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A COMBINED (SPLINE–) INTERPOLATION AND SMOOTHING METHOD FOR THE DETERMINATION OF THE EXTERNAL GRAVITATIONAL POTENTIAL FROM HETEROGENEOUS DATA

Abstract

The mathematical framework for a spline method combining interpolation and smoothing of heterogeneous data is presented. The method is demonstrated for a spherical earth model. A spline approximation for the gravitational field is obtained by using a Hilbert space with topology induced by the (Laplace–) Beltrami operator of the sphere.

1. Introduction

In connection with the determination of the earth's gravitational potential there exists the problem that terrestrial methods and space techniques are providing us with data of very heterogeneous character and non–uniform distribution. Besides that these data sets are of different accuracies and partially affected with irregularities. Methods to handle these problems are well–known, for instance, least squares collocation, least squares adjustment, combinations and modifications of both (cf., e.g., Krarup (1969), Moritz (1972/73), Meissl (1976)).

Recently Freeden (1981b) presented a new approach by interpreting such least squares techniques as special transcriptions of spline procedures (cf. Anselone–Laurent (1968)) into the geodetic nomenclature. In this concept both least squares collocation and adjustment can be recognized by considerations given in parallel to the classical (one–dimensional) results (cf., e.g., Schoenberg (1964), Greville (1969)) as particular kinds of interpolation and smoothing by splines respectively. Interpolating and smoothing spline functions are characterized by "energy" minimum properties in the framework of a (semi–) Hilbert space.

The purpose of this paper is to supplement these investigations by a method combining spline interpolation and smoothing. As usual we introduce a reproducing Hilbert space $\mathcal{H}$ of functions harmonic in the exterior of the earth. We assume that the gravitational potential V of the earth is an element of $\mathcal{H}$. The totality of bounded linear functionals defined on $\mathcal{H}$ forms a linear space called the dual space $\mathcal{H}^*$. Linear functionals which give mappings from $\mathcal{H}$ to the real numbers $\mathcal{R}$ are of great importance in geodetic approximation problems, because an observed quantity can be interpreted as the value of a linear functional applied on an element of the Hilbert space $\mathcal{H}$. Let us suppose that as a result of observation or experience we have obtained the set of real numbers $M_i V = \mu_i$, $i = 1, \ldots, p$; $N_j V = \nu_j$; $j = 1, \ldots, q$, corresponding to a system of $(p+q)-$ functionals $M_1, \ldots, M_p$, $N_1, \ldots, N_q$ of $\mathcal{H}^*$. A method is presented giving as approximation to V a function $S \epsilon \mathcal{H}$ "smooth" with respect to a predefined "energy"— norm in $\mathcal{H}$ and satisfying the following properties :

(i) $M_i S$ is "near" $M_i V = \mu_i$, $i = 1, \ldots, p$

(ii) $N_j S$ is equal to $N_j V = \nu_j$, $j = 1, \ldots, q$.

Therefore this approximation procedure can be regarded as a spline method combining interpolation and smoothing to get an approximating function S to the gravitational potential V without large oscillations and undulations.

As in the paper by Freeden (1981b) the treatment is again intended to be elementary in the sense that classical developments given in the one–dimensional spline theory (cf. in particular Greville (1969, Chapt. 14)) are adapted to geodetic requirements. From a numerical point of view this seems to be most important, because we are now immediately able to extend a great number of computational procedures available and well–proved for one–dimensional splines to the geodetic case.

2. Approximation Method

The set of functions defined and harmonic in the earth's exterior E and regular at infinity constitutes a linear space. As is well–known, a subset of this class may become a separable Hilbert space $\mathcal{H}$ with reproducing kernel $K(.,.)$ by the introduction of a suitable scalar product $(.,.)$.

Let the gravitational potential V of the earth be considered as an element of the space $\mathcal{H}$.

