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CHANGES OF DEFLECTIONS OF THE PLUMB-LINE BROUGHT ABOUT BY A CHANGE OF THE REFERENCE-ELLIPSOID

The preceding paper deals with the changes in the deflections of the plumb-line caused by a shift of the ellipsoid of reference. It seems useful to extend this along similar lines to the variations of the deflections caused by a change of the elements a and $\alpha = \frac{a-b}{a}$ of the ellipsoid (1). For both cases of course the change of the S-N component of the deflection, taken with the negative sign, is identical with the change in the geographical latitude φ of the station on the ellipsoid. The change of the geographical longitude λ is found by multiplying the negative value of the E-W component of the change of the deflection by sec φ.

Assuming a system of stations where the deflections of the plumb-line have been determined, our problem now is to derive the changes brought about in these deflections by a variation of the equatorial radius a and the flattening α of the ellipsoid of reference; we suppose the components ξ_0 and η_0 of the deflection of the plumb-line in the base-station of the system to remain the same as well as the distance N_0 in this point from the geoid to the ellipsoid. This condition, therefore, involves that the ellipsoids before and after the change must be tangent to each other in the projection P_0 of the base-station on the two ellipsoids (2).

To solve our problem we shall first determine the shift of the ellipsoid's centre which has to accompany the variations Δa and $\Delta\alpha$ of a and α for satisfying the condition of tangency in P_0. We shall then proceed by deriving separately the effect of the changes Δa and $\Delta\alpha$ on the deflection of the vertical and on N_1 in an arbitrary point P_1 of

(1) LAMBERT requested the writer to take this up and he was glad to do so; the present paper gives the result.

(2) We may state here once for all that for the values of N occurring in practice we need not distinguish between a projection of a station from the geoid to the ellipsoid along the normal to the geoid or along the normal to the ellipsoid or along the curve between both surfaces that is everywhere tangent to the direction of gravity; it can easily be seen that their differences may be neglected.

the system when we keep the ellipsoid's centre at the same place and secondly the effect on those quantities of this centre's shift. For deriving this last effect we may use the formulas of the preceding paper. The combination of both effects will give us the resulting changes of the deflection of the plumb-line and of N_1 in the station P_1 which it is our purpose to derive.

In making these deductions we shall assume that the changes of a and b of the ellipsoid do not materially exceed an order of magnitude of 1/20.000th part of a resp. b and that we thus may neglect the squares of $\frac{\Delta a}{a}$ and $\frac{\Delta b}{b}$ as well as $\alpha^2 \frac{\Delta a}{a}$ and $\alpha^2 \frac{\Delta b}{b}$. We shall not neglect the order of $\alpha \frac{\Delta a}{a}$ and $\alpha \frac{\Delta b}{b}$.

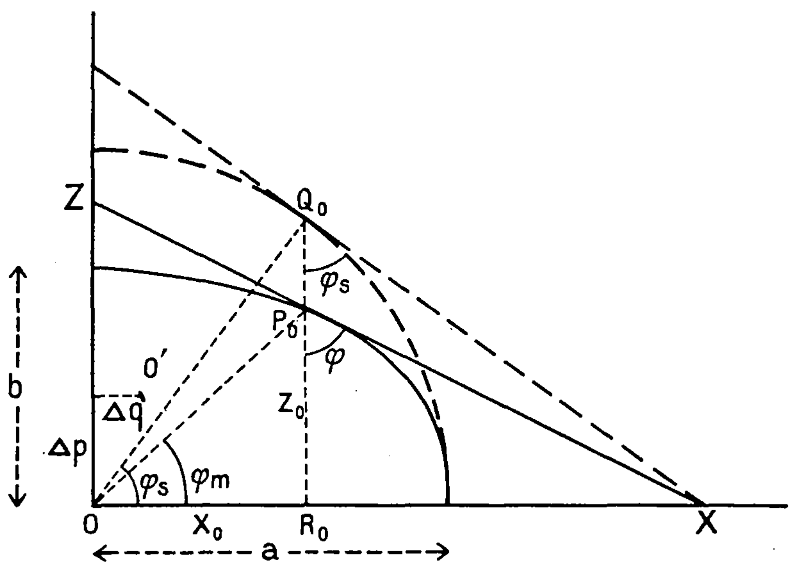

Fig. 1

The equation of the meridian through the base-point P_0 of the original ellipsoid having the equation :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

the line XZ tangent in P_0 has the equation :

