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ON THE INTEGRAL FORMULAS OF STOKES AND VENING MEINESZ

ABSTRACT

Recently much work has been done concerning the behavior of the truncation errors of the integral formulas of Stokes and Vening Meinesz. In our paper we examine the theoretical foundations of truncation error behavior.

In reference [1] (de Witte 1967), de Witte uses a technique which he attributes to Molodenskii to analyze the truncation errors of the integral formulae of Stokes (for geoidal height) and Vening Meinesz (for deflections of the vertical). The method of Molodenskii can be generally described as follows.

Suppose we have a complete set of orthonormal functions $\{\phi_n(\mathbf{x})\}$ together with a weight function $\mathbf{w(x)}$ on the interval $[\mathbf{a}, \mathbf{b}]$. Let

$$\mathbf{f(x)} = \sum_{n=1}^{\infty} a_n \phi_n(\mathbf{x}) \tag{1}$$

and let $\mathbf{K(x)}$ be an arbitrary continuous kernel function. Now we wish to evaluate the integral

$$\mathbf{J}(\eta) = \int_{\mathbf{a}}^{\eta} \mathbf{K(x)\, f(x)\, w(x)\, dx} \tag{2}$$

for various values of η between $\mathbf{a}$ and $\mathbf{b}$, i.e., $\mathbf{a} < \eta \leq \mathbf{b}$. We define the truncated kernel $\overline{\mathbf{K}}(\mathbf{x})$ as follows :

$$\overline{\mathbf{K}}(\mathbf{x}) = \begin{cases} \mathbf{K(x)} & \text{for } \mathbf{a} \leq \mathbf{x} \leq \eta \\ \mathbf{0} & \text{for } \eta < \mathbf{x} \leq \mathbf{b} \end{cases} \tag{3}$$

The truncated kernel is now expanded in an orthogonal series (generalized Fourier series)

$$\bar{K}(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \tag{4}$$

The integral (2) is now evaluated in the following manner :

$$J(\eta) = \int_a^{\eta} K(x) f(x) w(x) \, dx = \int_a^b \bar{K}(x) f(x) w(x) \, dx \tag{5}$$

$$= \int_a^b \left[\Sigma \, c_n \phi_n(x) \;\; \Sigma \, a_n \phi_n(x) \right] w(x) \, dx$$

If we formally integrate term by term and apply the orthogonality properties of the ϕ_n 's, (5) reduces to

$$J(\eta) = \sum_{n=1}^{\infty} c_n a_n \tag{6}$$

We see immediately that the method of Molodenskii is not rigorously valid since in general $\bar{K}(x)$ is discontinuous at the truncation point η, thus the series (4) is only pointwise convergent, hence the term by term integration in (5) is not justified. However, in practice, the series (1) and (4) are both truncated after a finite number of terms and hence the resulting Molodenskii approximation (6) is indeed valid.

Since the gravity anomaly Δg is required over the entire earth for the integral formulae mentioned above, de Witte applied the Molodenskii analysis to the respective kernels in hopes of finding a practical truncation logic for each of the integrals. After generating graphs of the Fourier coefficients $\{c_n\}$ in (4) as a function of truncation angle, he noted that all of the coefficients (except a few of lowest order) damp very nearly to zero at the zeros of the Stokes and Vening Meinesz kernels. He therefore recommends adoption of a high order reference model (in the third order – seventh order range) to eliminate the few significant low order coefficients, truncate the Stokes integration at the first zero of the Stokes kernel (approximately 39°), and compute the deflections of the vertical by differencing two Stokes integrations since the first zero of the Vening Meinesz kernel is significantly more remote than that of Stokes.

The main purpose of this note is to point out that de Witte's results could have been predicted by the theory of Legendre series expansions and to provide a complete theory of truncation error models.

Let $S(x)$ be the Stokes kernel and $V(x)$ be the Vening Meinesz kernel ; let $\bar{S}(x)$ and $\bar{V}(x)$ be the respective truncated kernels. The approximating functions for $\bar{S}(x)$ are the $\{P_n(x)\}$, the Legendre polynomials ; the approximating functions for $\bar{V}(x)$ are the $\{P_n^1(x)\}$, the associated Legendre functions of degree one.