In the Hilbert space $(\mathcal{H}, (.,.))$ any element F, especially the earth's gravitational potential V, may be represented by its expansion with respect to a complete, orthonormal system $\{Y_n\}_{n=0,1,2\ldots}$, i.e. :

$$F = \sum_{n=0}^{\infty} (F, Y_n) Y_n \tag{2.1}$$

in the sense of convergence in the Hilbert space topology, According to Parseval's identity we have

$$(F_1, F_2) = \sum_{n=0}^{\infty} (F_1, Y_n) \ (F_2, Y_n) \tag{2.2}$$

for all $F_1 \ F_2 \ \epsilon \ \mathcal{H}$.

Let h_m be the $(m+1)-$ dimensional linear space spanned by the functions $\{Y_n\}_{n=0,\ldots,m}$:

$$h_m = span \{ Y_o, \ldots, Y_m \}.$$

The linear space h_m is a Hilbert space with scalar product $(.,.)_{h_m}$ defined by

$$(F_1, F_2)_{h_m} = \sum_{n=0}^{m} (F_1, Y_n)(F_2, Y_n) \tag{2.3}$$

and reproducing kernel

$$k_m(x,y) = \sum_{n=0}^{m} Y_n(x) Y_n(y). \tag{2.4}$$

Let $h_m^{\perp}$ be the orthogonal complement of h_m in $\mathcal{H}$. The linear space $h_m^{\perp}$ is a Hilbert space with scalar product $(.,.)_{h_m^{\perp}}$ defined by

$$(F_1, F_2)_{h_m^{\perp}} = \sum_{n=m+1}^{\infty} (F_1, Y_n)(F_2, Y_n) \tag{2.5}$$

and reproducing kernel

$$k_m^{\perp}(x,y) = \sum_{n=m+1}^{\infty} Y_n(x) Y_n(y). \tag{2.6}$$

Hence, $\mathcal{H}$ is the orthogonal direct sum of h_m and $h_m^{\perp}$ with scalar product

$$(F_1, F_2) = (F_1, F_2)_{h_m} + (F_1, F_2)_{h_m^{\perp}} \tag{2.7}$$

and reproducing kernel

$$K(x,y) = k_m(x,y) + k_m^{\perp}(x,y). \tag{2.8}$$

Our approximation method now will be formulated for the linear space $\mathcal{H}$ equipped with the semi-inner product $(.,.)_{h_m^{\perp}}$ defined by (2.5), i.e., for the semi-inner product space $(\mathcal{H}\ (.,.)_{h_m^{\perp}})$.

Theorem 1 : Suppose that δ *and* $\beta_1^2, \dots, \beta_p^2$ *are prescribed positive weights. Let* $M_1, \dots, M_p$ *and* $N_1, \dots, N_q$ *be systems of bounded linear functionals on* $\mathcal{H}$ *such that the* $((m+1)+(p+q)) \times ((m+1)+(p+q))$ *– matrix*

$$\begin{pmatrix} \alpha & \beta & \xi \\ \beta' & \gamma & \zeta \\ \xi' & \zeta' & 0 \end{pmatrix} \tag{2.9}$$

is non–singular, where the matrices $\alpha, \beta, \gamma, \xi, \zeta$ *are given as follows*

$$\alpha = \left(M_i M_j k_m^{\perp}(.,.) + \delta \beta_i \delta_{ij}\right)_{\substack{i=1,\dots,p \\ j=1,\dots,p}} \quad (\delta_{ij} : \text{Kronecker symbol})$$

$$\beta = \left(M_i N_j k_m^{\perp}(.,.)\right)_{\substack{i=1,\dots,p\\ j=1,\dots,q}}$$

$$\gamma = \left(N_i N_j k_m^{\perp}(.,.)\right)_{\substack{i=1,\dots,q\\ j=1,\dots,q}}$$

$$\xi = \left(M_i Y_n\right)_{\substack{i=1,\dots,p\\ n=o,\dots,m}}$$

$$\zeta = \left(N_j Y_n\right)_{\substack{j=1,\dots,q\\ n=o,\dots,m}}$$

Suppose that $\mu \in \mathcal{R}^p$, $\mu' = (\mu_1, \dots, \mu_p)$; $\nu \in \mathcal{R}^q$, $\nu' = (\nu_1, \dots, \nu_q)$ *are given vectors, then the function* S *of the form*