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1. \tag{1}$$

Denoting the x and z components of the shift of the ellipsoid's centre which must accompany the changes Δa and $\Delta\alpha$ by Δq and Δp and giving the new coordinates with regard to the new centre O' an accent, the new shape of the equation of ZX becomes :

$$\frac{(x'+\Delta q)(x'_0+\Delta q)}{a^2}+\frac{(z'+\Delta p)(z'_0+\Delta p)}{b^2}=1. \qquad (2)$$

As ZX must be tangent in the same point P_0 to the new ellipsoid this equation must assume the same shape in the new coordinates as (1) if we introduce the new values $a'=a+\Delta a$ and $b'=b+\Delta b$ for a and b. Neglecting the squares of $\frac{\Delta a}{a}$, $\frac{\Delta b}{b}$, $\frac{\Delta q}{a}$ and $\frac{\Delta p}{b}$ we find this to be the case if

$$a'^2=a^2\left(1-\frac{x'_0}{a}\Delta q-\frac{z'_0}{b^2}\Delta p\right)\left(1-\frac{\Delta q}{x'_0}\right)$$

$$b'^2=b^2\left(1-\frac{x'_0}{a^2}\Delta q-\frac{z'_0}{b^2}\Delta p\right)\left(1-\frac{\Delta p}{z'_0}\right)$$

from which we derive :

$$\frac{\Delta a}{a}=-\frac{1}{2}\left(\frac{x'_0}{a}+\frac{a}{x'_0}\right)\frac{\Delta q}{a}-\frac{1}{2}\frac{z'_0}{b}\frac{\Delta p}{b} \qquad (3a)$$

$$\frac{\Delta b}{b}=-\frac{1}{2}\frac{x'_0}{a}\frac{\Delta q}{a}-\frac{1}{2}\left(\frac{z'_0}{b}+\frac{b}{z'_0}\right)\frac{\Delta p}{b} \qquad (3b)$$

and as :

$$\Delta\alpha=\alpha'-\alpha=\frac{b}{a}-\frac{b'}{a'}=\frac{b}{a}\left(\frac{\Delta a}{a}-\frac{\Delta b}{b}\right)=(1-\alpha)\left(\frac{\Delta a}{a}-\frac{\Delta b}{b}\right)$$

we find :

$$\Delta\alpha=\frac{1}{2}(1-\alpha)\left[-\frac{a}{x'_0}\frac{\Delta q}{a}+\frac{b}{z'_0}\frac{\Delta p}{b}\right] \qquad (3c)$$

Because of the negligibility of the above-mentioned second order quantities we can drop the accents of x_0 and z_0 in the equations (3) and we can neglect α^2 with regard to 1. Solving (3a) and (3c) with regard to $\frac{\Delta q}{a}$ and $\frac{\Delta p}{b}$ we obtain :

$$\frac{\Delta q}{a}=-\frac{x_0}{a}\left[\frac{\Delta a}{a}+(1+\alpha)\frac{z_0{}^2}{b^2}\Delta\alpha\right] \qquad (4a)$$

$$\frac{\Delta p}{b}=-\frac{z_0}{b}\left[\frac{\Delta a}{a}+(1+\alpha)\frac{z_0{}^2}{b^2}\Delta\alpha\right]+2(1+\alpha)\frac{z_0}{b}\Delta\alpha \qquad (4b)$$

We shall change now from x, y, z coordinates to latitude φ and longitude λ and in doing so we shall introduce temporarily three latitude-angles φ, viz. the geographical latitude φ, the geocentrical latitude φ_m and the corresponding spherical latitude φ_s (see fig. 1). We shall finally express the formulas only in φ. In fig. 1 three angles are indicated for the base-point P_0 but the following formulas are valid for all points of the meridian.

As : H

$$\frac{P_0R_0}{Q_0R_0} = \frac{b}{a}$$

we see that :

$$\operatorname{tg} \varphi_s = \frac{b}{a} \operatorname{tg} \varphi$$

and :

$$\operatorname{tg} \varphi_m = \frac{b}{a} \operatorname{tg} \varphi_s$$

and we easily derive for, the differences $\varphi - \varphi_s$ and $\varphi_s - \varphi_m$, breaking off after the square of the flattening :

$$\varphi - \varphi_s = \frac{1}{2}\alpha \sin 2\varphi_s + \frac{1}{2}\alpha^2 \cos^2 \varphi_s \sin 2\varphi_s + \ldots \qquad (5a)$$

or :

$$\varphi - \varphi_s = \frac{1}{2}\alpha \sin 2\varphi + \frac{1}{2}\alpha^2 \sin^2 \varphi \sin 2\varphi + \ldots \qquad (5b)$$

$$\varphi_s - \varphi_m = \frac{1}{2}\alpha \sin 2\varphi_s + \frac{1}{2}\alpha^2 \sin^2 \varphi_s \sin 2\varphi_s + \ldots \qquad (6a)$$

or :

$$\varphi_s - \varphi_m = \frac{1}{2}\alpha \sin 2\varphi_m + \frac{1}{2}\alpha^2 \cos^2 \varphi_m \sin 2\varphi_m + \ldots \qquad (6b)$$