Thus

$$\bar{S}(x) = \sum_{n=0}^{\infty} \alpha_n Q_n P_n(x), \tag{7}$$

$$\bar{V}(x) = \sum_{n=0}^{\infty} \beta_n q_n P_n^1(x), \tag{8}$$

where $\{Q_n\}$, $\{q_n\}$ are the Fourier coefficients and $\{\alpha_n\}$, $\{\beta_n\}$ are the required normalizing factors,

$$\alpha_n = \frac{2n+1}{2}, \quad \beta_n = \frac{2n+1}{2n(n+1)}. \tag{9}$$

The theory states that if $\bar{S}(x)$ (or $\bar{V}(x)$) is continuous and of bounded variation over the interval of interest, and differentiable at the endpoints, that the associated series (7) (or (8)) converges absolutely and uniformly on the interval. Thus, assuming continuity, one would expect the coefficients Q_n (or q_n) to damp rapidly and uniformly to zero with increasing order. (For the case of classical Fourier series (see (Courant 1963)), the associated series of amplitudes is majorized by a series of the form $M \sum \frac{1}{n^2}$, where M is a constant and n is the order. Thus it is clear that classical Fourier coefficients decrease extremely rapidly with increasing order). We note that $\bar{S}(x)$ and $\bar{V}(x)$ are continuous only in case the respective functions are truncated at a zero. In general the truncated kernels are discontinuous at the truncation point so that the convergence of the associated series is only pointwise and thus it is impossible to predict any local uniform behavior of the Fourier coefficients.

Another interesting observation derives from the fact that

$$\bar{V}(x) = \bar{S}'(x)\, dx \tag{10}$$

where (10) is assumed valid only at points where the differentiation makes sense (i.e., at points other than the truncation point). Since the zeros of $S(x)$ and $V(x)$ occur at different points, the series (7) and (8) satisfy the conditions of the classical theorem on differentiation of series (see (Courant 1963)) only at the endpoints. Thus

$$q_n(1) = -n(n+1)\, Q_n(1) \tag{11}$$

at the surface of the sphere : and

$$q_n(-1) = Q_n(-1) = 0. \tag{12}$$

On the interior of the interval $[-1, 1]$ differentiation of (7) to obtain (8) is invalid and hence no relations such as (11) and (12) exist.

Our final comment deals with use of Stokes formula in a finite difference approximation for the deflections of the vertical. The curves displayed in de Witte's paper which give the truncation error of the Molodenskii geoidal slope calculation and usual Vening Meinesz formulae are clearly not the same. The cause of this discrepancy will now be discussed and a complete theory of truncation error models given.

Let (ϕ, λ) be ordinary geographic latitude and longitude and (α, δ) a coordinate system centered at the nadir point (ϕ_0, λ_0) where α is an azimuth and δ a spherical range angle (colatitude). Define $\mathbf{N}_\mathrm{P}^\mathrm{Q}$ to be the height of the geoid at a point **P** on the reference surface (determined by Stokes) calculated in an (α, δ) coordinate system centered at **Q**. We now define two different types of Stokes differences :

$$\xi = \frac{\mathbf{N}_{\mathrm{P}+\Delta\phi}^{\mathrm{P}} - \mathbf{N}_{\mathrm{P}}^{\mathrm{P}}}{\Delta\phi}\left(\frac{1}{\mathbf{a}}\right)$$

$$\eta = \frac{\mathbf{N}_{\mathrm{P}+\Delta\lambda}^{\mathrm{P}} - \mathbf{N}_{\mathrm{P}}^{\mathrm{P}}}{\Delta\lambda}\left(\frac{1}{\mathbf{a}\cos\phi_0}\right) \qquad \text{(S1)}$$

$$\xi = \frac{\mathbf{N}_{\mathrm{P}+\Delta\phi}^{\mathrm{P}+\Delta\phi} - \mathbf{N}_{\mathrm{P}}^{\mathrm{P}}}{\Delta\phi}\left(\frac{1}{\mathbf{a}}\right)$$

$$\eta = \frac{\mathbf{N}_{\mathrm{P}+\Delta\lambda}^{\mathrm{P}+\Delta\lambda} - \mathbf{N}_{\mathrm{P}}^{\mathrm{P}}}{\Delta\lambda}\left(\frac{1}{\mathbf{a}\cos\phi_0}\right) \qquad \text{(S2)}$$

where **a** is the radius of spherical reference surface **S**, and where ξ, η as usual denote the prime and meridional deflections of the vertical. The method (S1) is equivalent to performing Stokes at two neighboring points and differencing, the integration in each case being about the same point (i.e., the integrating coordinate system is fixed at one of the points rather than varying with the nadir point). This means that the spherical truncation caps over which the integrations are performed are identical for identical truncation angles, The method (S2) is equivalent to performing Stokes at two neighboring points and differencing, the integration in each case being about the respective nadir point, This means that the spherical truncation caps are slightly different for truncation angles of less than $\mathbf{180^\circ}$.