$$S(x) = \sum_{i=1}^{p} a_i M_i k_m^{\perp}(.,x) + \sum_{j=1}^{q} b_j N_j k_m^{\perp}(.,x) + \sum_{n=0}^{m} c_n Y_n(x) \qquad (2.10)$$

with coefficients

$$a \in \mathcal{R}^p,\ a' = (a_1, \dots, a_p)\ ;\ b \in \mathcal{R}^q,\ b' = (b_1, \dots, b_q)\ ;\ c \in \mathcal{R}^{m+1},\ c' = (c_0, \dots, c_m)$$

uniquely determined by the linear equations

$$M_i S + \delta \beta_i^2 a_i = \mu_i \ , \qquad i = 1, \dots, p \qquad (2.11)$$

$$N_j S = \nu_j \ , \qquad j = 1, \dots, q \qquad (2.12)$$

$$\sum_{i=1}^{p} a_i M_i Y_n + \sum_{j=1}^{q} b_j N_j Y_n = 0 \ , \qquad n = o, \dots, m \qquad (2.13)$$

represents the only element of $\mathcal{H}$ *satisfying*

$$\sum_{i=1}^{p} \left(\frac{M_i S - \mu_i}{\beta_i}\right)^2 + \delta (S,S)_{h_m^{\perp}} \leqslant \sum_{i=1}^{p} \left(\frac{M_i F - \mu_i}{\beta_i}\right)^2 + \delta (F,F)_{h_m^{\perp}}$$

for all $F \in \mathcal{H}$ *with* $N_j F = \nu_j$, $j = 1, \dots, q$.

Proof :

Inserting the representation (2.10) into the equations (2.11–13) results in a linear system for the coefficients a_i, b_j, c_n , whose matrix is (2.9). Because the matrix (2.9) is assumed non-singular, the coefficients are uniquely determined.

Our purpose is to show that the function S such determined satisfies the property stated in Theorem 1.

For any function $F \in \mathcal{H}$ satisfying $N_j F = \nu_j$, $j = 1, \ldots, q$, it is easy to see that

$$(S, F)_{h\frac{\perp}{m}} = \sum_{i=1}^{p} a_i M_i F + \sum_{j=1}^{q} b_j \nu_j . \tag{2.14}$$

Solving (2.11) for a_i, multiplying by $M_i F$, and summing over $i = 1, \ldots, p$ gives

$$\sum_{i=1}^{p} a_i M_i F = \frac{1}{\delta} \sum_{i=1}^{p} M_i F \left(\frac{\mu_i - M_i S}{\beta_i^2} \right). \tag{2.15}$$

Combining with (2.14) yields

$$\delta (S, F)_{h\frac{\perp}{m}} = \sum_{i=1}^{p} M_i F \left(\frac{\mu_i - M_i S}{\beta_i^2} \right) + \delta \sum_{j=1}^{q} b_j \nu_j . \tag{2.16}$$

Now an elementary calculation gives

$$\begin{aligned} &\sum_{i=1}^{p} \left(\frac{M_i F - \mu_i}{\beta_i} \right)^2 + \delta (F, F)_{h\frac{\perp}{m}} \qquad (2.17) \\ &= \sum_{i=1}^{p} \left(\frac{M_i S - \mu_i}{\beta_i} \right)^2 + \delta (S, S)_{h\frac{\perp}{m}} \\ &+ \sum_{i=1}^{p} \left(\frac{M_i F - M_i S}{\beta_i} \right)^2 + 2 \sum_{i=1}^{p} \frac{(M_i F - M_i S)(M_i S - \mu_i)}{\beta_i^2} \\ &+ \delta (F - S, F - S)_{h\frac{\perp}{m}} + 2 \delta (S, F - S)_{h\frac{\perp}{m}} \end{aligned}$$

for every $F \in \mathcal{H}$.

According to (2.16) it follows that

$$2 \sum_{i=1}^{p} \frac{(M_i F - M_i S)(M_i S - \mu_i)}{\beta_i^2} + 2 \delta (S, F - S)_{h\frac{\perp}{m}} = 0 \tag{2.18}$$

provided that $N_j F = \nu_j$, $j = 1, \ldots, q$.