(5a) and (6a) give us :

$$\varphi - \varphi_m = \alpha \sin 2\varphi_s + \frac{1}{2}\alpha^2 \sin 2\varphi_s + \ldots \qquad (7a)$$

which leads to :

$$\varphi - \varphi_m = \alpha \sin 2\varphi_m + \alpha^2 \cos \varphi_m \sin 3\varphi_m + \ldots \qquad (7b)$$

The formulas (4) can now be written as follows; we shall separate the terms with the factor α and bring them together in the second part; terms with α^2 are again neglected. For the point P_0 we provide the angles φ with an extra sub-index zero. For changing over from the angle φ_{so} to φ_o we make use of (5b) and for simplification we introduce a quantity :

$$\Delta\beta = \frac{\Delta a}{a} + \sin^2 \varphi_0 \Delta\alpha \qquad (8a)$$

We thus find :

$$\frac{\Delta q}{a} = -\cos \varphi_{so} \left[\Delta\beta + \alpha \sin^2 \varphi_o \Delta\alpha - \frac{1}{2}\alpha \sin^2 2\varphi_o \Delta\alpha \right]$$

or :

$$\frac{\Delta q}{a} = -\cos\varphi_o\Delta\beta - \alpha\sin^2\varphi_o\cos\varphi_o\,(\Delta\beta - \cos 2\varphi_o\Delta\alpha) \qquad (8b)$$

and :

$$\frac{\Delta p}{b} = -\sin\varphi_{so}\left[\Delta\beta + \alpha\sin^2\varphi_o\Delta\alpha - \frac{1}{2}\alpha\sin^2 2\varphi_o\Delta\alpha\right] + 2\,(1+\alpha)\sin\varphi_{so}\Delta\alpha$$

r :

$$= -\sin\varphi_o\,(\Delta\beta - 2\Delta\alpha) + \alpha\sin\varphi_o\cos^2\varphi_o\,(\Delta\beta - \cos 2\varphi_o\Delta\alpha) + \alpha\sin\varphi_o\Delta\alpha \qquad (8c)$$

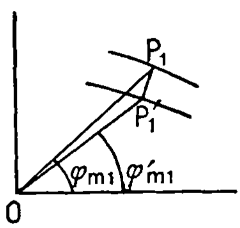

Fig. 2

We shall now determine the changes $\Delta\xi_1{}^I$, $\Delta\eta_1{}^I$, and $\Delta N_1{}^I$, of the meridian and prime vertical components ξ_1 and η_1 of the deflection of the plumb-line and of the distance N_1 between geoid and ellipsoid, which the changes Δa and $\Delta\alpha$ bring about in an arbitrary point N_1 of the original ellipsoid if during the change we keep the ellipsoid's centre at the same place. In doing so we assume that P_1 is projected on the new ellipsoid by a line in the sense of the normal on the ellipsoid in P_1; we denote this projection of P_1 on the new ellipsoid by P'_1. The line of projection $P_1P'_1$ has the length $\Delta N_1{}^I$. The geocentrical latitudes φ_{m1} and φ'_{m1} of P_1 and P'_1 differ only by a small amount caused by the small angle $\varphi_1 - \varphi_{m1}$ between the line of projection and the radius OP_1 of point P_1; this difference $\varphi_{m1} - \varphi'_{m1}$ is obviously given by :

$$\varphi_{m1} - \varphi'_{m1} = \frac{\Delta N_1{}^I}{a}(\varphi_1 - \varphi_{m1}) \qquad (9)$$

where the radius of P'_1 has been replaced by a.

As the angle $\varphi_1 - \varphi_{m1}$ has the order of magnitude of the flattening α the angle $\varphi_{m1} - \varphi'_{m1}$ has the order of magnitude of $\alpha\,\dfrac{\Delta N_1{}^I}{a}$, i. e. the smallest order of magnitude we do not neglect.