We now review the results of Vening Meinesz (Vening Meinesz 1928). The first technique of Vening Meinesz (the one which bears his name) obtains the deflections of the vertical as follows :

$$\xi = \frac{1}{2\pi\gamma} \int_S V(\delta)\, \Delta g\, (\alpha, \delta) \cos\alpha\, dS \qquad \text{(VM1)}$$

$$\eta = \frac{1}{2\pi\gamma} \int_S V(\delta)\, \Delta g\, (\alpha, \delta) \sin\alpha\, dS$$

where $V(\delta)$ is the Vening Meinesz kernel and γ is average gravity. Vening Meinesz' second method for obtaining the deflections is the following :

$$\xi = \frac{1}{2\pi\gamma} \int_S S(\delta) \frac{\partial \Delta g}{\partial \phi_0} dS$$

$$\eta = \frac{1}{2\pi\gamma} \int_S S(\delta) \frac{\partial \Delta g}{\partial \lambda_0} dS$$

where $S(\delta)$ is the Stokes kernel.

We now define two truncation error models for the deflections of the vertical. The first model, due to Cook (Cook 1950) and then de Witte, is given by :

$$\Delta\xi = \frac{KM}{2\gamma a^2} \sum_{n=0}^{\infty} (n-1)\, C_n^1\, q_n$$

$$\text{(M1)}$$

$$\Delta\eta = \frac{KM}{2\gamma a^2} \cos\phi_0 \sum_{n=0}^{\infty} (n-1)\, S_n^1\, q_n$$

where the $\{C_n^1\}$, $\{S_n^1\}$ are the spherical harmonic coefficients. The second model, due to Molodenskii, is given by :

$$\Delta\xi = \frac{1}{2\gamma} \sum_{n=2}^{\infty} Q_n \left.\frac{\partial \Delta g_n}{\partial \phi}\right|_{(\phi_0\, ,\, \lambda_0)}$$

$$\text{(M2)}$$

$$\Delta\eta = \frac{1}{2\gamma \cos\phi_0} \sum_{n=2}^{\infty} Q_n \left.\frac{\partial \Delta g_n}{\partial \phi}\right|_{(\phi_0\, ,\, \lambda_0)}$$

where Δg_n is the nth spherical harmonic component of Δg.

We will now show that (S1)$\Longleftrightarrow$(VM1)$\Longleftrightarrow$(M1), where the doubleheaded arrow symbolically indicates equivalent processes. The equivalence (VM1) $\Longleftrightarrow$ (M1) was shown by de Witte. The equivalence (S1) $\Longleftrightarrow$ (VM1) is shown in Figures 1 and 2. (For conciseness we limit ourselves to the meridional deflection, utilizing the same gravity model and nadir point used by de Witte). These two curves are identical up to numerical approximation.

We further demonstrate that (S2) $\Longleftrightarrow$ (VM2) $\Longleftrightarrow$ (M2). The equivalence (S2) $\Longleftrightarrow$ (VM2) is given in Figures 3 and 4, which coincide up to the accuracy of the computer program. The equivalence (VM2) $\Longleftrightarrow$ (M2) was conjectured by de Witte* and is demonstrated by Figure 5, which is a percentage error plot of Figure 4 and seen to be identical with the corresponding plot for (M2) which appears in de Witte's paper.

We have thus clarified de Witte's recommendation for obtaining deflections of the vertical by Stokes differences ; method (S2) must be used to obtain the truncation error behavior of Stokes formula. This author, however, recommends instead the process (VM2). The integration required, truncation behavior, and numerics are identical to a single Stokes integration rather than the two Stokes integrations required for (S2). The only additional work required is the differential filtering of the gravimetric data Δg, which only needs to be done once.

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* – Private communication.