Therefore,

$$\sum_{i=1}^{p} \left(\frac{M_i F - \mu_i}{\beta_i} \right)^2 + \delta (F, F)_{h\frac{\perp}{m}} \tag{2.19}$$

$$= \sum_{i=1}^{p} \left(\frac{M_i S - \mu_i}{\beta_i} \right)^2 + \delta (S, S)_{h_m^\perp}$$

$$+ \sum_{i=1}^{p} \left(\frac{M_i F - M_i S}{\beta_i} \right)^2 + \delta (F - S, F - S)_{h_m^\perp}$$

for all $F \in \mathcal{H}$ with $N_j F = \nu_j$, $j = 1, \ldots, q$.

But this implies that

$$\sum_{i=1}^{p} \left(\frac{M_i S - \mu_i}{\beta_i} \right)^2 + \delta (S, S)_{h_m^\perp} \tag{2.20}$$

$$\leqslant \sum_{i=1}^{p} \left(\frac{M_i F - \mu_i}{\beta_i} \right)^2 + \delta (F, F)_{h_m^\perp}$$

for all $F \in \mathcal{H}$ satisfying the interpolating constraints

$N_j F = \nu_j$, $j = 1, \ldots, q$.

Equality in (2.20) holds if and only if

$$\sum_{i=1}^{p} \left(\frac{M_i F - M_i S}{\beta_i} \right)^2 + \delta (F - S, F - S)_{h_m^\perp} = 0, \tag{2.21}$$

i.e. : $F - S \in h_m$ with $M_i (F - S) = 0$, $i = 1, \ldots, p$, and

$N_j (F - S) = 0$, $j = 1, \ldots, q$.

As a function of the space h_m, $F - S$ has the representation

$$F - S = \sum_{n=0}^{m} d_n Y_n \tag{2.22}$$

where the coefficients $d \in \mathbb{R}^{m+1}$, $d' = (d_0, \ldots, d_m)$, satisfy the linear equation system

$$\begin{pmatrix} \alpha & \beta & \xi \\ \beta' & \gamma & \zeta \\ \xi' & \zeta' & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.23}$$

Since the matrix (2.9) is invertible it is obvious that

$d_o = \ldots = d_m = 0$.

Therefore we have $F = S$. The element $S \in \mathcal{H}$ is the only function in $\mathcal{H}$ having the minimal property (2.20) and satisfying interpolating constraints. Our approximation method can be regarded as a compromise between interpolating and smoothing. These two concepts were considered separately in Freeden (1981b).

The values $\mu_1, \ldots, \mu_p, \nu_1, \ldots, \nu_q$ are the observed quantities. In Theorem 1 we have shown that there is a unique function $S \in \mathcal{H}$ satisfying

$M_i S$ be "near" $M_i V = \mu_i$, $i = 1, \ldots, p$

and

$N_j S$ be equal to $N_j V = \nu_j$, $j = 1, \ldots, q$.

The "nearness" of the values $M_i S$ to μ_i, $i = 1, \ldots, q$, can be controlled by choosing the constant δ in a suitable way. A small value of δ emphasizes fidelity to the observed data at the expense of smoothness, while a large value does the opposite.

Taking $\delta = 0$ yields $M_i S = \mu_i$, $i = 1, \ldots, p$, i.e., the combined smoothing and interpolation procedure leads back to strict interpolation.

For numerical purposes it is advantageous to adapt the quantities $\beta_1^2, \ldots, \beta_p^2$ to the standard deviations of the measured values.

Our investigations have been formulated under pre–defined Hilbert space topology. As in collocational theory (cf., e.g., Eeg–Krarup (1975), Sjöberg (1975) Tscherning (1977)), it remains a challenge in both theory and practice to select Hilbert spaces of simple nature and practical value. Physical interpretations of the (semi–) topology induced by $(.,.)_{h_m^\perp}$ in $\mathcal{H}$ as mentioned in the following example might give a deeper insight into this problem.

Similar methods have been proposed by Moritz (1973), Tscherning (1974), Tscherning–Rapp (1974).