We can now write :

$$\Delta\xi_1{}^I = \varphi_1 - \varphi'_1 = \varphi_1 - \varphi_{m1} + \frac{\Delta N_1{}^I}{a}(\varphi_1 - \varphi_{m1}) - (\varphi'_1 - \varphi'_{m1}) \quad (10)$$

By making use of formula (7b) and of (11c) (to be derived below) we obtain :

$$\Delta\xi_1{}^I = -\sin 2\varphi_{m1}\Delta\alpha - 2\alpha\cos\varphi_1\sin 3\varphi_1\Delta\alpha - \alpha\sin 2\varphi_1\left(\frac{\Delta a}{a} - \sin^2\varphi_1\Delta\alpha\right)$$

giving :

$$\Delta\xi_1{}^I = -\sin 2\varphi_1\Delta\alpha - \alpha\sin 2\varphi_1\left(\frac{\Delta a}{a} + \cos^2\varphi_1\Delta\alpha\right) \quad (11a)$$

As it is clear that the change of the ellipsoid, as long as its centre is kept at its place, does not affect the longitude, we have :

$$\Delta\eta_1{}^I = 0 \quad (11b)$$

For deriving $\Delta N_1{}^I$, we make use of the development of the radius OP_1 :

$$r_1 = a\left[1 - \alpha\sin^2\varphi_{m1} - \frac{3}{2}\alpha^2(\sin^2\varphi_{m1} - \sin^4\varphi_{m1}) + \ldots\right]$$

and we obtain :

$$\frac{\Delta N_1{}^I}{a} = \frac{r_1 - r'_1}{a} = -\frac{\Delta a}{a} + \frac{\Delta a - \Delta b}{a}\sin^2\varphi_{m1} + 3\alpha\sin^2\varphi_1\cos^2\varphi_1\Delta\alpha$$

which leads to :

$$\frac{\Delta N_1{}^I}{a} = -\frac{\Delta a}{a} + \sin^2\varphi_1\Delta\alpha + \alpha\sin^2\varphi_1\left[\frac{\Delta a}{a} - \cos^2\varphi_1\Delta\alpha\right] \quad (11c)$$

The first two terms have been used in the above deduction of $\Delta\xi_1{}'$.

For deriving the second part of $\Delta\xi$, $\Delta\eta_1$ and $\frac{\Delta N_1}{a}$ which is caused by the shift of the ellipsoid's centre we may use the formulas 2A, 2B and 3A, 3B, 3C of the preceding paper, where $-p$, $-q$ and $-r$ represent the components of the shift. We have, therefore, to introduce $p = -\Delta p$, $q = -\Delta q$ and $r = 0$. We shall, furthermore, make use of :

$$e^2 = 2\alpha - \alpha^2$$

and, therefore, if neglecting α^2 and higher powers of α with regard to the unity :

$$\frac{a}{\rho_m} = \frac{W^3}{1-e^2} = \frac{(1-2\alpha\sin^2\varphi_1)^{\frac{3}{2}}}{1-2\alpha} = 1 + \alpha(2 - 3\sin^2\varphi_1) \quad (12a)$$

$$\frac{a}{\rho_m} = W = 1 - \alpha\sin^2\varphi_1 \quad (12b)$$

$$\frac{\Delta p}{a} = (1-\alpha)\frac{\Delta p}{b}. \quad (12c)$$

Whe thus obtain :

$$\Delta\xi_1{}^{II} = \left[1 + \alpha(2 - 3\sin^2\varphi_1)\right]\left[\cos\varphi_1 \frac{\Delta p}{b} - \sin\varphi_1 \frac{\Delta q}{a} + \right.$$

$$\left. + 2\sin\varphi_1 \sin^2\frac{1}{2}(\lambda_1 - \lambda_0)\frac{\Delta q}{a} - \alpha\cos\varphi_1 \frac{\Delta p}{b}\right]$$

$$\Delta\eta_1{}^{II} = \left[1 - \alpha\sin^2\varphi_1\right]\sin(\lambda_1 - \lambda_0)\frac{\Delta q}{a} \tag{13}$$

$$\frac{\Delta N_1{}^{II}}{a} = -\sin\varphi_1 \frac{\Delta p}{b} - \cos\varphi_1 \frac{\Delta q}{a} +$$

$$+ 2\cos\varphi_1 \sin^2\frac{1}{2}(\lambda_1 - \lambda_0)\frac{\Delta q}{a} + \alpha\sin\varphi_1 \frac{\Delta p}{b}$$

in which we have to introduce the values of $\frac{\Delta p}{b}$, and $\frac{\Delta q}{a}$ given by the formulas (8). Adding the formulas for $\Delta\xi_1{}^I$, $\Delta\eta_1{}^I$ and $\frac{\Delta N_1{}^I}{a}$ we obtain the final results of our deductions.