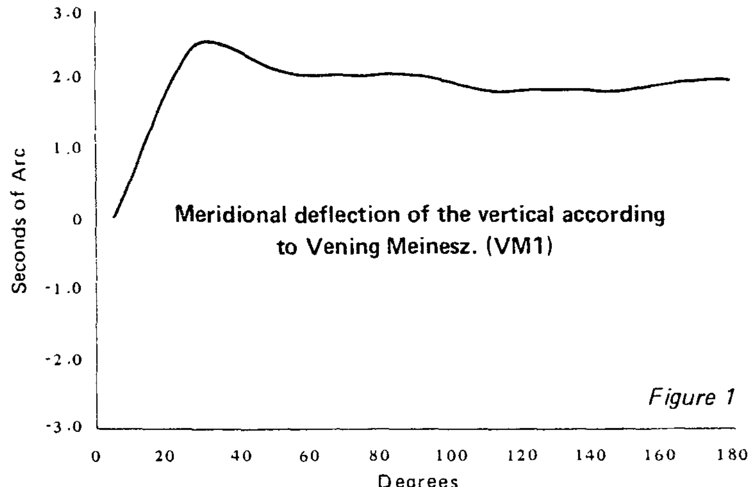

Figure 1

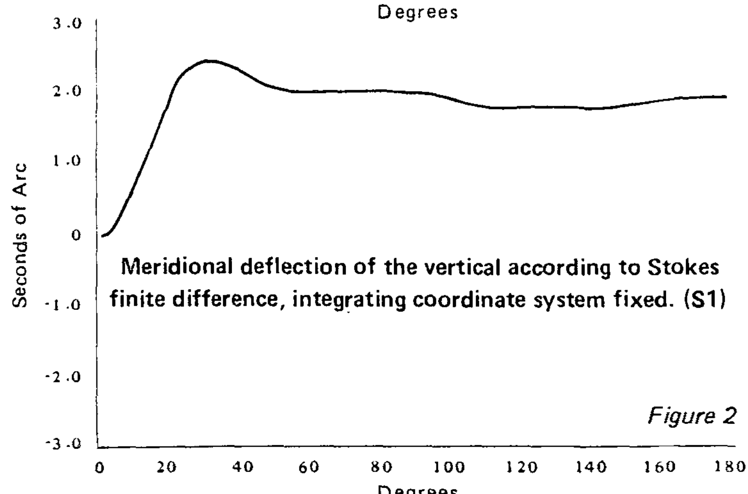

Figure 2

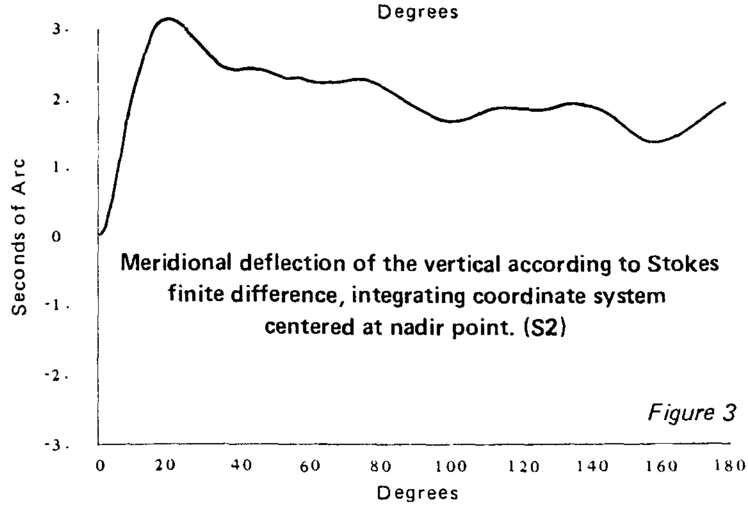

Figure 3

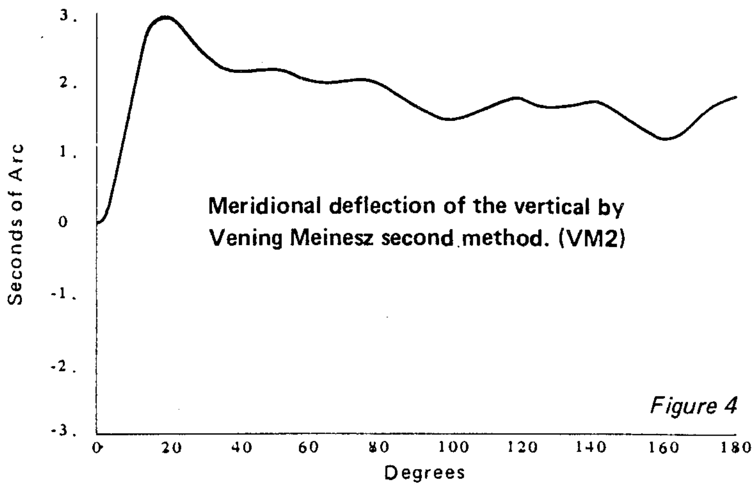

Figure 4

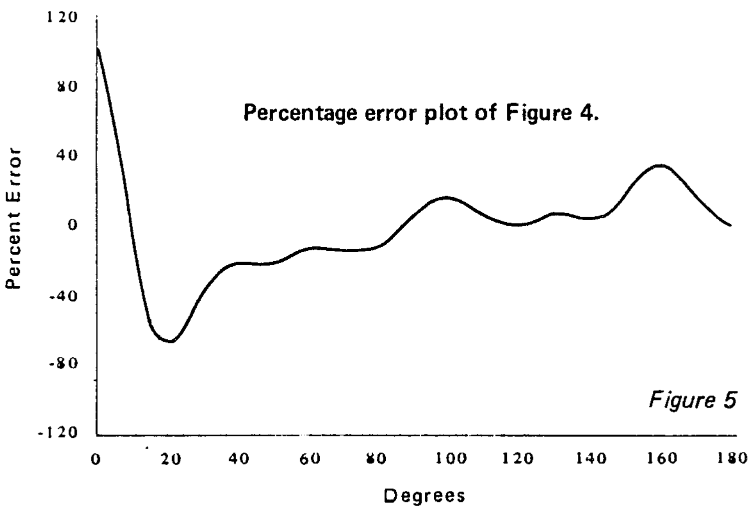

Figure 5