3. A simple example for a spherical earth model

Let E be the "outer space" of the unit sphere Ω in Euclidean space $\mathbb{R}^3$. As usual, denote by $S_{n,1}, \ldots, S_{n,2n+1}$ a (maximal) system of spherical harmonics of order n orthonormalized in the sense of the metric of the Hilbert space $L^2(\Omega)$ of square–integrable functions. We set

$$H_{n,j}(x) := |x|^{-(n+1)} \; S_{n,j}(z), \quad x = |x| \, z, \quad z \in \Omega \tag{3.1}$$

$$(n = 0, 1, \ldots, \; j = 1, \ldots, 2n+1).$$

Let us construct a Hilbert space $\mathcal{H}$ by choosing the scalar product

$$(F_1, F_2) = (F_1, F_2)_{h_0} + (F_1, F_2)_{h_0^\perp} \tag{3.2}$$

where

$$(F_1, F_2)_{h_0} = \int_\Omega F_1(x) H_{0,1}(x)\, d\omega \int_\Omega F_2(x) H_{0,1}(x)\, d\omega$$

and

$$(F_1, F_2)_{h_0^\perp} = \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} [n(n+1)]^2 \int_\Omega F_1(x) H_{n,j}(x)\, d\omega \times$$

$$\times \int_\Omega F_2(x) H_{n,j}(x)\, d\omega . \qquad (d\omega : \text{surface—element})$$

Since the spherical harmonics are the eigenfunctions of the Beltrami operator Δ^* with respect to the eigenvalues $\lambda_n = n(n+1)$ it follows that

$$-n(n+1) \int_\Omega F_i(x) H_{n,j}(x)\, d\omega = \int_\Omega F_i(x) [\Delta_x^* H_{n,j}(x)]\, d\omega \qquad (3.3)$$

holds for $i = 1, 2$.

By Green's surface identity we obtain

$$\int_\Omega F_i(x) [\Delta_x^* H_{n,j}(x)]\, d\omega = \int_\Omega [\Delta_x^* F_i(x)] H_{n,j}(x)\, d\omega . \qquad (3.4)$$

According to Parseval ' s identity in $L^2(\Omega)$ we find

$$(F_1, F_2)_{h_0^\perp} = \int_\Omega [\Delta_x^* F_1(x)][\Delta_x^* F_2(x)]\, d\omega . \qquad (3.5)$$

It should be noted that

$$(F, F)_{h_0^\perp} = \int_\Omega [\Delta_x^* F(x)]^2\, d\omega \qquad (3.6)$$

may be physically interpreted (at least under some simplifying assumptions) as the bending energy of a thin membrane spanned wholly over the (unit) sphere, F denoting the deflection normal to the rest position supposed of course to be spherical.

This model is obviously suggested by the interpretation of the integral over the square of the (linearized) curvature as the potential energy of a statically deflected thin beam.

In $\mathcal{H}$ the functions

$$Y_{n,j} = \begin{cases} H_{n,j} & \text{for } n = o, j = 1 \\ -\dfrac{1}{n(n+1)} H_{n,j} & \text{for } n > o, j = 1, \ldots, 2n+1 \end{cases}$$

constitute a Hilbert—basis (cf. Meissl (1976) , chapt. 11) .

The reproducing kernel has the representation

$$K(x,y) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} Y_{n,j}(x)\, Y_{n,j}(y) \tag{3.7}$$

i.e. :

$$K(x,y) = k_0(x,y) + k_0^{\perp}(x,y)\ , \tag{3.8}$$

where

$$k_0(x,y) = \frac{1}{4\pi\, |x|\ |y|} \tag{3.9}$$

and

$$k_0^{\perp}(x,y) = \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{[n(n+1)]^2}\, H_{n,j}(x)\, H_{n,j}(y)\,. \tag{3.10}$$

For elements $x\,,\ y \in \Omega$, the kernel $k_0^{\perp}(x,y)$ is the first iterated Green function of the unit sphere with respect to the Laplace–Beltrami operator Δ^* on Ω and the eigenvalue $\lambda_0 = 0$ (cf. Freeden (1979 / 1980)).

Using the terminology given above our results can be summarized as follows : Let $M_1, \ldots, M_p$ and $N_1, \ldots, N_q$ be systems of bounded linear functionals on $\mathcal{H}$ satisfying the assumptions of Theorem 1 .