$$\Delta\xi_1 = \left[\sin(\varphi_1 - \varphi_0) - 2\cos\varphi_0 \sin\varphi_1 \sin^2\frac{1}{2}(\lambda_1 - \lambda_0)\right]\Delta\beta -$$

$$4\cos\varphi_1 \cos\frac{1}{2}(\varphi_1 + \varphi_0)\sin\frac{1}{2}(\varphi_1 - \varphi_0)\,\Delta\alpha +$$

$$+\left[\left(2 - 3\sin^2\varphi_1 - \cos(\varphi_1 + \varphi_0)\right)\sin(\varphi_1 - \varphi_0) - \right.$$

$$- 2\cos\varphi_0 \sin\varphi_1\left(2 + \sin^2\varphi_0 - 3\sin^2\varphi_1\right)\sin^2\frac{1}{2}(\lambda_1 - \lambda_0) -$$

$$\left. - \sin 2\varphi_0 \sin^2\frac{1}{2}(\varphi_1 - \varphi_0) - 2\cos\varphi_1 \cos\frac{1}{2}(\varphi_1 + \varphi_0)\sin\frac{1}{2}(\varphi_1 - \varphi_0)\right]\alpha\Delta\beta +$$

$$+\left[\frac{1}{2}\sin 4\varphi_0\left(\sin^2\frac{1}{2}(\varphi_1 - \varphi_0) + \sin\varphi_0 \sin\varphi_1 \sin^2\frac{1}{2}(\lambda_1 - \lambda_0)\right) - \right.$$

$$- \sin\varphi_0 \sin\frac{3}{2}(\varphi_1 + \varphi_0)\sin\frac{3}{2}(\varphi_1 - \varphi_0) -$$

$$- \sin\varphi_0 \sin\frac{1}{2}(\varphi_1 + \varphi_0)\sin\frac{1}{2}(\varphi_1 - \varphi_0) -$$

$$\left. \cos\varphi_1 \cos\frac{1}{2}(\varphi_1 + \varphi_0)\sin\frac{1}{2}(\varphi_1 - \varphi_0) + 4\sin 2\varphi_1 \cos^2\frac{1}{2}(\varphi_1 + \varphi_0)\sin^2\frac{1}{2}(\varphi_1 - \varphi_0)\right]\alpha\Delta\alpha \tag{14a}$$

$$\Delta\eta_1 = -\cos\varphi_0 \sin(\lambda_1 - \lambda_0)\,\Delta\beta\; +$$

$$+\cos\varphi_0 \sin(\lambda_1-\lambda_0)\left[\sin(\varphi_1-\varphi_0)\sin(\varphi_1+\varphi_0)\,\alpha\Delta\beta + \frac{1}{4}\,\mathrm{tg}\,\varphi_0 \sin 4\varphi_0\,\alpha\Delta\alpha\right] \tag{14b}$$

$$\frac{\Delta N_1}{a} = -2\left[\sin^2\frac{1}{2}(\varphi_1-\varphi_0) + \cos\varphi_0\cos\varphi_1\sin^2\frac{1}{2}(\lambda_1-\lambda_0)\right]\Delta\beta\; +$$
$$+\,4\cos^2\frac{1}{2}(\varphi_1+\varphi_0)\sin^2\frac{1}{2}(\varphi_1-\varphi_0)\,\Delta\alpha\; +$$
$$+\left[-\frac{1}{2}\sin 2\varphi_0\left(\sin(\varphi_1-\varphi_0) + 2\sin\varphi_0\cos\varphi_1\sin^2\frac{1}{2}(\lambda_1-\lambda_0)\right)+\right.$$
$$\left. +\,2\sin\varphi_1\cos\frac{1}{2}(\varphi_1+\varphi_0)\sin\frac{1}{2}(\varphi_1-\varphi_0)\right]\alpha\Delta\beta\; +$$
$$+\left[\frac{1}{2}\sin 4\varphi_0\left(\sin(\varphi_1-\varphi_0) + 2\sin\varphi_0\cos\varphi_1\sin^2\frac{1}{2}(\lambda_1-\lambda_0)\right)+\right.$$
$$+\,2\sin\varphi_0\cos\frac{1}{2}(\varphi_1+\varphi_0)\sin\frac{1}{2}(\varphi_1-\varphi_0)\; -$$
$$\left. -\cos^2\varphi_1\sin(\varphi_1+\varphi_0)\sin(\varphi_1-\varphi_0)\right]\alpha\Delta\alpha. \tag{14c}$$

