodesy and is more concerned with those of astronomy. He appears to show that the ellipticity cannot be well determined from existing gravity data.

2. The geodesist desires to know N, the rise of the geoid above his reference figure — a spheroid, uniquely located. He requires N at a limited number of points, specifically at, or in relation with, the origins of each detached survey. At other points of these surveys changes of N can be found from deflexion surveys. Measured base-lines require the local N for proper reduction. Detached surveys are on different scales unless bases are reduced for heights above a common spheroid.

The geodesist also desires to know the deflexions of the vertical at the origin of each independent survey, in terms of a universal reference figure. Only when these two needs are met is he in a position to compute correctly the length and direction of a straight line joining two surveyed points belonging to different surveys.

3. The geodesist uses an ellipsoïd of revolution (spheroid) as basis of computations. This is defined by major axis length, $2a$, and flattening or ellipticity, ε. From it departures of the actual geoid, or even of the actual ground surface, can be expressed.

At present detached surveys, while employing spheroids of the same a and ε, unavoidably use spheroids having parallel rotation axes but different centres. Geodesists seek to overcome this fault by use of STOKES's formula.

4. The geodesist *need* not seek « improved » values of a and ε, even if these represent a spheroid closer to the geoid. If the geoid were an exact tri-axial ellipsoid it would be computationally inconvenient for use. The geoid can be delineated with all the accuracy observations allow, by reference to any reasonably chosen spheroid. It would be well if all great surveys were expressed in terms of the same elements. Unfortunately this is not the case; but appropriate allowances can be and are made as needed.

The geodesist is painfully conscious of the incompleteness not only of gravity data, but also of ground heights and ocean depths which affect his reductions. Efforts are continuously made to lessen these defects.

5. Meantime STOKES's formula has remained unused for over a century. Extended to take account of matter external to the geoid it may be written

Geoidal rise, $N' = N + \delta N$:

$$N = \frac{a}{4\pi G}\int(\Delta g_o + 4\pi\rho h)\, f d\omega = \frac{a}{2G}\int \Delta g_m f \sin\psi\, d\psi$$

where

$$2\pi\Delta g_m = \int_o^{2\pi} (\Delta g_o + 4\pi\rho h)\, d\chi$$

δN is a small correction, often negligible, due to external matter less its condensation on the geoid;

Δg_o is g_o, the actual gravity at geoidal level, less γ_o, the theoretical value consistent with the spheroid of reference;

h is the spirit levelled topographical height above the geoid (positive values only).

Clearly each region makes its contribution ∂N to N. Where data of either g or h are inadequate, the corresponding ∂N cannot be surely found. Clearly also, for determination of N at a particular point, some regions are of less importance than others. There must be considerable cancellation in the formation of Δg_m.

6. Observation yields point-values of g at ground (or submarine) level. Values at geoidal level have to be inferred; and for the quadratures of STOKES's integral representative values, freed of « station error », are needed.

No useful values of deflexions can be obtained from N gradients ignoring topography. Comparison has to be made with actual value at a point of observation. « Free Air » reduction will not serve for geodesy.

Various systems of reduction have been employed, depending on fictitious transfer of topography. Their use need not imply a slavish belief in some special form of isostasy; nor anything more than admission that they have been observed to lead to smaller anomalies than those found on the Free Air system ignoring topographic mass, and that they are more tractable for interpolation. Thus, on system (x) may be found the value of Δg_{ox} at geoidal level. Then

$$\Delta g_o = \Delta g_{ox} - X_o$$

in which $-X_o$ represents the effect at geoidal level of the restoration of topography or the reversal of its transfer, and the virtual elimination of the reduction system. Then

$$N_c = \frac{a}{4\pi G} \int g_{ox} f d\omega$$

and

$$N_s = \frac{a}{4\pi G} \int (4\pi\rho h - X_o) f d\omega$$

and

$$N = N_c + N_s$$

in which N_s is dependent wholly on topography transferred on the (x) system; and Δg_{ox}, freed from topographic disturbances, actual and hypothesised on the (x) system, should be more representative and more suited to interpolation than Δg_o. It should also benefit by the curvature of the (x) system level surfaces being less anomalous than those in nature.

7. In the light of wide experiences gained in gravity surveys limits which, on reasonable probability, Δg_{ox} will not exceed can be laid down; and these limits will be sensibly smaller than similar limits for Δg_o. From these an upper limit for ∂N_c over an area A can be estimeted.

As regards $\int (4\pi\rho h - X_o) f d\omega$, on some systems this becomes negligible beyond quite moderate ranges.

8. The upshot of this is that at a particular point and for a specified accuracy

(a) $\partial_a N$ can be properly computed for the area A,
(b) $\partial_b N$ is almost certainly negligible for area B,
(c) $\partial_c N$ is not calculable adequately for lack of data and its estimated upper limit is not negligible, for area C. The relative importance of subdivisions of C can be assessed.

Clearly data in parts of C should be strengthened by more observation of g, until they can be transferred to area A.

Alternatively an upper limit to the likely effects of unknown gravity anomalies over any specified area at various points of interest could be assessed.

It is important that points at which N is to be calculated from gravity residuals should be located most favourably from the point of view of available g-data. Differences of N can be found by deflexion traverses and the value of N brought up to any required point (Survey origin, for example).