Then there exists a unique function $S \in \mathcal{H}$ which minimizes

$$\sum_{i=1}^{p} \left(\frac{M_i F - \mu_i}{\beta_i} \right)^2 + \delta \int_{\Omega} [\Delta_x^* F(x)]^2\, d\omega \tag{3.11}$$

subject to the constraints

$$N_j F = \nu_j \ , \quad j = 1, \ldots, q\,.$$

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REFERENCES

P.M. ANSELONE, P.J. LAURENT : A general method for the construction of interpolating and smoothing spline functions, Num. Math., *12,* 66–82 (1968).

A. BJERHAMMAR : Discrete approaches to the solution of the boundary value problem in physical geodesy. Boll. Geod. Sci. Aff., *34,* 185–240 (1974).

K. BÖHMER : Spline–Funktionen, Stuttgart : B.G. Teubner (1974).

C. DE BOOR, R.E. LYNCH : On splines and their minimum properties. J. Math. Mech., *15,* 953–969 (1966).

J. EEG, T. KRARUP : Integrated geodesy, Methoden und Verfahren der math. Physik, (editors : B. Brosowski, E. Martensen). B.I. Wissenschaftsverlag, Bd. 13, 77–123 (1975).

W. FREEDEN : Über eine Klasse von Integralformeln der mathematischen Geodäsie. Veröff. Geod. Inst. RWTH Aachen, Report No. 27 (1979).

W. FREEDEN : On integral formulas of the unit sphere and their application to numerical computation of integrals. Computing, *25,* 131–146 (1980).

W. FREEDEN : On spherical spline interpolation and approximation. Math. Meth. in the Appl. Sci., *3,* 551–575, (1981a).

W. FREEDEN : On approximation by harmonic splines. Manuscripta geodaetica, 193–244 , (1981b).

T.N.E. GREVILLE : Introduction to spline functions in : Theory and applications of spline functions. Academic Press, (1969).

T. KRARUP : A contribution to the mathematical foundation of physical geodesy. Geod. Inst. København, Meddelelse No. 44, (1969).

P. MEISSL : Elements of functional analysis, in : Methoden und Verfahren der math. Physik (editors : B. Brosowski, E. Martensen) B.I. Wissenschaftsverlag, Bd. 12, 19–78 (1975).

P. MEISSL : Hilbert spaces and their application to geodetic least squares problems. Boll. Geod. Sci. Aff., Vol. XXXV, No. 1, 181–201 (1976).

H. MORITZ : Advanced least–squares methods. Dept. of Geod. Science, OSU, Report No. 175 (1972).

H. MORITZ : Stepwise and sequential collocation. Dept.of Geod. Science, OSU, Report No. 203, (1973).

H. MORITZ : Interpolation and approximation, in : Approximation methods in geodesy (editors : H. Moritz, H. Sünkel), Sammlung Wichmann, Bd. 10, 1–45, (1978).

H. MORITZ : Advanced physical geodesy. Sammlung Wichmann, Bd. 13, (1980).

I.J. SCHOENBERG : On trigonometric spline interpolation. J. Math. Mech., *13,* 785–825 (1964).

L. SJÖBERG : On the discrete boundary value problem of physical geodesy with harmonic reductions to an internal sphere. The Royal Institute of Technology, Division of Geodesy, Stockholm (1975).

C.C. TSCHERNING : A FORTRAN IV program for the determination of the anomalous potential using stepwise least squares collocation.Dept. of Geodetic Science, OSU, Report No. 212 (1974).

C.C. TSCHERNING : A note on the choice of norm when using collocation for the computation of approximations to the anomalous potential. Bull. Geod., *51,* 137–147 (1977).

C.C. TSCHERNING, R.H. RAPP : Closed covariance expressions for gravity anomalies, geoid undulations and deflections of the vertical implied by anomaly degree–variance models. Dept. of Geodetic Science, OSU, Report No. 208, (1974).

E.T. WHITTAKER : On a new method of graduation. Proc. Edinburgh Math. Soc., *41.* 63–75 (1923).

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