As it ought to be, these formula's are annulled for $\varphi_1 = \varphi_0$ and $\lambda_1 = \lambda_0$. For $\Delta\xi_1$ and $\frac{\Delta N_1}{a}$ the terms containing the flattening α are complicated. For systems of no world-wide extent we can simplify them considerably. As $\alpha\Delta\alpha$ and $\alpha\Delta\beta$ are assumed to have an order of magnitude of 1:6.000.000 or less we may usually neglect in the factors by which they are multiplied the terms containing the squares and products of $\varphi_1 - \varphi_0$ and $\lambda_1 - \lambda_0$. If these angles e.g. are less than one tenth, which corresponds to a distance of some 600 km from the base-station, the squares and products obviously amount to an order of magnitude of 1:600.000.000.

For the case we may neglect them the factors of $\alpha\Delta\beta$ are annulled and those of $\alpha\Delta\alpha$ are reduced to a simple shape; for $\frac{\Delta N_1}{a}$ it is likewise annulled. We thus get for the complete formulas :

$$\Delta\xi_1 = \left[\sin(\varphi_1-\varphi_0) - 2\cos\varphi_0\sin\varphi_1\sin^2\frac{1}{2}(\lambda_1-\lambda_0)\right]\Delta\beta\; -$$
$$-\,4\cos\varphi_1\cos\frac{1}{2}(\varphi_1+\varphi_0)\sin\frac{1}{2}(\varphi_1-\varphi_0)\,\Delta\alpha\; - \tag{15a}$$
$$-\left(2+\frac{3}{4}\,\mathrm{tg}\,\varphi_0\sin 4\varphi_0\right)\sin(\varphi_1-\varphi_0)\,\alpha\Delta\alpha.$$

$$\Delta\eta_1 = -\cos\varphi_0 \sin(\lambda_1 - \lambda_0)\,\Delta\beta + \frac{1}{4}\sin\varphi_0 \sin 4\varphi_0 \sin(\lambda_1 - \lambda_0)\;\alpha\Delta\alpha. \quad (15b)$$

$$\frac{\Delta N_1}{a} = -2\left[\sin^2\frac{1}{2}(\varphi_1 - \varphi_0) + \cos\varphi_0 \cos\varphi_1 \sin^2\frac{1}{2}(\lambda_1 - \lambda_0)\right]\Delta\beta +$$

$$+ 4\cos^2\frac{1}{2}(\varphi_1 + \varphi_0)\sin^2\frac{1}{2}(\varphi_1 - \varphi_0)\,\Delta\alpha \quad (15c)$$

with :

$$\Delta\beta = \frac{\Delta a}{a} + \sin^2\varphi_0\Delta\alpha. \quad (8a)$$

The factors of $\sin(\varphi_1 - \varphi_0)$ resp. $\sin(\lambda_1 - \lambda_0)$ of the $\alpha\Delta\alpha$ terms may be tabulated for a special case in order to facilitate their computation.

As we have already mentioned in the beginning, the value of $\Delta\xi_1$ with the negative sign represents the change in geographical latitude of P_1 on the ellipsoid caused by the change of the elements a and α of the ellipsoid. In the same way the value of $\Delta\eta_1$, multiplied by $-\sec\varphi_1$ gives the change of the geographical longitude of P_1 on the ellipsoid by this same cause.

The first term of the formula for $\frac{\Delta N_1}{a}$ can be written in a simpler way if we plot the points P_0 and P_1 on a sphere with radius a at the same geographical latitudes and longitudes as they have on the ellipsoid and if we denote the geocentric angle between the radii to those images by (P_1P_0) and the length of the chord between the images by P_1P_0. It is easy to derive :

$$-2\left[\sin^2\frac{1}{2}(\varphi_1 - \varphi_0) + \cos\varphi_0 \cos\varphi_1 \sin^2\frac{1}{2}(\lambda_1 - \lambda_0)\right]\Delta\beta =$$

$$= -2\sin^2\frac{1}{2}(P_1P_0)\,\Delta\beta = -\frac{1}{2}\left(\frac{P_1P_0}{a}\right)^2\Delta\beta. \quad (15c')$$