It is desirable to inter-relate the origins of

North America
Europe and N. Africa
India and S. Asia
Central and S. Africa.

For this purpose classifications of the earth's surface as sketched above should be made for these domains; and from these the order of urgency of gravity survey would be apparent.

The degree of uncertainty in N as found from existing data could also be assessed.

9. Sir Harold JEFFREYS advocates the removal of harmonics up to order 4 from the STOKES function f. Thus the « Summary » ends;

> « — the difficulties arising from the use of the complete
> « STOKES formula would be greatly reduced by omission of the
> « terms in P_2, P_3, P_4 (Zonal harmonics). »

There is no doubt that what the geodesist wants is the full N. The correction applicable to a base line to reduce it to spheroidal level is the same whether the value of N derives from the second harmonic or the hundredth harmonic. The harmonic division is merely a matematical artifice until some physical cause is shown. A great merit of STOKES's formula is that no harmonic analysis of the gravity results is needed.

It would be computationally possible to obtain the partial value of N resulting from the process suggested; and then to complete N by addition of $a(v_2 + \frac{1}{2} v_3 + \frac{1}{3} v_4)$ *provided* the low harmonics of g, $G(v_2 + v_3 + ...)$, had been found. But I pose the question; — if gaps in gravity data make the evaluation of the STOKES integral imperfect, can it reasonably be expected that the same data can yield the few low order harmonics of g whereby these imperfections can be corrected ? In the first place geodesists require to find N at a handful of points and this makes a smaller demand on the data than would the adequate assessment of the low harmonics. If these points can be successfully treated they can be supplemented by geoidal sections based on deflexion observations, and the geoid can thus be determined over all land areas.

10. In a statement of the geodesist's use of STOKES's formula, fuller reference must be made to deflexions. Every independent survey

requires values of absolute deflexions relative to a uiversal spheroid, at one point of its triangulation network; and also the value of N. Equations for the two components of deflexions have been derived from the STOKES formula. If these are applied to gravity residuals on the (x) system, the derived deflexions ξ_c, η_c will require supplementing by ξ_s, η_s, the effects of the reversal of the topographic transfer of the (x) system. Thus the complete deflexions will be

$$\xi = \xi_c + \xi_s \qquad \eta = \eta_c + \eta_s.$$

These are point-values, suitable for direct comparison with astronomical observations of latitude and longitude if computation is made for the station or ground level, as may best be done. This allows two disconnected surveys to be put into the same terms; so that, if extended to meet, any discrepancy will be merely due to error of observation and not to reference system.

It is clear that deflexions are related to horizontal gradients of gravity anomaly. It is clear that harmonics of low order, unless of large amplitude, have small gradients.

The calculation of absolute deflexions from gravity anomalies in some geodetic cases is more important than the calculation of N. These absolute deflexions are also important in relation to lunar parallax observations.

11. The geoid needs to be much more completely surveyed, combining absolute determiations of N, ξ, η with geoidal sections from observed deflexions. Early efforts may be tentative and partial. They should at least show where data is insufficient and so guide the framing of observational programmes.

Merely to wait, inactive, until data happens to be fully abundant would be deplorable. On the contrary it is desirable to calculate results and to assess the precision which can be obtained from existing data, and to plan for improvement.

RESUMEN

El cálculo de las discordancias entre el geoide y una superficie de referencia única debe ser efectuado por la fórmula de STOKES, con origenes locales. Estos resultados son necesarios para permitir el enlace de levantamientos aislados.

Los errores que provengan de lagunas en los datos gravimétricos deberán ser examinados regionalmente, de modo que permita el establecimiento de un plan eficaz de medidas.

RÉSUMÉ

Le calcul des discordances entre le Géoïde et une surface de référence unique devrait être entrepris par la formule de STOKES aux origines des levés. Ces résultats sont nécessaires pour permettre le rattachement de levés isolés.

Les erreurs qui proviendraient des lacunes dans les données gravimétriques devraient être examinées régionalement, de manière à permettre l'établissement d'un plan efficace de mesures.

ZUSAMMENFASSUNG

Für die örtlichen Zentralpunkte sollten die Abstände zwischen dem Geoid und einer einheitlichen Referenzfläche mit Hilfe der STOKES'schen Formel berechnet werden. Man braucht diese Ergebnisse, um nicht zusammenhängende Landesaufnahmen miteinander zu verbinden.

Die Fehler, die durch Lücken im gravimetrischen Beobachtungs material verursacht werden, sollten gebietsweise untersucht werden, so daß man zweckmäßige Vorschläge für auszuführende Messungen machen könnte.

SUMMARY

The calculation of the elevations and inclinations of the geoid in relation to a unique international reference spheroid should be pursued by means of STOKES's formula at survey origins. Such results are required to enable detached geodetic surveys to be properly related and so to fulfil their comprehensive functions. Errors likely to arise from incompleteness of gravity data should be examined regionally so that supplementary observations of gravity may be planned and carried out with greatest effect for the aim in view.

SOMMARIO

Il calcolo degli scostamenti fra il geoide ed una superficie unica di riferimento dovrebbe essere intrapreso mediante la formula di Stokes con origini locali, e ciò onde consentire il raccordo di reti geodetiche isolate.

Gli errori che provengono da lacune nei dati gravimetrici dovrebbero essere esaminati regionalmente, in modo da consentire la preparazione di un efficiente piano di misure.